MATH4994 — Capstone Projects in Mathematics and Economics

Topic One: Fair divisions and allocation schemes

1.1 Criteria for fair divisions

- Proportionality, envy-freeness, equitability and efficiency

1.2 Procedures for two-player and multi-player cake-cutting and chore division

- Discrete cut-and-choose procedures
- Continuous moving-knife procedures

1.3 Adjusted winner for two-party allocation of discrete goods

- Point allocation procedures
- Pareto efficiency

1.1 Criteria for fair divisions

- Fair division is the problem of dividing a set of goods or resources between several people who have an entitlement to them, such that each person receives his due share. This problem arises in cake-cutting, divorce settlements, etc.

Theory of fair division procedures

- Provide explicit criteria for various different types of fairness.
- Provide efficient procedures (algorithms) to achieve a fair division; desirable to require the least number of steps (minimum cuts in cake cutting).
- Study the properties of such divisions both in theory and in real life. Understand the impossibility of achieving "fairness" based on certain criteria and/or within a given allowable set of procedures.

There is a set X of goods and a group of n players. A division is a partition of X into n disjoint subsets: $X = X_1 \cup X_2 \cup ... \cup X_n$, where subset X_i is allocated to player i, $i = 1, 2, \ldots, n$.

The set X can be of two types: indivisible items or divisible resource.

- X may be a finite set of *indivisible* items. For example, $X =$ {piano, car, apartment}, such that each item should be given entirely to a single person.

- X may be an infinite set representing a *divisible* resource, for example: money or a cake. For example, the section $[0, 1]$ may represent a long narrow cake, that has to be cut into parallel pieces.

The set to be divided may be

- *homogeneous* - such as money, where only the amount matters.
- *heterogeneous* - such as a cake that may have different ingredients, different icings, etc. In the general case, different parts may be valued differently by different people.

The items to be divided may be

- *desirable* - such as a car or a cake.
- *undesirable* - such as house works (cleaning floor, washing dishes).

Desirability, divisibility and homogeneity properties of items

- When dividing inheritance, or dividing household property during divorce, it is common to have *desirable indivisible heterogeneous* property such as houses, and *desirable divisible homogeneous* property such as money.

- In the housemates problem, several friends rent a house together, and they have to both allocate the rooms in the apartment (a set of *indivisible, heterogeneous, desirable* goods), and divide the rent to pay (*divisible, homogeneous, undesirable* good). This problem is also called *the room assignment-rent division*. The players may set different rents for different rooms.

Subjective value functions

There cannot be an objective measure of the value of each item as different people may assign different values to each item. The presence of different measures of values opens a vast potential for many challenging questions.

The i^{th} person in the group of n persons is assumed to have a personal subjective *value function*, V_i, which assigns a numerical value to each subset of X. Usually the value functions are assumed to be normalized, so that every person values the empty set as 0 [$V_i(\emptyset) = 0$ for all i], and the entire set of items as 1 [$V_i(X) = 1$ for all i] if the items are desirable.

- The cake-cutting procedure assumes that each player knows his own valuation function but not others. In a continuous procedure, like the moving knife procedure, the valuation of a specific piece of cake by other players may be revealed during the procedure by the actions taken by others.

Properties of a value function for divisible resources

- Non-negativity: $V_i(B) \geq 0$ for all $B \subseteq [0,1]$
- Normalization: $V_i(\emptyset) = 0$ and $V_i([0,1]) = 1$
- Additivity: $V_i(B \cup B') = V_i(B) + V_i(B')$ for disjoint B, $B' \subseteq [0,1]$
- V_i is continuous: Single points do not have any value. In such case, the Intermediate-Value Theorem is applicable.

The value function resembles the probability density function defined in $[0,1]$. Single points have probability measure zero.

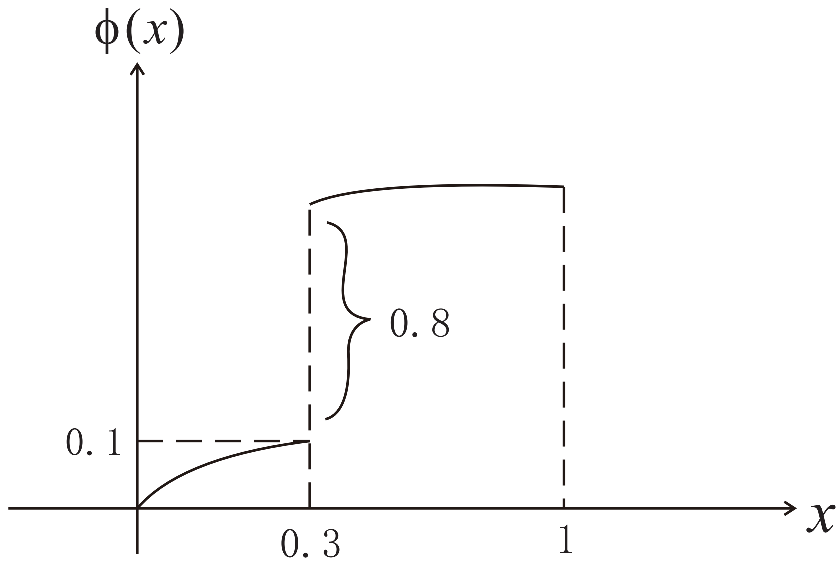
The Intermediate-Value Theorem states that if a continuous function f with an interval $[a, b]$ as its domain takes values $f(a)$ and $f(b)$ at each end of the interval, then it also takes any value between $f(a)$ and $f(b)$ at some point within the interval.

Application: Suppose player i assigns a subinterval S with a value less than $1/n$. Recall $V_i(S) < \frac{1}{n}$ and $V_i([0, 1]) = 1$. By the Intermediate-Value Theorem, there exists a larger subinterval that is obtained by enlarging continuously from S whose value to player i becomes exactly equal to $1/n$. This continuity property is essential in the construction of the moving knife procedure.

Example of discontinuous value function

Define $\phi(x) = V([0, x])$, where $x \in [0, 1]$.

Suppose there is a strawberry of infinitesimally small size placed at $x = 0.3$, which gives a jump of value of 0.8.



It is not possible to find a subinterval $B \in [0, 1]$ such that $V(B) = 0.5$ due to loss of continuity of the value function. We observe $V(B) \leq 0.1$ if B does not contain $x = 0.3$ and $V(B) \geq 0.8$ if B contains $x = 0.3$. In this case, the Intermediate-Value Theorem is not applicable since $V(B) = 0.5$ cannot be achieved by any B.

Examples

1. For the set of indivisible items {piano, car, apartment}, Alice may assign a value of $\frac{1}{3}$ to each item, which means that each item is important to her just the same as any other item. Bob may assign the value of 1 to the set $X = \{$car, apartment$\}$, and the value 0 to all other subsets except X. This means that he wants to get only the car and the apartment together. The car alone or the apartment alone, or each of them together with the piano, is worthless to him. This value function violates the additivity property since

   $$V(\{\text{car}\}) + V(\{\text{apartment}\}) = 0 + 0 < 1 = V(\{\text{car, apartment}\}).$$

2. If X is a long narrow cake (modeled as the interval $[0, 1]$), then Alice may assign each subset a value proportional to its length, which means that she wants as much cake as possible, regardless of the icings. Bob may assign value only to subsets of $[0.4, 0.8]$ since this part of the cake contains cherries and Bob only cares about cherries.

Notions of fair divisions

1. A *proportional division*, also called simple fair division, means that every person gets at least his due share according to *his own value function*. That is, each of the n people gets a subset of X which he values as at least $\frac{1}{n}$: $V_i(X_i) \geq \frac{1}{n}$ for all i. It is said to be super-proportional (strongly fair) division if $V_i(X_i) > \frac{1}{n}$ for all i. Obviously, super-proportional division is not possible if all value functions of the players are the same.

2. An *envy-free division* guarantees that no-one will want somebody else's share more than their own. That is, every person gets a share that he values at least as much as all other shares: $V_i(X_i) \geq V_i(X_j)$ for all i and j. In simple language, envy-free means each player receives a piece he or she would not swap for that received by any other players.

- Webster dictionary defines envyness as a "painful or resentful awareness of an advantage enjoyed by another joined with a desire to possess the same advantage".

Proposition

Suppose $X = (X_1, X_2, ..., X_n)$ is a complete allocation, where $X_1 \cup X_2 \cup \ldots \cup X_n = X$ and X_j's are disjoint. If an allocation is envy-free, then it must also be proportional. In other words, envy-freeness is the stronger notion of fairness.

envy-freeness $\Rightarrow$ proportional division

We prove by contrapositive argument. Suppose that $V_i(X_i) < \frac{1}{n}$ for some i. Since the allocation is complete, by virtue of additivity of the value function, we deduce that

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} V_i(X_j) = V_i(X - X_i) > \frac{n-1}{n},$$

then we cannot have $V_i(X_j) \leq \frac{1}{n}$ for all j, $j \neq i$. This implies that $V_i(X_j) > \frac{1}{n}$ for some $j \neq i$. This would give $V_i(X_i) < \frac{1}{n} < V_i(X_j)$, indicating failure of envy-freeness.

For two agents, proportionality and envy-freeness are equivalent. Suppose $V_i(X_i) \geq \frac{1}{2}$, $i = 1, 2$, then $V_1(X_2) \leq \frac{1}{2}$ and $V_2(X_1) \leq \frac{1}{2}$, so it is envy-free.

Note that there is no such equivalence when there are three or more players. It is still possible that player i may think player j receives X_j where $V_i(X_j) > V_i(X_i)$ (failure of envyfreeness) while $V_i(X_i) \geq \frac{1}{n}$, $n \geq 3$. See the discussion of the proportional discrete cuts procedure for 3-player division later.

A division is *super-envy-free* if

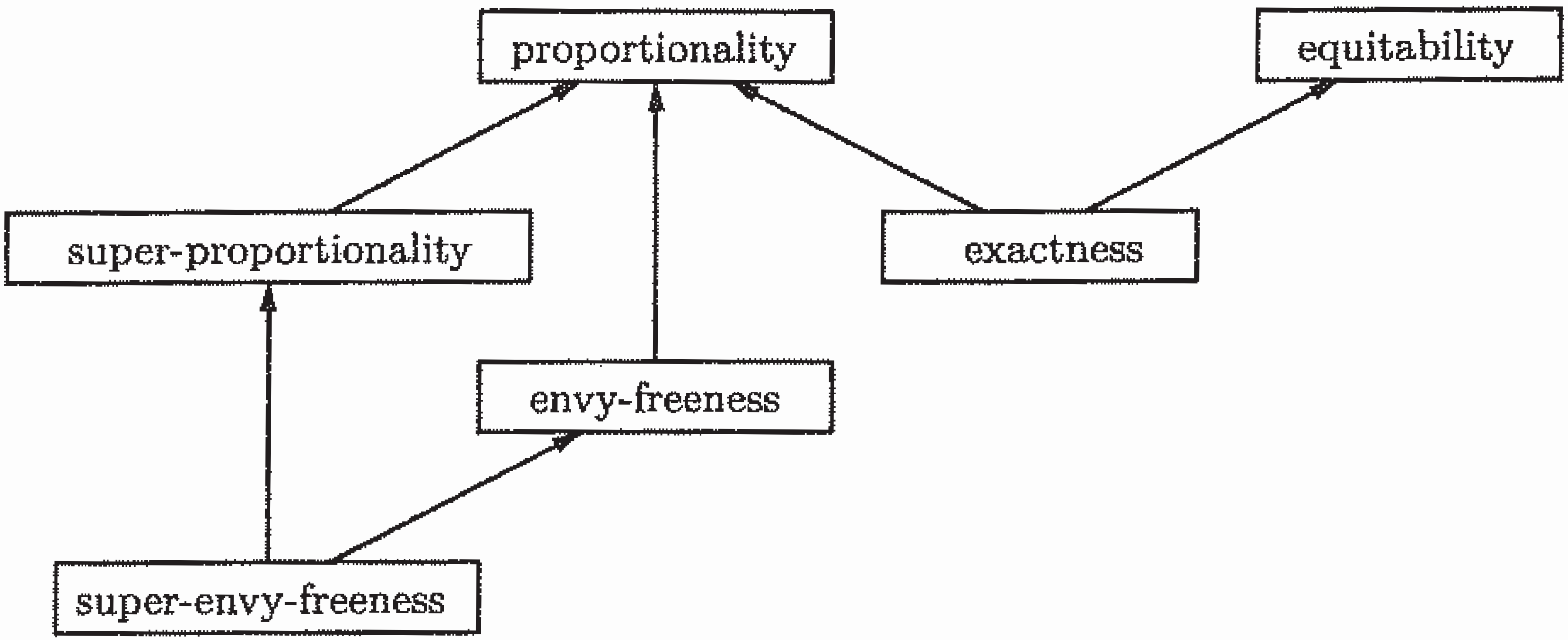
$$V_i(X_j) < \frac{1}{n} \text{ for each } j,\ 1 \leq j \leq n \text{ and } j \neq i.$$

We then deduce that $\sum_{\substack{j=1 \\ j \neq i}}^{n} V_i(X_j) < \frac{n-1}{n}$, so $V_i(X_i) > \frac{1}{n}$. It is seen that super-envy-freeness gives $V_i(X_i) > \frac{1}{n} > V_i(X_j)$, $j \neq i$, thus implying super-proportionality and envy-freeness.

