MATH4999 — Capstone Projects in Mathematics and Economics

Solution to Homework Four

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1. (a) When $S = 2$, due to the satisfaction of the quota property, one state is rounded up while the other state is rounded down (the very special case of both q_1 and q_2 being integer valued can be easily dealt with as a separate case). Without loss of generality, suppose
$$a_1^h = \lceil q_1^h \rceil \quad \text{and} \quad a_2^h = \lfloor q_2^h \rfloor,$$
where the superscript represents the house size. Alabama paradox occurs if and only if
$$a_1^{h+1} = \lceil q_1^h \rceil - 1 \quad \text{and} \quad a_2^{h+1} = \lfloor q_2^h \rfloor + 2.$$
In order to secure two additional seats for state 2, the new fractional remainder for state 2 has to be larger than that of state 1. The occurrence of the Alabama paradox would imply
$$q_2^{h+1} - q_2^h > 1.$$
However, with an increase of only one seat in the house size, we observe
$$\begin{aligned} &q_1^{h+1} - q_1^h > 0, \quad q_2^{h+1} - q_2^h > 0 \text{ and} \\ &(q_1^{h+1} - q_1^h) + (q_2^{h+1} - q_2^h) = 1 \end{aligned}$$
so that
$$0 < q_2^{h+1} - q_2^h < 1 \quad \text{and} \quad 0 < q_1^{h+1} - q_1^h < 1.$$
A contradiction is encountered.

   (b) With an increase in the house size, q_i increases so that
$$q_i^{old} < q_i^{new} \text{ which implies } \lfloor q_i^{old} \rfloor \leq \lfloor q_i^{new} \rfloor.$$
A loss of more than one seat would imply
$$\begin{aligned} a_i^{new} &\leq a_i^{old} - 2 \leq \lfloor q_i^{old} \rfloor + 1 - 2 \\ &= \lfloor q_i^{old} \rfloor - 1 \leq \lfloor q_i^{new} \rfloor - 1. \end{aligned}$$
This leads to a violation of the lower quota property.

2. (a) It suffices to show that if an optimal choice has been made under Hill's method, then interchanging a single seat between 2 states r and s reduce $\displaystyle\sum_{i=1}^{S} \frac{1}{a_i}(a_i - q_i)^2$. We prove by contradiction. Suppose an interchange is possible from state r with $a_r > 0$ to state s with $a_s \geq 0$, then
$$\begin{aligned} &\frac{(a_r - 1 - q_r)^2}{a_r - 1} + \frac{(a_s + 1 - q_s)^2}{a_s + 1} < \frac{(a_r - q_r)^2}{a_r} + \frac{(a_s - q_s)^2}{a_s} \\ \Leftrightarrow \quad &\frac{q_r^2}{a_r - 1} + \frac{q_s^2}{a_s + 1} < \frac{q_r^2}{a_r} + \frac{q_s^2}{a_s} \\ \Leftrightarrow \quad &\frac{q_r}{\sqrt{a_r(a_r - 1)}} < \frac{q_s}{\sqrt{a_s(a_s + 1)}}. \end{aligned}$$

This is a violation to the property that

$$\max_i \frac{q_i}{\sqrt{a_i(a_i+1)}} \le \min_i \frac{q_i}{\sqrt{a_i(a_i-1)}}.$$

(b) In the lecture note, Webster's Method has been shown to minimize

$$\begin{aligned}
\overline{s} &= \sum_{i=1}^{S} p_i \left(\frac{a_i}{p_i} - \frac{h}{P}\right)^2 \\
&= \sum_{i=1}^{S} p_i \left(\frac{a_i - q_i}{p_i}\right)^2 \quad \left(\text{since } q_i = p_i \frac{h}{P}\right) \\
&= \frac{h}{P} \sum_{i=1}^{S} \frac{(a_i - q_i)^2}{q_i},
\end{aligned}$$

so Webster's Method also minimizes $\sum_{i=1}^{S} \frac{1}{q_i}(a_i - q_i)^2$, which is a scalar multiple of $\overline{s}$.

3. (a) To observe the minimum requirement that every state receives at least one seat, we take the maximum between 1 and $\left\lfloor \left\lfloor \frac{p_i}{\lambda_i} \right\rfloor \right\rfloor$. It may occur that

$$\sum_{i=1}^{S} \left\lfloor \frac{p_i}{\lambda_i} \right\rfloor = h$$

does not have a solution for any positive λ due to the occurrence of a tie between two or more states, where $\frac{p_i}{\lambda}$ happen to be integer valued in two or more states at some value of λ. Note that $\sum_{i=1}^{S} \left\lfloor \frac{p_i}{\lambda} \right\rfloor$ is a non-increasing step function of λ. When a tie occurs, its value may jump across a particular integer h, say from $h+1$ to $h-1$ without taking the value h for any choice of positive λ.