3. An *equitable division* means each person's subjective valuation of the piece that he receives is the same as the other person's subjective valuation. $V_i(X_i) = V_j(X_j)$ for all i and j. A more stringent criterion is *exactness*, where $V_i(X_i) = \frac{1}{n}$, for all i.

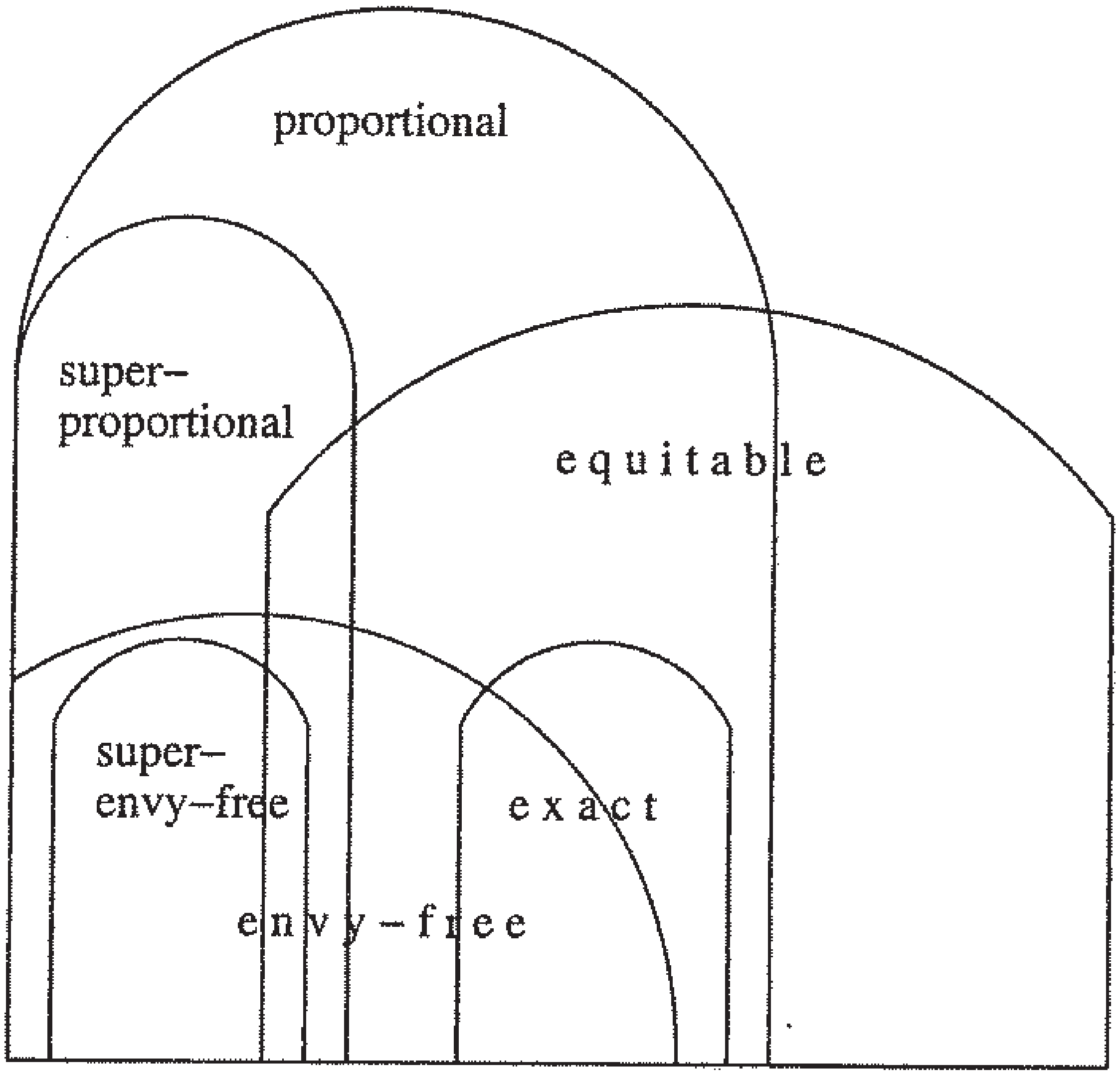
Equitability may not imply envyfreeness since the subjective valuations of the players may differ. There is no guarantee that player i's value of other piece held by another player always observes $V_i(X_i) \geq V_i(X_j)$, $j \neq i$.

Most envy free allocations (hence proportional) would not satisfy the stringent equality constraint that equitability requires.



Implications between the valuation criteria for divisions

- Even $V_i(X_i) \geq \frac{1}{n}$ for all i, it is not guaranteed to observe $V_i(X_i) = V_j(X_j)$ for any i and j.
- It is still possible to have $V_i(X_i) = V_j(X_j)$ for all pairs of i and j while $V_i(X_i) < \frac{1}{n}$, for all i.



Inclusions between the valuation criteria for divisions as a Venn diagram

Efficient allocation (Pareto optimal)

An allocation is efficient or Pareto optimal if there is no other allocation that is strictly better for at least one player and as good for all the others.

For example, a division where one player gets the whole set and attaches value to any portion of the set is Pareto optimal. If any portion is given to another player, then the value function of this particular player on the reduced set will be lowered due to additivity property. An efficient (in Pareto sense) allocation needs not be proportional, envy-free or equitable.

Example of an allocation that is not Pareto optimal

Suppose player A places no value at all on a portion to which some other player B attaches some value. Taking away that portion from A would keep the same valuation value for the new allocation for A but the new piece received by the other player B gives a higher value for B. This allocation is not Pareto optimal since the new allocation is strictly better for B while it is as good for A.

1.2 Procedures for two-player and multi-player cake-cutting

- Based on the given fairness criterion, a fair division procedure lists the actions to be performed by the players based on available set of items and their valuations. A valid procedure is one that guarantees a fair division for every player who acts rationally according to their valuations.

- A procedure is said to be finite if it always (independently of the players' valuation functions) terminates with a solution after only a finite number of decisions has been made.

- Continuous procedures may involve one player moving a knife along the side of a cake and some other players saying stop and cutting the cake at the spot. The players are making decision continuously.

Operational properties

- Does the procedure guarantee that each agent receives a single continguous slice (rather than the union of several pieces)? We prefer continguous procedures, which also minimize the number of cuts to be made. Note that a procedure for n players will require at least $n-1$ cuts.

- If the number of cuts is not minimal, can we provide an upper bound on the number of cuts?

- Does the procedure require an active referee, or can all actions be performed by the players themselves?

Maximin criterion

Each player follows the strategy that maximizes the value of the minimum size cake (maximin piece) that he can guarantee, regardless of what the other players do.

Risk aversion

Each player is risk averse in the sense that he will never choose a strategy that may yield a more valuable piece of cake if it entails the possibility of getting less than a maximin piece. The maximin piece serves as the benchmark.

Strategy vulnerable

A procedure is said to be strategy vulnerable (player can game around) if a maximin player can assuredly do better by misrepresenting its value function. A procedure that is not strategy vulnerable is strategy proof, resulting maximin players always play truthfully.

Cake-cutting problems

"Cake-cutting" is the problem of fair division of a single *divisible* and *heterogeneous* good between n players.

The *cake* is represented by the unit interval $[0, 1]$:

Each agent i has a value function V_i defined for each subinterval of $[0, 1]$.

Two-agent discrete cut-and-choose procedure

One agent (chosen at random) *cuts* the cake in two pieces (she considers to be of equal value based on her valuation), and the other *chooses* one of them (the piece he prefers). The chooser always takes the piece with higher or at least equal valuation. The cutter is indifferent to the two pieces. Therefore, the procedure is not equitable. However, it satisfies

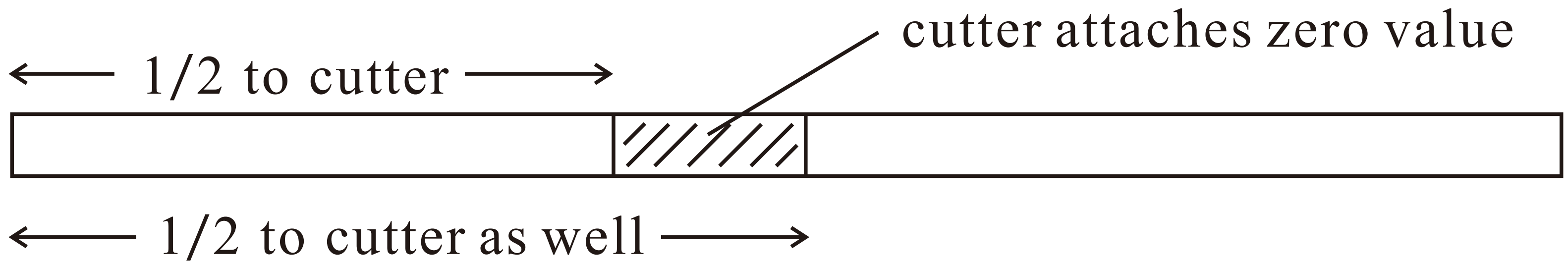
- *Envy-freeness*: No agent will envy the other. Proportionality is satisfied automatically in a two-player division.
- Even if the role of the cutter is determined by the flip of a coin, a fair procedure not favoring either player, the cutter would think that the chooser has a definite advantage. This is the *failure of equitability* since $1/2 = V_{\text{cutter}}(X_{\text{cutter}}) \leq V_{\text{chooser}}(X_{\text{chooser}})$.
- If the cutter knows the valuation of the chooser, he may generally obtain more. Without the knowledge of the valuation of the chooser, by risk aversion, the cutter always cuts the cake in two equal pieces based on his personal valuation.

- *Pareto optimality is observed*

  We assume that the two value functions attach value to any finite piece of the cake. For any allocation other than the two halves (according to the cutter's value function), we would like to show that no other allocation achieves "at least as good for all players and better for at least one player".

  Note that any new allocation would have one piece that is more to the cutter while other piece is less to the cutter.

  1. If the larger piece viewed by the cutter is taken by the chooser, the cutter is worse off.
  2. Suppose the smaller piece viewed by the cutter is chosen by the chooser, though the cutter is better off but the chooser is worse off. This is because this smaller piece has lower value to the chooser when compared with the original equal half piece (the smaller piece is a proper subset of the equal half piece).



Note that Pareto optimality may fail for the "equal halves allocation" when we allow the cutter's value function to attach zero value to some portion of the cake. Based on the cutter's equal halves procedure, it may still be possible that the chooser can get a more valuable piece while the cutter is indifferent. Under this very specialized scenario, Pareto optimality may fail.

Recall $\phi(x) = V([0, x])$, which is a nondecreasing function of x. When $\phi(x)$ is constant over $[x_1, x_2]$, this means the player assigns zero value to the interval $[x_1, x_2]$. When the value function attaches value to any finite piece of the cake, then $\phi(x)$ is strictly increasing.

Nonproportional cut-and-choose procedure for three players

Given: Cake $X = [0, 1]$ and players p_1, p_2, and p_3 with valuation functions v_1, v_2, and v_3.

Step 1: p_1 cuts X into three pieces of equal value, S_1, S_2, and S_3, i.e.,

$$V_1(S_1) = V_1(S_2) = V_1(S_3) = \frac{1}{3}.$$

Step 2: p_2 chooses one of the three pieces that is most valuable to her.

Step 3: p_3 chooses one of the remaining two pieces.

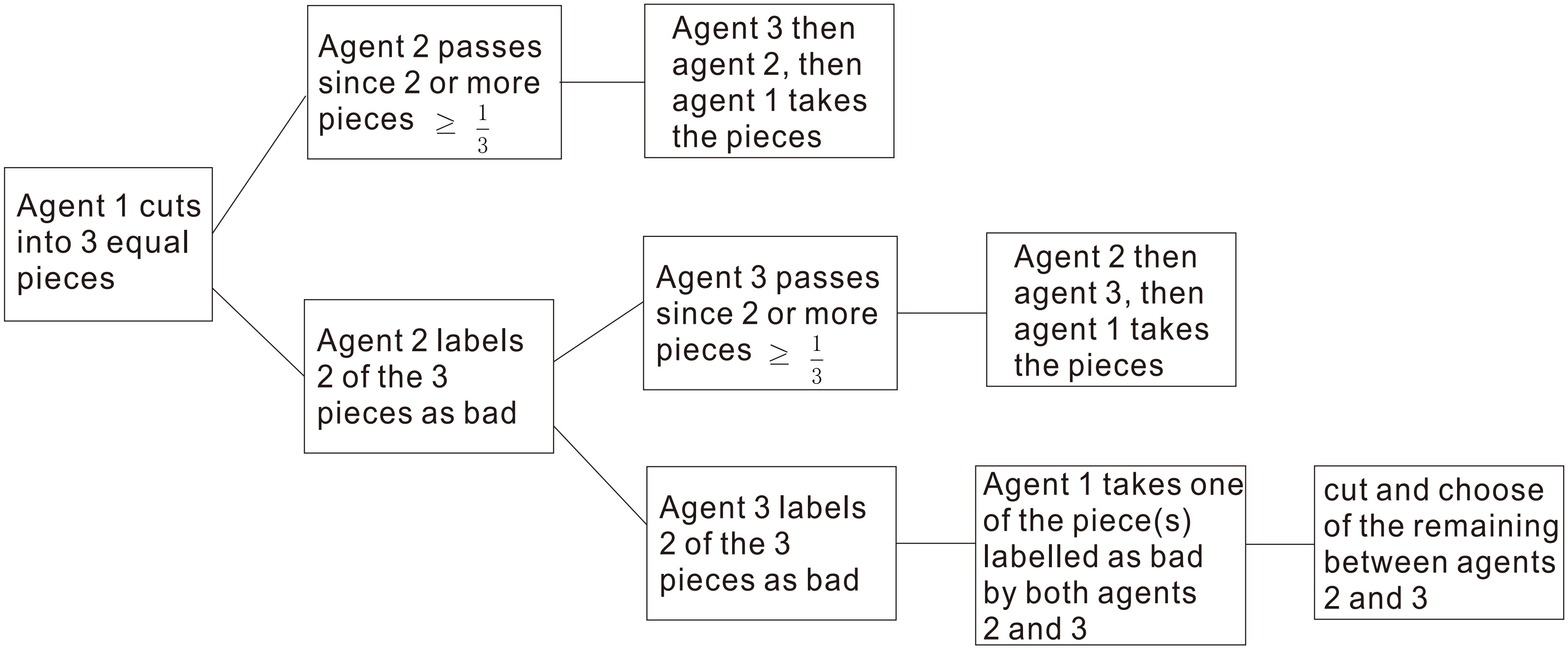
Step 4: p_1 receives the remaining piece.

The procedure does not fulfill many fairness criteria.

- It is not proportional: Though players p_1 and p_3 each receive proportional shares (p_1 because he cuts the cake into three pieces of equal value, and p_2 because she is the first one to choose), player p_2 is not guaranteed to receive a proportional share, for example not in the case when $V_3(S_1) = 1/2$ and $V_3(S_2) = V_3(S_3) = 1/4$ and player p_2 happens to choose S_1 in Step 2. In this case, player p_3 has the choice between two pieces both of which are worth less than one third of the cake to her.

- Since it is nonproportional, so it is not envy-free.

Proportional discrete cuts procedure for 3-player division



Can Agent 2 game around by always reporting 2 of the 3 pieces as bid? This game around strategy is ruled out by risk aversion since he may be worst off if Agent 1 takes his second largest piece (which may have valuation above 1/3).

- All valuations of the divisions are personal, which may differ among the 3 players. It guarantees a *proportional* division of the cake, where $V_i(X_i) \geq \frac{1}{3}$, $i = 1, 2, 3$. In the case where Agent 2 is the second chooser (scenario shown at the top of the flow chart), he is guaranteed to receive a piece with value $\geq \frac{1}{3}$ since he has valued two pieces with values $\geq \frac{1}{3}$.

- It is *not envy-free*. (i) Suppose Agent 2 passed, it may be possible that Agent 2 may envy Agent 3 if Agent 3 may choose the larger of the two pieces that Agent 2 considered acceptable. (ii) In another case, the cut-and-choose played by Agent 2 and 3 may not be $50-50\%$ in Agent 1's own valuation. In this case, there exists another piece received by Agent 2 or Agent 3 that has player 1's valuation higher than $1/3$. This violates envy-free division.

- The resulting pieces might *not* be *continguous*. As one example, suppose both Agents 2 and 3 label the middle piece as "bad" and Agent 1 takes it. One additional cut is required if the cut-and-choose cut is different from Agent 1's original cut.

Proportional procedure with trimming for arbitrary n players (Last Diminisher procedure)

One of the n players cuts a piece that she considers to be worth exactly $1/n$ in her measure. All other players (still) in the game then value this piece in turn according to their valuation functions. If any of these players considers the piece to be super-proportional (to be worth more than $1/n$), this player trims the piece to exactly $1/n$ according to his measure before passing it on to the next player.

When the last player has evaluated this piece, it is given to the player who was the last trimming it, or to the player who cut it in the first place if no trimmings have been made. The player receiving the piece leaves the game and the trimmings are reassembled with the remainder of the cake. By assumption of additivity, the values of the pieces are not lowered by the trimming operations.

The same procedure is then applied to the $n-1$ remaining players and the reassembled remainder of the cake. This process is repeated until only two players remain who then apply the simple cut-and-choose procedure.

Given: Cake $X = [0, 1]$ and players $p_1, p_2, \ldots, p_n$, where p_i has valuation function V_i, $1 \leq i \leq n$.

Step 1: Player p_1 cuts piece S_1 such that $V_1(S_1) = 1/n$.

Step 2: The cut piece is passed successively to $p_2, p_3, \ldots, p_n$, each of whom trims the piece as appropriate. In more detail, let S_{i-1}, $2 \leq i \leq n$, be the piece player p_{i-1} passes on to p_i.

- If $V_i(S_{i-1}) > 1/n$, player p_i trims piece S_{i-1} and passes on the trimmed piece S_i where $V_i(S_i) = 1/n$.

- If $V_i(S_{i-1}) \leq 1/n$, player p_i passes on the untrimmed piece $S_i = S_{i-1}$.

The last player that either cut or trimmed this piece receives S_n and drops out.

Step 3: Reassemble the remainings $X - S_n$ of the cake and rename the remaining players to be $p_1, p_2, \ldots, p_{n-1}$.

Step 4: Repeat Steps 1 to 3 until $n = 2$. The remaining two players, p_1 and p_2, apply the cut-and-choose procedure.

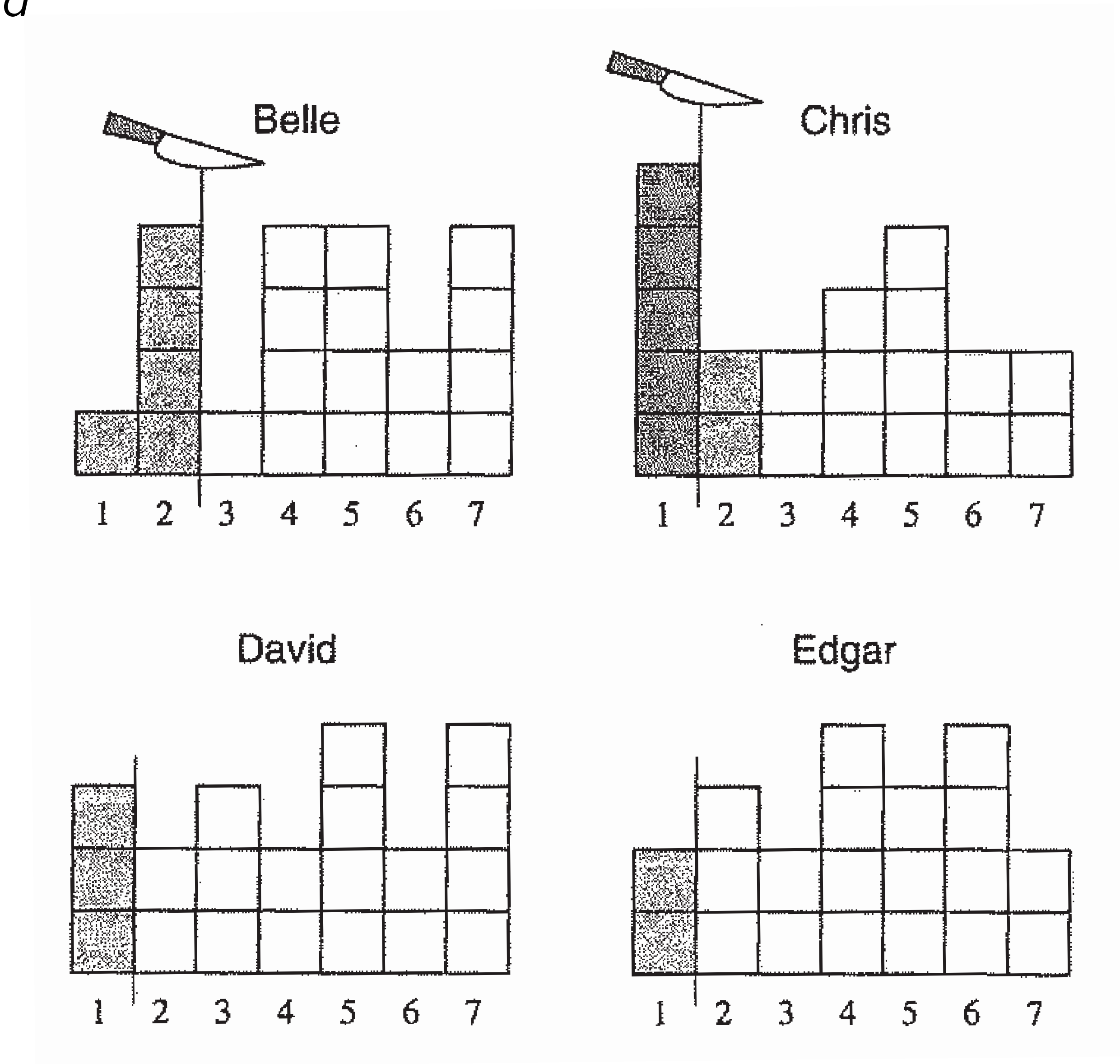
The preferences of the four players will be given using the box representation and the players will take their turn in alphabetical order. The valuation functions of the players are represented using 20 boxes each.

In the first round, Belle cuts a slice of the pizza that she values exactly $1/4$. This is represented by the five light gray boxes in the first two columns of Belle's box representation. She then passes the slice on to Chris who values it to be $7/20 > 1/4$. Thus, Chris trims the slice to exactly $1/4$ according to his measure and passes on the trimmed slice (the five dark gray boxes in the first column of Chris' box representation) to David.

Since David values the trimmed slice only $3/20 < 1/4$ according to his measure, he passes the slice on as it is (without trimming it). Edgar is the last one to evaluate the trimmed slice and considers it to be worth $2/20 < 1/4$.