As a numerical example, take $p_1 = p_2 = 90,000$ and $h = 19$. When $\lambda = 9,000$, $\sum_{i=1}^{2} \left\lfloor \frac{p_i}{\lambda} \right\rfloor = 20$; and when $9,000 < \lambda \le 10,000$, $\sum_{i=1}^{2} \left\lfloor \frac{p_i}{\lambda} \right\rfloor = 18$.

(b) For each $i \in \mathcal{S}$, we have $1 < a_i = \frac{p_i}{\lambda} - y_i$, when $0 \le y_i \le 1$. Note that $y_i = 0$ when $\frac{p_i}{\lambda}$ happens to be an integer. Since $\lambda = \frac{p_i}{a_i + y_i}$ and so

$$\lambda \le \left.\frac{p_i}{a_i}\right|_{i\in\mathcal{S}} \quad \text{and} \quad \lambda \ge \left.\frac{p_i}{a_i+1}\right|_{i\in\mathcal{S}}$$

so

$$\max_{i\in\mathcal{S}} \frac{p_i}{a_i+1} \le \lambda \le \min_{i\in\mathcal{S}} \frac{p_i}{a_i}.$$

It may occur that $\frac{p_i}{\lambda} < 1 \Leftrightarrow \lambda > p_i$ but a_i is set equal to 1 due to the minimum requirement. Therefore, the right side inequality has to exclude the case $a_i = 1$. However, the left side inequality remains valid for all i.

4. Given the choice of the rank index $r(p,a) = \frac{p}{2a(a+1)/(2a+1)}$, suppose a_i seats have been allocated to state i and a_j seats have been allocated to state j, an additional seat will be allocated to state i if and only if

$$\begin{aligned} & \frac{p_i(2a_i+1)}{2a_i(a_i+1)} \geq \frac{p_j(2a_j+1)}{2a_j(a_j+1)} \\ \Leftrightarrow \quad & \frac{p_j}{a_j} - \frac{p_i}{a_i+1} \leq \frac{p_i}{a_i} - \frac{p_j}{a_j+1}. \end{aligned}$$

We deduce that the corresponding test of inequality is

$$\frac{p_i}{a_i} - \frac{p_j}{a_j}.$$

5. Consider the following population data

State	A	B	C	D	E
Population	246	1771	1529	6521	6927
q_i	0.1737	1.2505	1.0796	4.6046	4.8913

with house size $h = 12$

(i) Hamilton's method

$$a_A = 1, a_B = 1, a_C = 1, a_D = 4, a_E = 5.$$

(ii) Jefferson's method

Take $\lambda = 1200$, $a_i \left\lfloor \left\lfloor \frac{p_i}{\lambda} \right\rfloor \right\rfloor$, then $a_A = 0, a_B = 1, a_C = 1, a_D = 5, a_E = 5$.

(iii) Webster's method

Take $\lambda = 1300$; $a_A = 0, a_B = 1, a_C = 1, a_D = 5, a_E = 5$

(iv) Hill's method: $a_A = 1, a_B = 1, a_C = 1, a_D = 4, a_E = 5$.

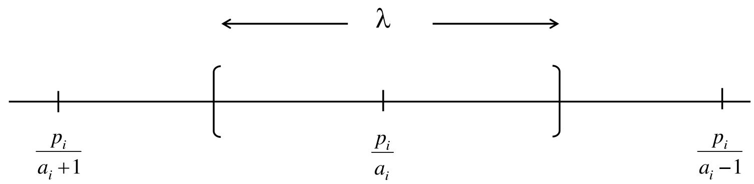
Note that

$$\begin{aligned} &\frac{6927}{\sqrt{4\times 5}} > \frac{6521}{\sqrt{4\times 5}} > \frac{1771}{\sqrt{1\times 2}} \quad \text{so} \quad a_E = 5 \\ &\frac{6521}{\sqrt{3\times 4}} > \frac{1771}{\sqrt{1\times 2}} \quad \text{so} \quad a_D = 4. \end{aligned}$$

(v) Quota method

In this example, it happens that Jefferson's method does not violate the upper quota. Hence, the apportionment based on the Quota method agrees with that of Jefferson's apportionment.