As Chris is the last one trimming the slice, he is the one receiving it and drops out.

First round



We represent the portion a player is receiving by dark gray boxes.

Summary

1. In the first round, we observe $S_n \subseteq S_{n-1} \subseteq \cdots \subseteq S_1$ due to potential trimming of the piece when passing between successive payers, and $S_n = \frac{1}{n}$ to the one who receives the final piece.

2. The same procedure is repeated with the remainder part $\geq \frac{n-1}{n}$ to every one staying behind in the division game.

3. Repeating the procedure until down to 2 players, which is then finally settled by the cut-and-choose procedure.

Proportionality is guaranteed since every player receives a piece that he thinks to be of size at least $1/n$. Equitability is not under consideration in this procedure.

By risk aversion, for $i = 2, 3, \ldots, n-1$, player i will not pass S_{i-1} without trimming when $V_i(S_{i-1}) > \dfrac{1}{n}$ since the piece may be received by another player. The last player n may game around by trimming only a small piece ϵ when $V_n(S_{n-1}) > \dfrac{1}{n}$. As a result, he will receive the trimmed piece with value larger than $1/n$.

Luckily, this does not violate proportionality. The last trimmed piece has value less than $1/n$ for all earlier $n-1$ players.

As a remark, if we assume that the n^{th} player is not being informed to be the last player, then the above strategy based on risk aversion remains valid.

Failure of envy-freeness

None of the players already dropped out is involved in any way in the assignment of future portions. Players already dropped out cannot raise an objection if they consider any of the future pieces to be worth more than $1/n$.

Envy-free discrete cuts procedure for 3-player division

Initialization:

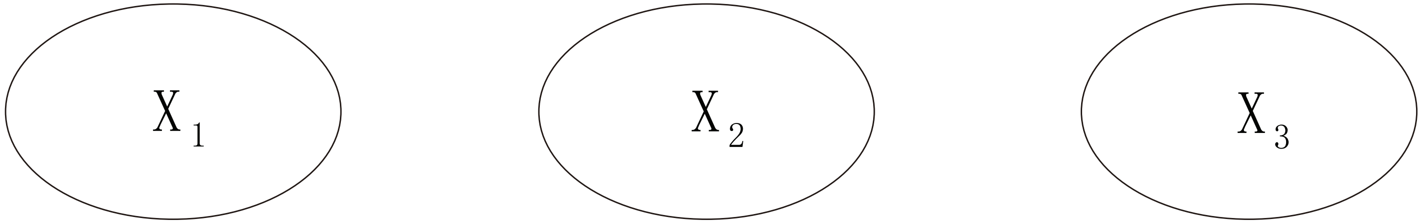
1. Agent 1 divides the cake into three equally-valued pieces X_1, X_2, X_3 : $V_1(X_1) = V_1(X_2) = V_1(X_3) = 1/3$.

2. Agent 2 trims the most valuable piece according to V_2 to create a tie for most valuable. For example, if $V_2(X_1) > V_2(X_2) \geq V_2(X_3)$, agent 2 removes $X' \subseteq X_1$ such that $V_2(X_1 \backslash X') = V_2(X_2)$. We call the three pieces – one of which is trimmed – *cake 1* ($X_1 \backslash X'$, X_2, X_3 in the example), and we call the trimmings *cake 2* (X' in the example).

Remark If $V_2(X_1) = V_2(X_2) \geq V_2(X_3)$, then $X' = \phi$.

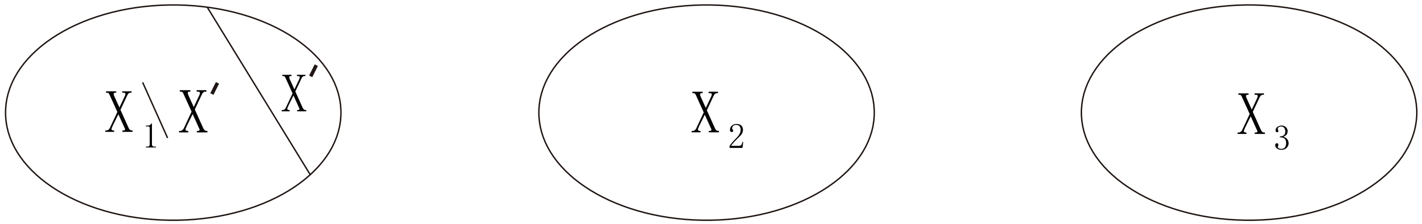
Exact division for Agent 1

$$V_1(X_1) = V_1(X_2) = V_1(X_3) = \frac{1}{3}$$



$V_2(X_1) > V_2(X_2) \geq V_2(X_3)$

Trim X' from X_1 so that $V_2(X_1 \backslash X') = V_2(X_2) \geq V_2(X_3)$



Cake 1 $X_1 \backslash X'$, X_2, X_3

Cake 2 X'

Division of cake 1:

Agent 3 chooses first from one of the three pieces of cake 1.

- If Agent 3 chose $X_1 \backslash X'$, then Agent 2 chooses X_2. Otherwise, Agent 2 receives $X_1 \backslash X'$.

Between Agents 2 and 3, we call them T and $\overline{T}$ according to

Agent T – takes $X_1 \backslash X'$;   Agent $\overline{T}$ – the other person.

Agent 1 receives the remaining piece of cake 1 (always an untrimmed piece) since $X_1 \backslash X'$ is guaranteed to be received by either Agent 2 or Agent 3.

Division of cake 2:

Agent $\overline{T}$ divides cake 2 into three equally-valued pieces.

Agents $T, 1$ and $\overline{T}$ select a piece of cake 2 each, in that order. Agent T definitely gets less than X_1, so she will not be envied by Agent 1.

Proof of envy-freeness

- The division of cake 1 is clearly envy free: Agent 3 chooses first; agent 2 receives one of the two pieces that she views as tied for largest; and agent 1 definitely receives an untrimmed piece, which he also views as tied for largest.
- Now consider the division of cake 2. Agent T (who received the trimmed piece) chooses first, and agent $\overline{T}$ is indifferent between the three pieces. Agent 1 will never envy the combined pieces received by T (even if T received all of cake 2). Note the clever assignment of $\overline{T}$ to do the cutting of cake 2 and T to choose first is aimed to avoid the potential envy of agent 1 against the combined piece received by agent T. This is because the combined piece received by T is less than X_1. Obviously, $\overline{T}$ would not envy T and Agent 1 since $\overline{T}$ does the cutting into 3 equal pieces according to his valuation, though he is the last one to choose.
- Combining envy-free divisions of the two disjoint pieces of cake yields an envy-free division of the combined cake. Hence, agents T and $\overline{T}$ are not envious overall.

Continuous moving knife procedures

1. **Single-knife procedure: proportional but not envy-free**

   Suppose there are three kids who are to split the cake. One strategy is for Mom (referee) to place a knife over one corner of the cake and begin to move it slowly across the cake. When any of the kids says "stop", that kid (K1 let's say) gets the piece. Presumably K1 thinks she got $\frac{1}{3}$ of her valuation and K2 and K3 (who did not speak up) believe that remainder is at least $\frac{2}{3}$.

   By risk aversion, K1 should be refrained from saying "stop" too late. This is because she runs into the risk of not being able to achieve 1/3 with the remaining portion with valuation less than 2/3 when K2 or K3 initiates the first call of "stop".

Mom keeps on moving the knife, until K2 says "stop". K2 calls before K3 when the knife hits K2's valuation of $1/2$ of the remaining cake earlier than K3. Lucky for both K2 and K3, they are likely to receive more than $\frac{1}{3}$ of their valuations.

The division is seen to be proportional where every player envisions to receive at least $1/3$. Envy may exist since K1 might think the piece received by K3 or K2 has a value more than $\frac{1}{3}$.

Extension to n players

Given: Cake $X = [0, 1]$ and players $p_1, p_2, \ldots, p_n$ with continuous valuation functions.

Step 1:

- A knife is being held above the cake and is moved gradually and continuously from the left to the right until any of the players calls “Stop!” because she considers the piece of cake to the left of the knife to be worth $1/n$.

- In the case that several players call “Stop!” at the same time, a random tie-breaker is applied (for the sake of simplicity, we assume the outcome to be in favor of player p_i, where i is the smallest index among all players calling “Stop!”).

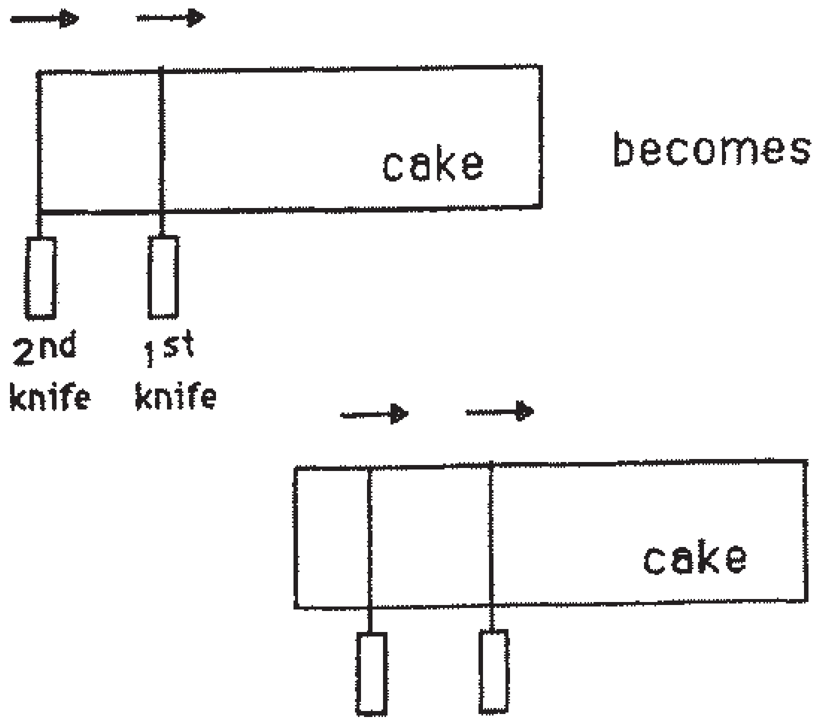
- The knife cuts the cake at the current position.

- The player who called "Stop!" receives the left piece of the cake and drops out.

Step $2, 3, \ldots, n-1$: Repeat Step 1 for all remaining players and the remaining cake.

Step n: The last remaining player receives the remaining cake.

2. **Austin's two-knife procedure: equitable (exact) division for two players ($V_i(X_i) = 1/2$, for all i)**



There is a single knife that moves slowly across the cake from the left edge toward the right edge, until one of the players (say, player 1) calls "stop" (at the point when the piece so determined is of value exactly 1/2). Note that $\phi(x) = V([0, x])$ starts at zero value at $x = 0$ and equals one at $x = 1$. There exists x^* such that $\phi(x^*) = 1/2$.

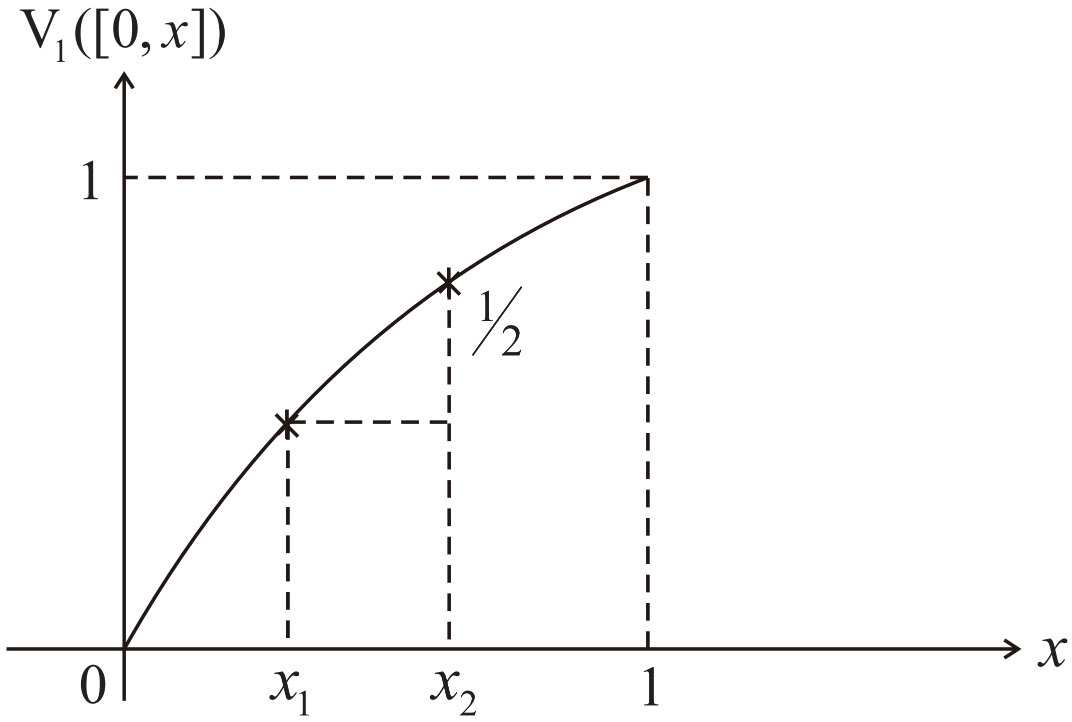
At this time, a second knife is placed at the left edge of the cake. Player 1 then moves both knives across the cake in parallel fashion (in such a way that the piece between the two knives remains to have value exactly 1/2 in player 1's measure).

When the knife on the right arrives at the right-hand edge of the cake, the left-hand knife lines up with the position that the first knife was in at the moment when player 1 first called "stop". Let the position of the left and right knife to be x_1 and x_2, respectively. Suppose Player 1 shouts at x^*, then $V_1([0, x^*]) = \frac{1}{2} = V_1([x^*, 1])$.

While the two knives are moving, player 2 can call "stop" at any time (which he does precisely when the value of the piece between the two knives is exactly 1/2 in his measure).

Player 1 divides the cake into exactly one half to his valuation within the two knives at all times. Player 2 waits for a particular subinterval between the two knives that is exactly one half. The piece between the two knives has valuation equals $\frac{1}{2}$ for both players (same for the remaining piece of the cake).

The mover (Player 1) always maintains $V_1([x_1, x_2]) = 1/2$.



Note that x_1 can be determined as a function of x_2 via $V_1([x_1, x_2]) = 1/2$ (see the above plot). Hence, $V_2([x_1, x_2])$ can be visualized to have continuous dependence on x_2. We may write it as $V_2([g(x_2), x_2])$.

How can one guarantee that there will be a point where player 2 thinks the piece between the knives is of value exactly 1/2?

When the two knives start to move, $x_2 = x^*$, $V_2([0, x^*]) < 1/2$ since player 2 does not initiate "stop". Now, x_2 increases from x^* to 1 while keeping $V_1([x_1, x_2]) = 1/2$. With $x_2 = 1$, we have

$$1 - V_2([0, x^*]) = V_2([x^*, 1]) > 1/2.$$

Note that $V_2([g(x^*), x^*]) < 1/2$ while $V_2([g(1), 1]) > 1/2$. By the Intermediate Value Theorem, we deduce that there exists $\widehat{x} \in (x^*, 1)$ such that

$$V_2([g(\widehat{x}), \widehat{x}]) = 1/2.$$

Gaming around

If one player is assigned to receive the piece between the knives, then she can game around the procedure by delaying the call to stop and waiting for the other player to call first. The piece between the two knives has evaluation higher than $1/2$ since the point of evaluation of $1/2$ has passed. By risk aversion, the player calls "stop" right after the two parallel knives start moving since there exists possibility that the piece between the knives stays lower in valuation in all later moments.

To avoid gaming around by delaying the "stop" call, the piece to be received by either player is based on the throw of a fair coin after the cutting has been done.

Equitable division among two players into more than two pieces

Two players cut cake into k pieces, k is any integer ≥ 2. Each piece is valued at $\frac{1}{k}$ by both players.

Firstly, the two partners obtain a single piece of cake that both of them value as exactly $\frac{1}{k}$, for any integer $k \geq 2$. We call this procedure $\text{Cut}_2(\frac{1}{k})$.

As the starting procedure, Alice makes $k-1$ parallel marks on the cake such that k pieces so determined have a value of exactly $\frac{1}{k}$.

- If there is a piece that George also values as $\frac{1}{k}$, then we are done.
- Otherwise, there must be a piece that George values as less than $\frac{1}{k}$, and an adjacent piece that George values as more than $\frac{1}{k}$. It is not possible to have all adjacent pairs to be more than $\frac{1}{k}$ or less than $\frac{1}{k}$.

- Let Alice place two knives on the two marks of one of these pieces, and move them in parallel, keeping the value between them at exactly $\frac{1}{k}$, until they meet the marks of the other piece. There must be a point at which George agrees that the value between the knives is exactly $\frac{1}{k}$.

- Use $\text{Cut}_2(\frac{1}{k})$ to cut a piece which is worth exactly $\frac{1}{k}$ for both partners.

- The remaining reassembled cake is worth exactly $\frac{k-1}{k}$ for both partners; use $\text{Cut}_2(\frac{1}{k-1})$ to cut another piece worth exactly $\frac{1}{k-1}$ of the remaining cake, or $\frac{1}{k}$ of the original cake for both partners.

- Continue the procedure until there are k pieces.

By recursively applying Cut_2, the two partners can divide the entire cake to k pieces, each of which is worth exactly $\frac{1}{k}$ for both of them.

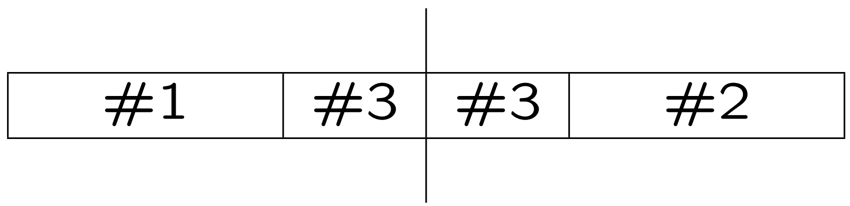
Extension to n players

It is possible to divide a cake to n players, such that each player receives a piece worth exactly $\frac{1}{n}$ for him.

3-player case

- Player #1 and #2 use $\mathrm{Cut}_2(\frac{1}{2})$ to give each one of them a piece worth exactly $\frac{1}{2}$ for them. The two separate pieces received by one of the players are now put together as single piece.

- Player #3 uses $\mathrm{Cut}_2(\frac{1}{3})$ with player #1 to get exactly $\frac{1}{3}$ of player #1's share and then $\mathrm{Cut}_2(\frac{1}{3})$ with player #2 to get exactly $\frac{1}{3}$ of player #2's share. Player #1 remains with exactly $\frac{1}{3}$; the same is true for player #2. As for player #3, he gets exactly $\frac{1}{3}$ of the entire cake since he gets 1/3 of the two pieces that have combined value of one.



Envy-free moving-knife procedure for three players

1. Webb's procedure (combined with Austin's procedure)

Step 1: A referee slowly moves a knife across the cake until someone yells "Cut!" to indicate that she values the piece to be cut off at one-third of the cake. Suppose that Annie is the one who yells "cut", and let P_1 represent the piece of cake that is cut off.

Step 2: Annie and Ben (chosen based on the throw of a fair coin among Ben and Chris) now use Austin's procedure to divide the remaining cake into two pieces that they both consider equally valuable. Let P_2 and P_3 denote these two pieces.

1/3 to Annie upon her shout	equitable division between Annie and Ben by Austin's procedure

Step 3: Chris chooses first from the three pieces P_1, P_2, and P_3. Ben chooses next, and Annie chooses last.

Envy freeness (3 cuts for 3 players)

Chris envies no one since he gets to choose first. Since Annie yelled "cut" the first time, she believes that P_1 is exactly one-third of the cake. She thinks that P_2 and P_3 are equally valuable and together are worth two-thirds of the cake, so she thinks P_2 and P_3 are each exactly one-third of the cake as well. Since she considers each of the three pieces to be equally valuable, she envies no one.

Finally, Ben considers P_1 to be less than one-third the cake since he was not the one to yell "cut". So he thinks P_2 and P_3 together make up more than two-thirds of the cake. So Ben values P_2 and P_3 equally, and strictly more than P_1. Since Ben chooses second, at least one of P_2 and P_3 will be available, so he envies no one.

- If Annie delays her shout, then she may receive a piece less than 1/3 of her valuation. Imagine that she may be the second person to choose and the largest piece has been taken. She can choose among the two equal pieces with valuation less than 1/3.

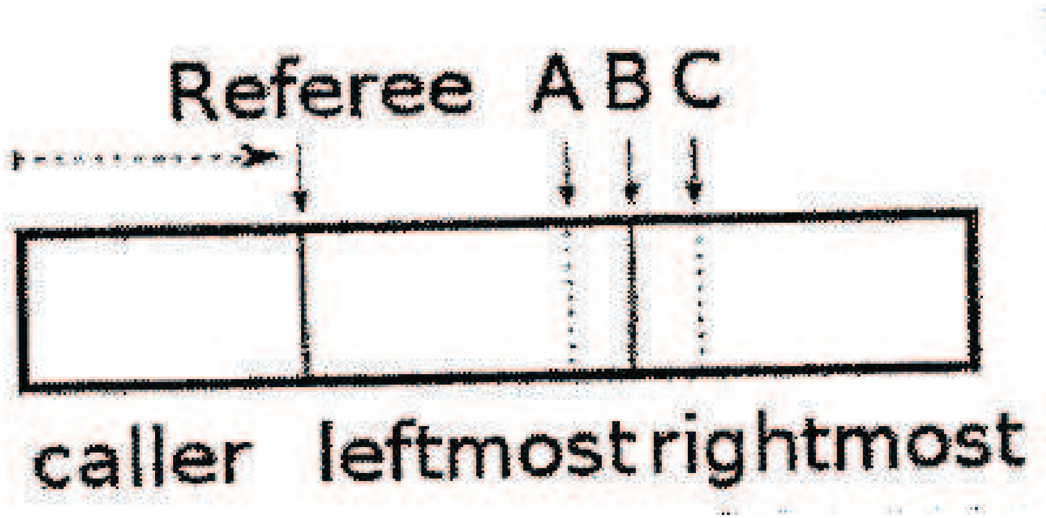
2. Stromquist's procedure with only two cuts

It requires only two cuts, the minimum for three pieces. There is no natural generalization to more than three players which divides the cake without extra cuts. For example, 11 cuts are required for 4-player envy free moving knife procedure.

The resulting partition is not necessarily efficient. For example, it cannot produce the efficient allocation of cutting a cake when the vanilla strips are on the two edges, while the chocolate strip and banana strip are in the middle. Suppose Alice only favors chocolate, Ben only favors banana and Chris favors only vanilla. The allocation that allocates the parts according to the sole flavor is the only efficient allocation since players' valuations of all other allocations can always be improved by choosing this efficient allocation, where $V_i(X_i) = 1$, $i = 1, 2, 3$. However, this efficient allocation cannot be achieved by this 3-player moving knife procedure with 2 cuts.

vanilla	banana	chocolate	vanilla

The sword and 3 knives move simultaneously.



- A referee moves a sword from left to right over the cake in a continuous manner, dividing it into a small left piece and a large right piece.

- Each player moves his knife continuously that cuts the right portion (right to the sword) into halves according to his valuation.

- When any player shouts “cut”, the cake is cut by the sword and by whichever of the players’ knives happens to be the central one of the three knives (that is, the second in order from the sword).

The cake is divided in the following manner:

- The piece to the left of the sword, which we denote *Left*, is given to the player who first shouted "cut". We call this player the "shouter" and the other two players the "quieters".

- The piece between the sword and the central knife, which we denote *Middle*, is given to one of the remaining players whose knife is closest to the sword. The remaining piece, *Right*, is given to the third player.

Summary

The shouter always receives the LEFT piece.

(i) Suppose A shouts, then B receives *Middle* and C receives *Right*.