6. (a) We apportion a_i seats to state i such that p_i/a_i is closest to λ (see figure):



Thus, we observe

$$\begin{aligned} & \frac{p_i}{a_i} - \lambda \leq \lambda - \frac{p_i}{a_i+1} \quad \text{and} \quad \lambda - \frac{p_i}{a_i} \leq \frac{p_i}{a_i-1} - \lambda \quad \text{for all } i \\ \Leftrightarrow \quad & \frac{a_i + \frac{1}{2}}{a_i(a_i+1)} p_i \leq \lambda \leq \frac{a_i - \frac{1}{2}}{a_i(a_i-1)} p_i \quad \text{for all } i. \end{aligned}$$

Hence, we have

$$\max_i \frac{p_i}{d(a_i)} \leq \lambda \leq \min_i \frac{p_i}{d(a_i - 1)},$$

where $d(a_i) = (a_i+1)a_i / \left(a_i + \frac{1}{2}\right)$.

(b) The rank index is $r(p,a) = \dfrac{p}{d(a)}$. Since $d(0) = 0$ for Dean's method, so those states which have not been assigned any seat would have rank index value being infinite. Before allocating the second seat to any state, every state must be allocated at least one seat.

7. (a) Recall that Jefferson's Method observes

$$\begin{aligned} \max_i \frac{p_i}{a_i^{Jeff} + 1} &\leq \min_i \frac{p_i}{a_i^{Jeff}} \\ \frac{1}{\min_i \frac{p_i}{a_i^{Jeff}}} &\leq \frac{1}{\max_i \frac{p_i}{a_i^{Jeff}+1}} \\ \Leftrightarrow \quad \min_i \frac{a_i^{Jeff} + 1}{p_i} &\geq \max_i \frac{a_i^{Jeff}}{p_i}. \end{aligned}$$

The last inequality is established by observing

$$\min \left\{ \frac{1}{x_1}, \ldots, \frac{1}{x_N} \right\} = \frac{1}{\max\{x_1, \ldots, x_N\}},$$

x_i's are all positive.

Consider another method other than Jefferson, there exists state k such that $a_k^{other} = a_k^{Jeff} + n$, for some positive integer n. Note that

$$\frac{a_k^{other}}{p_k} = \frac{a_k^{Jeff} + n}{p_k} \geq \frac{a_k^{Jeff} + 1}{p_k} \geq \min_i \frac{a_i^{Jeff} + 1}{p_i} \geq \max_i \frac{a_i^{Jeff}}{p_i}.$$

Hence, the Jefferson Method observes $\min_a \max_i \dfrac{a_i}{p_i}$.

(b) Given h seats and S states, the Adams apportionment observes

$$\max_i \frac{p_i}{a_i^{Adams}} \leq \min_i \frac{p_i}{a_i^{Adams} - 1}.$$

For another apportionment solution other than the Jefferson appointment, there exists a state k such that

$$a_k^{other} = a_k^{Adams} - n, \quad n = 1, 2, \cdots .$$

Consider

$$\frac{p_k}{a_k^{other}} = \frac{p_k}{a_k^{Jeff} - n} \ge \frac{p_i}{a_i^{Adams} - 1} \ge \min_i \frac{p_i}{a_i^{Adams} - 1}$$

so that

$$\max_i \frac{p_i}{a_i^{other}} \ge \min_i \frac{p_i}{a_i^{Adams} - 1} \ge \max_i \frac{p_i}{a_i^{Adams}}.$$

Hence, the Adams apportionment observes the mini-max property.

$$\min_a \max_i \frac{p_i}{a_i}.$$

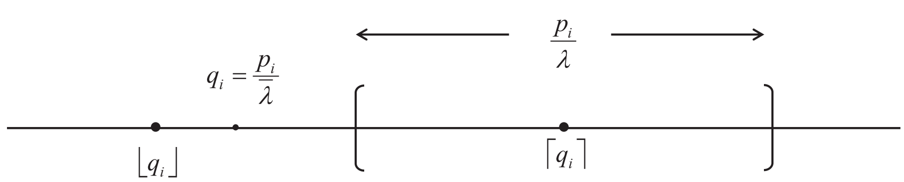
8. In general, for a given population $\boldsymbol{p}$, the apportionment solutions obtained from $M^\alpha(\boldsymbol{p}, h)$ and $M^\beta(\boldsymbol{p}, h)$ differ. In this problem, given that the apportionment solutions from $M^\alpha(\boldsymbol{p}, h)$ and $M^\beta(\boldsymbol{p}, h)$ for a given $\boldsymbol{p}$ *agree for all* house size h, then the orderings of the sequences $\left\{\dfrac{p_i}{a_i + \alpha}\right\}$ and $\left\{\dfrac{p_i}{a_i + \beta}\right\}$ in the recursive scheme of apportioning the seats are identical. Since for $\alpha < \delta < \beta$, we have

$$\frac{p_i}{a_i + \alpha} > \frac{p_i}{a_i + \delta} > \frac{p_i}{a_i + \beta}$$

so that the orderings of the three sequences are identical. Hence, the same apportionment solution is resulted for the 3 parametric methods.