(ii) Suppose B shouts, then A receives *Middle* and C receives *Right*.

(iii) Suppose C shouts, then A receives *Middle* and B receives *Right*.

Strategy to ensure envyfreeness

Each player can act according to his own measure that guarantees no other player receives more than him based on his personal valuation. Precisely, this is envyfreeness.

- Always hold your knife such that it divides the part to the right of the sword to two pieces that are equal in your eyes (hence, your knife initially divides the entire cake to two equal parts and then moves rightwards as the sword moves rightwards).
- Shout "cut" when Left becomes equal to the piece you are about to receive if you remain quiet. That is, if your knife is leftmost, shout "cut" if Left = Middle; if your knife is rightmost, shout if Left = Right; if your knife is central, shout "cut" if Left = Middle = Right.

Gaming around

If the player does not shout according to the above strategy, she may receive a smaller piece with certain probability though delaying shout might yield a larger piece.

Take A as an example, if the sword has passed the point where Left = Middle, A may be forced to take Middle (smaller than Left due to delayed shout) when someone shouts subsequently. Based on the "risk averse" assumption of the players, the players should play honestly.

On the other hand, A will not shout too early (prior to Left = Middle) since A can wait to receive larger piece of Left. For example, A should remain not to shout even at $V_A(\text{Left}) = \frac{1}{3}$ since $V_A(\text{Middle}) > \frac{1}{3}$ (since A's knife position is still within the Middle piece).

How about A games around by choosing not to cut the portion on the right of the sword into equal halves? Suppose A now becomes the rightmost player by this gaming strategy, he will receive RIGHT upon someone's shout. However, $V_A(\text{RIGHT}) < V_A(\text{MIDDLE})$ since he places higher valuation on the MIDDLE as A's truthful knife position should be more to the left.

Proof of envy-free share

First, consider the two quieters. Each of them receives a piece that contains his knife, so they do not envy each other. Additionally, since they remained quiet, the piece they receive is larger in their eyes then Left, so they also do not envy the shouter.

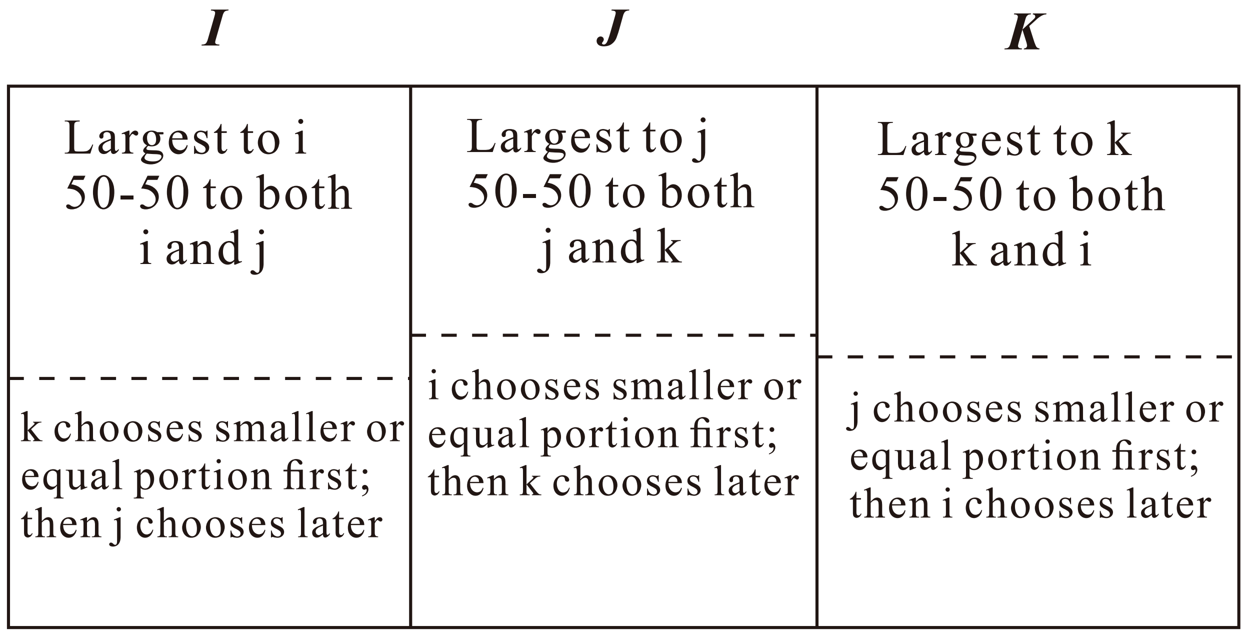
The shouter receives Left, which is equal to the piece he could receive by remaining silent and larger than the third piece. For example, suppose A is the shouter, then LEFT (piece received) = MIDDLE (piece to be received if remains silent) > RIGHT (third piece). Hence, the shouter does not envy any of the quieters.

3-person chore (dirty work) division procedure (each getting as little as possible of the share)

Step 1. Divide the chores into three *portions* using any 3-person envy-free cake-division procedure (that guarantees players a piece they think is the largest), such as the Stromquist moving-knife procedure. Now label each portion by the name of the player to whom the cake-division procedure would assign that portion (this player believes that portion is the *largest*).

Step 2. Arrange players i and j to divide portion I into 2 pieces. This can be achieved via Austin's procedure: letting player i and one other player, say j, agree on a 50-50 split, let the remaining player k choose the half she thinks is smallest, and give the other half to j.

Step 3. Repeat Step 2 for each player, then end the procedure.



This procedure requires at most 8 cuts (Step 1 uses 2 cuts, and Austin's procedure uses at most 2 cuts each time it is applied).

Player i participates in cutting portions I and K.

- i and j cut portion I so that both portions are 50-50 to both i and j; k picks the one that is smaller or equal, the remaining portion I goes to j.
- k and i cut portion K so that both portions are 50-50 to both k and j; j picks the one that is smaller or equal, the remaining portion K goes to i.

Proof of envy-freeness

Consider i, the pieces labelled J and K are smaller or equal to the piece labelled I. He receives portions of pieces from J and K, avoiding I (the piece that he considers to be the largest).

Observe the careful design of the cutters and orders of choosing the pieces:

- For K, i is the late chooser but i serves as the cutter. There is no harm to serve as the second chooser.

- For J, though i does not serve as the cutter, i is the first chooser.

- For I, i serves as the cutter to ensure that both pieces are larger than (or at least equal) the two pieces received by i.

(i) i does not envy j

We compare the two portions of pieces received by i and j.
For piece K, though j receives the portion earlier than i, i does not envy since i does the cutting into equal halves to his own valuation.
For piece I, j receives 50% of I to i's valuation. Also, i chooses the smaller or equal portion from piece J, and piece J is smaller or equal to piece I.

(ii) i does not envy k

For piece J, i receives smaller or equal portion compared to that of k.
For portion of piece I received by k, that portion is larger or equal to that portion of K received by i.

1.3 Adjusted winner procedure for two-player allocation of discrete goods

The adjusted winner procedure is a method of dispute resolution (division of indivisible/discrete goods) for two players that guarantee an outcome that is envy-free, equitable and efficient.

Suppose that Annie and Ben are getting divorced. Each player has 100 points to distribute over all the items according to which they value most. Annie and Ben's point distribution are below.

Annie	*Item*	*Ben*
35	House	15
20	Investments	**25**
10	Piano	**25**
5	TV	**15**
25	Dog	10
5	Car	**10**
100	Total	100

Criteria of a good division procedure

- Fairness, like observing equitability, efficiency and envy-freeness
- Difficulty of manipulating a procedure that produces a division (providing intrinsic incentive to be truthful about one's evaluations of item values)

Two-stage division

During the first stage, each item is initially awarded to the person who values it most. So Annie receives the house and the dog, and Ben receives the investment account, baby grand piano, plasma TV, and the car. At this point, Annie has 60 points, and Ben has 75 points. Since Ben has more points, we say that Ben is the initial winner.

- In case there is a tied (equal point) item, the tied item is given to whomever has fewer points at the time.

Ben is the initial winner, some items are taken away from Ben to Annie

The next stage is the *equitability adjustment*. We need to transfer items, or fractions thereof, from Ben to Annie until the point totals of each are equal and the allocation is thus equitable. To determine the order, for each of Ben's items, we consider the ratio of the points assigned by Ben to the item to the points assigned by Annie to the item. Note that each of these ratios will be at least 1, since Ben received the items to which he had assigned more points.

The ratios for each of Ben's items are as follows:

$$\begin{aligned} \text{Investment} : \frac{25}{20} &= 1.25 \\ \text{Piano} : \frac{25}{10} &= 2.5 \\ \text{TV} : \frac{15}{5} &= 3 \\ \text{Car} : \frac{10}{5} &= 2 \end{aligned}$$

How to achieve Pareto efficiency?

The taking away of items (whole or partial) from the initial winner starts with the item for which the ratio above is the smallest, then the next smallest, and so on. Intuitively, this is the sensible way to achieve Pareto efficiency since the “cost” to Ben per point transferred to Annie is smallest.

For example, transferring the TV requires lowering Ben’s point total by 3 points for every 1 point transferred to Annie, while transferring the car would only lower Ben’s point total by 2 for every 1 point transferred to Annie.

We start with the ratio for the Investment, since it has the smallest ratio. If we were to transfer the entire Investment to Annie, then Annie would have more points than Ben.

Let x be the fraction of the Investment transferred from Ben to Annie, so that $1 - x$ is the fraction retained by Ben. After the transfer, Annie will have 60 points (from the house and dog) plus $20x$ (her portion of the investments), while Ben will have 50 points (from the piano, TV, and car) plus $25(1 - x)$ (his portion of the investments).

To guarantee that the resulting point totals are equal, we need to ensure that $60 + 20x = 75 - 25x$. Solving the equation, we obtain $x = \frac{1}{3}$.

Annie receives the house, the dog, and one-third of the investment portfolio, while Ben keeps the piano, TV, car and two-thirds of the investments.

Each person walks away with an impressive total of $66\frac{2}{3}$ points, well over half the total value.

How to split indivisible goods?

If we had needed to split the piano, it certainly would not be simple since a third of a piano is not very valuable to anyone!

Together, then Annie and Ben might decide to sell the piano and split the profits according to the prescribed proportions. Or if Ben receives the larger share, he may choose to own the whole piano by buying out Annie's share (challenge: how to set the price if it is not in the market).

- If after the whole portion of an item is transferred, the initial winner still has the larger point total, then the next item is transferred. The procedure terminates until transferring an item results in equal point total.

- The procedure can be easily modified in the case of unequal entitlements, for instance if a prenuptial agreement indicated that the shared property be divided 60%-40%.

Summary of the Adjusted Winner procedure

Suppose that Ann and Bob are each given 100 points to distribute among n goods as he/she sees fit. In other words, Ann and Bob each select a valuation, $\boldsymbol{\alpha} = (A_1, \ldots, A_n)$ and $\boldsymbol{\beta} = (B_1, \ldots, B_n)$, respectively. For convenience rename the goods so that

$$A_1/B_1 \geq A_2/B_2 \geq \cdots A_r/B_r \geq 1 > A_{r+1}/B_{r+1} \geq \cdots A_n/B_n.$$

Let $\boldsymbol{\alpha}/\boldsymbol{\beta}$ be the above vector of real numbers (after renaming of the goods). Notice that this renaming of the goods ensures that Ann, based on her valuation $\boldsymbol{\alpha}$, values the goods $G_1, \ldots, G_r$ at least as much as Bob; and Bob, based on his valuation $\boldsymbol{\beta}$, values the goods $G_{r+1}, \ldots, G_n$ more than Ann does.

The Adjusted Winner algorithm proceeds as follows:

1. Give all the goods $G_1, \ldots, G_r$ to Ann and $G_{r+1}, \ldots, G_n$ to Bob. Let X, Y be the number of points received by Ann and Bob respectively. Assume for simplicity that $X \geq Y$.

2. If $X = Y$, then stop. Otherwise, transfer a portion of G_r from Ann to Bob which makes $X = Y$. If equitability is not achieved even with all of G_r going to Bob, transfer $G_{r-1}, G_{r-2}, \ldots, G_1$ in that order to Bob until equitability is achieved.

Israeli-Palestinian conflict in the Middle East

Five major issues to be negotiated:

1. *West Bank*: Several areas of the West bank are inhabited by Israelis who have no desire to leave their homes. The Palestinians, however, believe that these settlements are illegal, and that the Israelis should evacuate.

2. *East Jerusalem*: In 1967, Israel unified control over all the Jerusalem by defeating Jordanian forces in the Six Days War. A majority of the residents of east Jerusalem are Palestinian, however, and both Israelis and Palestinians argue that East Jerusalem is central to their sovereignty.

3. *Palestinian Refugees*: Israel has refused to recognize that its establishment and expansion in 1948 and 1967 displace Palestinian villages and communities. The Palestinians insist that Israel recognizes the refugees "right to return" to Israel, and provides compensation for the refugees and to Arab states that have hosted the refugees.

4. *Palestinian Sovereignty*: Israel does not recognize Palestine as a sovereign nation.

5. *Security*: Some Israelis fear that terrorism would flourish under a Palestinian state that lacks the means to effectively fight terrorism. Specific security issues include: border control, control of airspace, security in Jerusalem, and "early warning stations" in the West Bank and Gaza.

Point allocation

By examining the expert opinions and interim agreements, we arrive at the following reasonable estimates of possible point allocations by each side.

Israel	*Item*	*Palestine*
22	West Bank	21
25	East Jerusalem	23
12	Palestinian Refugees	18
15	Palestinian Sovereignty	24
26	Security	14
100	Total	100

In the first stage of the adjusted winner procedure, Israel wins the issues of the West Bank, East Jerusalem and security, while Palestine wins the issues of refugees and sovereignty.

After the first stage, Israel has 73 points and Palestine has 42 points. Since Israel is the initial winner, then we look at the ratios of points for the issues won by Israel:

$$\begin{aligned} \text{West Bank} &: \quad \frac{22}{21} \\ \text{East Jerusalem} &: \quad \frac{25}{23} \\ \text{Security} &: \quad \frac{26}{14} \end{aligned}$$

The equitability adjustment begins with the West Bank since $\frac{22}{21} < \frac{25}{23} < \frac{26}{14}$. Not the whole West Bank will be transferred since transferring the entire West Bank would give the Palestinians more points than the Israelis.

To determine the percentage x of the West Bank retained by Israel, we solve for x in the following equation:

$$\begin{aligned} 51 + 22x &= 42 + 21(1 - x) = 63 - 21x \\ 43x &= 12 \\ x &= \frac{12}{43} \approx \frac{2}{7}. \end{aligned}$$

The Israelis are left with the issues of East Jerusalem, security, and roughly $\frac{2}{7}$ of the issue of the West Bank. The Palestinians are left with the issues of refugees, sovereignty, and roughly $\frac{5}{7}$ of the issue of the West Bank.

Remark The division of land is easier though different parts of the land may have different individual valuations. The division of sovereignty would require political interpretation of varying degrees of autonomy.

Equitability, efficiency and envy-freeness

Equitability: The procedure is equitable by design. The procedure ends when the point totals of each party are equal.

Efficiency (in Pareto sense): There exist no other allocations that give higher point to one player and at least as good for the other player when compared to the allocations based on the adjusted winner procedure.

Envy-freeness: This property follows from the other two when exactly two parties are involved.

Since this is a two-player allocation, we have envyfreeness $\Leftrightarrow$ proportionality. It is highly desirable to observe that all of the four fairness criteria are met in the adjusted winner procedure.

We prove efficiency later. The next key result:

$$\text{equitability} + \text{efficiency} \Rightarrow \text{envyfreeness}.$$

Proof of envy-freeness

We prove by contradiction. Suppose that the allocation is equitable and efficient, but not envy-free. Since envy-freeness and proportionality are equivalent for two players, by virtue of failure of proportionality, it must be the case that at least one of the players received less than half according to his own valuation. The assumption of equitability implies that both players received less than half.

This allocation is not Pareto efficient because we can find another division in which both players do better: give each player's share to the other player (swapping the allocation). If each player originally received x points, where $x < 50$, then now each receives $100 - x > 50$ points, so this allocation is strictly better for both players involved, contradicting efficiency of the original allocation.

Pareto efficiency in the adjusted winner procedure

Parties: Annie and Ben.
Items: $G_1, ..., G_n$ to be divided between Annie and Ben.

In the first stage of the adjusted winner procedure, every item is first given to the person who values it most. Items are then transferred from the initial winner to the other player until both have an equal number of points.

The proof of efficiency hinges on the order in which the items are transferred: the transfer begins with the item with the smallest ratio of points given by the initial winner to points given by the other player. In this way, we minimize the effective cost to the initial winner for all points transferred to the other player.

$A_i =$ points allocated by Annie for G_i
$B_i =$ points allocated by Ben for G_i

Lemma 1

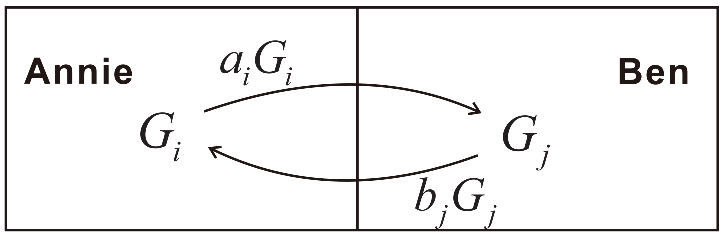
Suppose that we have an allocation of the items in which

(i) Annie values item G_i at least as much as Ben does: $A_i \geq B_i$;

(ii) Ben values item G_j at least as much as Annie does: $B_j \geq A_j$.

Suppose that Annie trades her portion of G_i for Ben's portion of G_j.

Transfer of goods



Transfer of points

Annie	**Ben**
$+b_j A_j$	$+a_i B_i$
$- a_i A_i$	$- b_j B_j$

If this trade is strictly better for one player, then it is strictly worse for the other. That is, Pareto improvement of allocation is NOT possible.

Proof

Recall that $A_i \geq B_i$ and $B_j \geq A_j$. During the trade, Annie gives away a total of $a_i A_i$ points, and gains a total of $b_j A_j$ points. If the trade is strictly better for Annie, then

$$b_j A_j > a_i A_i. \tag{1}$$

We compare Ben's points before or after trade, where

$$\begin{aligned} & \text{Ben's points after trade} \ - \ \text{Ben's point before trade} \\ = \ & a_i B_i - b_j B_j \\ \leq \ & a_i A_i - b_j A_j \text{ since } B_j \geq A_j \text{ and } B_i \leq A_i \\ < \ & 0 \text{ by virtue of } (1), \end{aligned}$$

so Ben is strictly worse off after the trade.

Similarly, if the trade is strictly better for Ben, then it is strictly worse for Annie.

It is not surprising that the trade cannot achieve one player strictly better and the other player at least as good since both players give away portions of goods that worth more to the donor than the receiver.

Lemma 2

Suppose that we have transfer of portion of a pair of items that observe $\frac{A_j}{B_j} \leq \frac{A_i}{B_i}$. If Annie trades her portion of G_i for Ben's portion of G_j, and this trade is strictly better for one player, then the trade is strictly worse for the other.

Proof
If the trade is better for Annie, then $b_j A_j > a_i A_i$. Since $\frac{A_j}{B_j} \leq \frac{A_i}{B_i}$, then $A_j B_i \leq A_i B_j$. Now, consider

$$\begin{aligned}
& \text{Ben's points after trade} \;-\; \text{Ben's point before trade} \\
= \;& a_i B_i - b_j B_j \\
< \;& B_i \left(\frac{b_j A_j}{A_i}\right) - b_j B_j \text{ since } b_j A_j > a_i A_i \\
= \;& b_j \left(\frac{B_i A_j - B_j A_i}{A_i}\right) \\
\leq \;& 0 \text{ since } A_j B_i \leq A_i B_j,
\end{aligned}$$

so Ben is strictly worse off after the trade.

If the trade strictly benefits Ben, it follows that $a_i B_i > b_j B_j$. We then have

$$\begin{aligned} & \text{Annie's points after trade} - \text{Annie's point before trade} \\ = \ & b_j A_j - a_i A_i \\ < \ & b_j A_j - A_i \left(\frac{b_j B_j}{B_i} \right) \text{ since } a_i B_i > b_j B_j \\ = \ & b_j \left(\frac{A_j B_i - A_i B_j}{B_i} \right) \\ \leq \ & 0 \text{ since } A_j B_i \leq A_i B_j, \end{aligned}$$

so Annie is strictly worse off after the trade.

Remark

Note that $A_i \geq B_i$ and $B_j \geq A_j \Rightarrow \dfrac{A_j}{B_j} \leq \dfrac{A_i}{B_i}$. Lemma 2 uses the less stringent condition and arrives at the same result that if the trade is strictly better off for one player, then it is strictly worse for the other.

Lemma 3

If a given allocation is not efficient, then there exist single item goods G_i and G_j and some portions thereof such that if Annie exchanges her fraction a_i of G_i for Ben's fraction b_j of G_j, the resulting trade yields an allocation that is at least as good for both players and strictly better for at least one of the players.

Proof

Since the given allocation is not efficient, there exist disjoint sets S and T of goods belonging to Annie and Ben, respectively, such that an exchange of S and T makes Annie better off without hurting Ben. We start with

$$S \underset{\text{Ben}}{\succeq} T \quad \text{and} \quad T \underset{\text{Annie}}{\succ} S.$$

The key in the proof of Lemma 3 is to show that S and T can each be taken to be a fraction of a **single item**.

Assumption of *weak additivity* of preferences: If A and B are disjoints sets of goods, and the player values A at least as much as some set X of goods and B at least as much as some set Y of goods, then she must value $A \cup B$ at least as much as $X \cup Y$. In other words, given $A \succeq X$ and $B \succeq Y$, then $A \cup B \succeq X \cup Y$.

Write $S = S_1 \cup ... \cup S_n$, where S_i's are pairwise disjoint, and S_i is a fraction α_i of a single item, where $0 \leq \alpha_i \leq 1$. Ben can now break up T into a disjoint union $T = T_1 \cup ... \cup T_n$ (not necessarily subsets of a single item) such that an exchange of S_i for T_i yields an allocation that is no worse for him than the current allocation. This is always possible since goods are assigned with known values. That is, Ben splits T into $T_1, T_2, \ldots, T_n$ such that

$$S_i \underset{\text{Ben}}{\succeq} T_i, \quad i = 1, 2, \ldots, n.$$

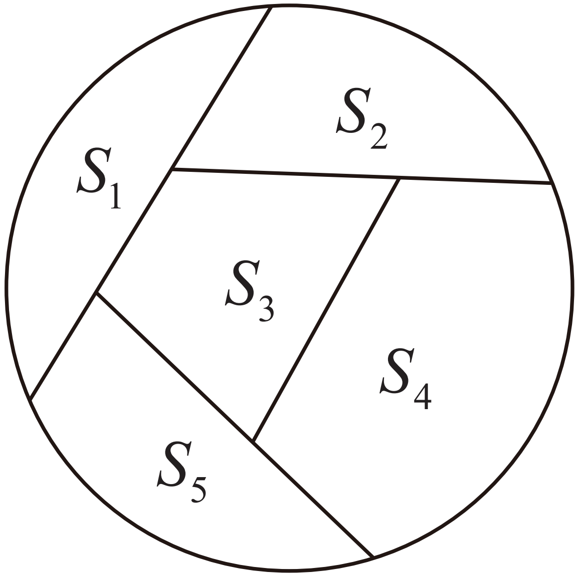
Remark: Each S_i is single item but T_i is not.