9. Let $\overline{\lambda} = p/h$ denote the average constituents per seat. Under the Webster apportionment, if a_i seats are allocated to state i, then p_i/λ lies inside $\left[a_i - \dfrac{1}{2}, a_i + \dfrac{1}{2}\right]$.

   - Consider the scenario where rounding up for q_i occurs even $q_i - \lfloor q_i \rfloor < 0.5$. The corresponding p_i/λ would lie inside $\left[\lceil q_i \rceil - \dfrac{1}{2}, \lceil q_i \rceil + \dfrac{1}{2}\right]$.



     Since $q_i = \dfrac{p_i}{\overline{\lambda}}$ and q_i lies on the left side of the interval $\left[\lceil q_i \rceil - \dfrac{1}{2}, \lceil q_i \rceil + \dfrac{1}{2}\right]$, then $\overline{\lambda} > \lambda$.

   - Consider the other scenario where rounding down for q_j occurs even $q_i - \lfloor q_i \rfloor > 0.5$. The corresponding p_j/λ would lie inside $\left[\lfloor q_i \rfloor - \dfrac{1}{2}, \lfloor q_i \rfloor + \dfrac{1}{2}\right]$ and q_j lies on the right side of this interval, we have $\overline{\lambda} < \lambda$. This contradicts $\overline{\lambda} > \lambda$.

- If both rounding up for q_i with $q_i - \lfloor q_i \rfloor < 0.5$ and rounding down for q_j with $q_j - \lfloor q_j \rfloor > 0.5$ occur, then we cannot find a divisor that is common for all states in the Webster apportionment.

10. Using the hint, the population $(p_1 \quad p_2)$ apportions to $(a_1 + 1, \quad a_2 - 1)$ provided

$$a_1 + \frac{1}{2} < q_1' < a_1 + \frac{3}{2} \quad \text{and} \quad a_2 - \frac{3}{2} < q_2' < a_2 - \frac{1}{2}.$$

Furthermore,

$$\frac{q_1}{q_2} = \frac{\frac{p_1}{p_1+p_2}(h - a_3)}{\frac{p_2}{p_1+p_2}(h - a_3)} = \frac{p_1}{p_2}$$

so that

$$\frac{2a_1 + 1}{2a_2 - 1} < \frac{p_1}{p_2} < \frac{2a_1 + 3}{2a_2 - 3}.$$

11. Consider the following 3 apportionment solutions:

(i) $(a_1^{(1)}, a_2^{(1)}, a_3^{(1)}) = (7, 5, 4)$

(ii) $(a_1^{(2)}, a_2^{(2)}, a_3^{(2)}) = (7, 6, 3)$

(iii) $(a_1^{(3)}, a_2^{(3)}, a_3^{(3)}) = (8, 5, 3)$,

and the use of the inequity measure $\frac{a_i}{a_j} - \frac{p_i}{p_j}$, we observe the following cycling phenomenon when we compare various pairs.

(a) comparison of apportionments (i) and (ii) for States 2 and 3
In Apportionment (i), State 3 is the favored state, so the inequity measure is

$$\frac{a_3^{(1)}}{a_2^{(1)}} - \frac{p_3}{p_2} = 0.231.$$

In Apportionment (ii), State 2 is the favored state, so the inequity measure is

$$\frac{a_2^{(2)}}{a_3^{(2)}} - \frac{p_2}{p_3} = 0.243.$$

Since the inequity measure is less in Apportionment (i), so $(7, 6, 3) \to (7, 5, 4)$.

(b) comparison of apportionments (ii) and (iii) for States 1 and 2
Similarly, we compare the inequity measures between States 1 and 2 for Apportionments (ii) and (iii). We obtain

$$\frac{a_2^{(2)}}{a_1^{(2)}} - \frac{p_2}{p_1} = 0.156, \quad \frac{a_1^{(3)}}{a_2^{(3)}} - \frac{p_1}{p_2} = 0.173.$$

Since the inequity measure is less in Apportionment (ii), so $(8, 5, 3) \to (7, 6, 3)$.

(c) comparison of apportionments (iii) and (i) for States 1 and 3
Again, we compare the inequity measures between States 1 and 3 for Apportionments (i) and (iii). We obtain

$$\frac{a_1^{(3)}}{a_3^{(3)}} - \frac{p_1}{p_3} = 0.160, \quad \frac{a_3^{(1)}}{a_1^{(1)}} - \frac{p_3}{p_1} = 0.172.$$

Since the inequity measure is less in Apportionment (iii), so $(7,5,4) \to (8,5,3)$. Combining the results, we observe cycling between apportionments since

$$(7,5,4) \to (8,5,3) \to (7,6,3) \to (7,5,4).$$