Starting from

$$S \underset{\text{Ben}}{\succeq} T \qquad \text{and} \qquad T \underset{\text{Annie}}{\succ} S,$$

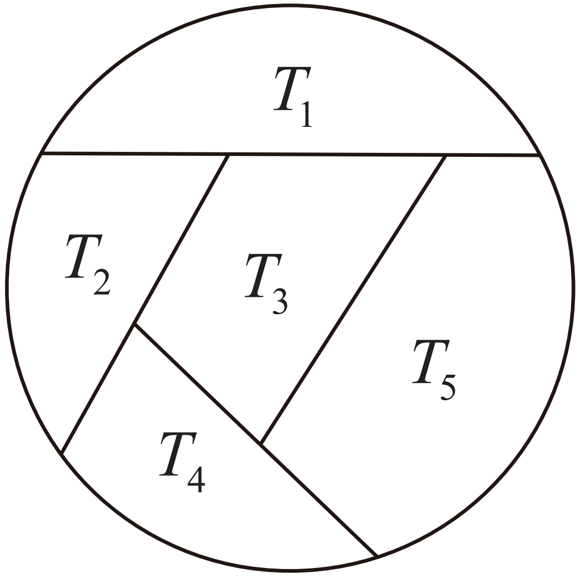
then perform the division:

$$S = S_1 \cup \ldots \cup S_5$$



S_i is a fraction α_i of single item i, $i = 1, 2, \ldots, 5$.

$$T = T_1 \cup \ldots \cup T_5$$



$S_i \underset{\text{Ben}}{\succeq} T_i$, $i = 1, 2, \ldots, 5$

T_i is not a fraction of single item

Recall a simpler algebraic result. Suppose we have

$$x_1 + x_2 + \cdots + x_n > y_1 + y_2 + \cdots + y_n,$$

then there exists at least one i such that $x_i > y_i$.

In the current context, for a given i, there is no guarantee that $T_i \underset{\text{Annie}}{\succ} S_i$. The next step in the proof is to show that there exists at least one i such that $T_i \underset{\text{Annie}}{\succ} S_i$.

Assume contrary, suppose $S_i \underset{\text{Annie}}{\succeq} T_i$ for all i, by the weak additivity of preferences, then the existing allocation is at least as good for Annie as the one obtained by exchanging $S_1 \cup ... \cup S_n = S$ for $T_{\cup} ... \cup T_n = T$, contradicting the assumption that an exchange of S and T makes Annie better off without hurting Ben.

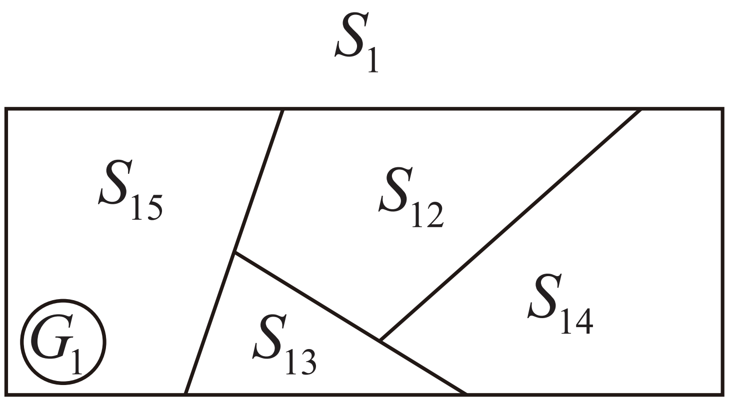
Relabeling if necessary, suppose that Annie prefers the allocation obtained by exchanging S_1 for T_1 to the existing allocation. Now S_1 consists of some portion of a single good, but T_1 may consist of portions of several goods.

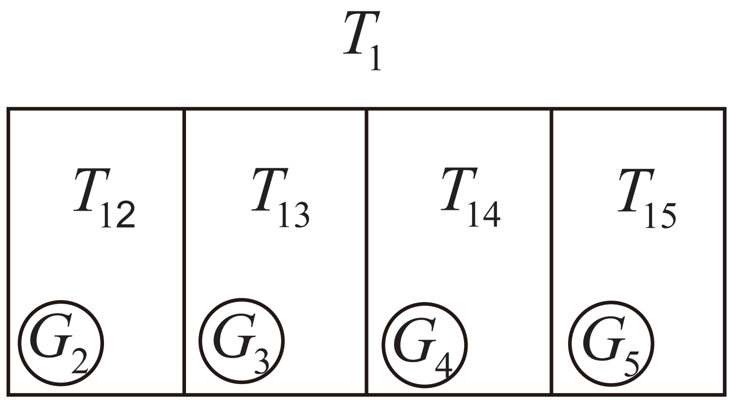
Division of T_1 into a disjoint union

Now $T_1 \underset{\text{Annie}}{\succ} S_1$, then Annie splits S_1 such that $T_{1j} \underset{\text{Annie}}{\succ} S_{1j}$, $j = 2, 3, \ldots, m$, where $T_1 = T_{12} \cup \ldots \cup T_{1m}$ and $S_1 = S_{12} \cup \ldots \cup S_{1m}$, $m \leq n$, and T_{1j} is some portion of the single j^{th} good. Recall that $S_{12}, \ldots, S_{1m}$ belong to the same item S_1, while T_{1j} is portion of the j^{th} item.

Though $T_{1j} \underset{\text{Annie}}{\succ} S_{1j}$, $j = 2, 3, \ldots, m$, there is no guarantee that $S_{1j} \underset{\text{Ben}}{\succeq} T_{1j}$ for given j. However, it is not possible to have $S_{1j} \underset{\text{Ben}}{\prec} T_{1j}$ for all j. If so, this would give $S_1 \underset{\text{Ben}}{\prec} T_1$ by weak additivity, a contradiction to the earlier result (see p.85). Therefore, there exists j^* such that $S_{1j^*} \underset{\text{Ben}}{\succeq} T_{1j^*}$. Note that each S_{1j^*} and T_{1j^*} is coming from single item, where S_{1j^*} comes from item 1 and T_{1j^*} comes from item j^*, $j^* \neq 1$. Since S and T are disjoint, when S contains a fraction of the first item, T does not contain any fraction of the first item.

In conclusion, we have found S_{1j^*} and T_{1j^*}, each is a portion of a single item for which an exchange of S_{1j^*} for T_{1j^*} yields an allocation that is strictly better for Annie and no worse for Ben than the existing allocation.





Split S_1 into $S_1 = S_{12} \cup \ldots \cup S_{15}$ such that $T_{1j} \underset{\text{Annie}}{\succ} S_{1j}$, $j = 2, \ldots, 5$.

There exists j^* such that $S_{1j^*} \underset{\text{Ben}}{\succeq} T_{1j^*}$.

Split T_1 into $T_1 = T_{12} \cup \ldots \cup T_{15}$ where T_{1j} is a fraction β_j of single item j, $0 \leq \beta_j \leq 1$, $j = 2, \ldots, 5$. T_1 does not contain any fraction of G_1.

Summary

Starting from $S \underset{\text{Ben}}{\succeq} T$ and $T \underset{\text{Annie}}{\succ} S$, where S and T generally involve multiple items. We would like to achieve $S_{1j^*} \underset{\text{Ben}}{\succeq} T_{1j^*}$ and $T_{1j^*} \underset{\text{Annie}}{\succ} S_{1j^*}$, where each S_{1j^*} and T_{1j^*} is a fraction of distinct single item.

1. Split S such that $S = S_1 \cup \cdots \cup S_n$, where S_i is a fraction of item i (single), $i = 1, 2, \ldots, n$. Ben cuts T into $T = T_1 \cup \cdots \cup T_n$ such that $S_i \underset{\text{Ben}}{\succeq} T_i$ for all i.

   By weak additivity, we can show that there exists some i such that $T_i \underset{\text{Annie}}{\succ} S_i$. For notional convenience, write these particular S_i and T_i as S_1 and T_1, respectively.

2. Split T_1 such that $T_1 = T_{12} \cup T_{13} \cdots \cup T_{1n}$, where T_{1j} is a fraction of single item, $j = 2, 3, \ldots, n$. Note that T_1 does not contain any fraction of item 1 since S and T are disjoint sets.

   Annie cuts S_1 into $S_1 = S_{12} \cup \cdots \cup S_{1n}$ such that $T_{1j} \underset{\text{Annie}}{\succ} S_{1j}$, $j = 2, 3, \ldots, n$. Again, by weak additivity, we can show that there exists j^* such that $S_{1j^*} \underset{\text{Ben}}{\succeq} T_{1j^*}$ while $T_{1j^*} \underset{\text{Annie}}{\succ} S_{1j^*}$, $j^* \neq 1$.

We have found S_{1j^*} and T_{1j^*}, each consisting of a portion of a single item for which an exchange of S_{1j^*} for T_{1j^*} yields an allocation that is strictly better for Annie and no worse for Ben than the existing allocation.

Proof of Pareto efficiency of the adjusted winner procedure

We prove by contradiction. Assume failure of Pareto efficiency, then Pareto improvement as exemplified by Lemma 3 can be stated as follows: there exist *single item* goods G_i and G_j and portions thereof such that if Annie exchanges her fraction a_i of G_i for Ben's fraction b_j of G_j, the resulting trade yields an allocation that is at least as good for both and strictly better for at least one.

In the following steps of proof, details of the adjusted winner procedure are incorporated in the arguments. Without loss of generality, we assume that Annie was the initial winner after the first step of the adjusted winner procedure. Since Annie still has at least a_i of item G_i, then Annie (initial winner) must value item G_i at least as much as Ben does, so $A_i \geq B_i$. Indeed, any item held by Annie (initial winner) observes the property that Annie gives higher value (or at least the same value) to the item than Ben.

(i) $B_j \geq A_j$
Now if Ben values item G_j at least as much as Annie does, $B_j \geq A_j$, then Lemma 1 implies that the trade will not give both parties at least as good and one party is strictly better as we are assuming. This leads to a contradiction.

(ii) $B_j < A_j$
Since $B_j < A_j$, Ben values item G_j less than Annie does. Ben does not receive any portion of G_j in the first stage of allocation. Since Ben has part of G_j, so he must have received that during the *transfer stage* of the adjusted winner procedure.

Comparing G_i and G_j, G_i is still held by Annie (initial winner) while G_j (whole or part) is received by Ben (initial loser) in the transfer stage. As a consequence of G_j being chosen in the transfer stage, we deduce that $\frac{A_i}{B_i} \geq \frac{A_j}{B_j}$. By Lemma 2, this contradicts our assumption that the trade benefits at least one and at least as good for the other.

Manipulability

Determining point totals is itself not an easy task. The situation is still more stressful if the parties involved need to worry about strategies as well, especially in the case of a divorce where each party has in depth knowledge of the other's like and dislike.

It is natural to wonder whether this knowledge would enable one party to manipulate the system, and achieve a better outcome by submitting dishonest point allocations.

Unless knowledge of the other's party's valuations is strictly one-sided, then honesty is the best policy in the adjusted winner procedure.

Example: Honesty is the best policy

Suppose that Annie and Ben are getting a divorce, and currently share the following items: a townhouse in Central Square, season passes to the Red Sox, and a painting.

They value the items as follows:

Annie	*Item*	*Ben*
50	Townhouse	30
20	Red Sox Tickets	50
30	painting	20
100	Total	100

Applying the adjusted winner procedure, we see that Annie is initially awarded the townhouse and the painting, while Ben gets the Red Sox tickets.

Annie currently has 80 points, while Ben has 50, so Annie is the initial winner. The ratio of points for the townhouse is $\frac{5}{3}$, while the ratio for the painting is $\frac{3}{2}$, so the painting needs to be divided. Solving for x in the following equation gives the fraction of the painting that Annie keeps:

$$50 + 30x = 50 + 20(1 - x) = 70 - 20x \text{ giving } x = \frac{2}{5}.$$

Annie ends up with the townhouse and $\frac{2}{5}$ of the painting (Annie and Ben decide that she will buy out his share of the painting), and Ben gets the Red Sox tickets and $\frac{3}{5}$ of the painting. Each with a total of 62 points.

Knowledge on one side

Annie is confident that she can estimate Ben's point allocations fairly well, and decides to submit the following false valuations, rather than her true preferences given above

Annie's fake valuations	*Item*	*Ben's sincere valuations*
32	Townhouse	30
48	Red Sox Tickets	50
20	painting	20
100	Total	100

- Lower her value of Townhouse to be slightly above that of Ben.
- Set her value of painting to be the same as that of Ben so that the tied item is given to her since she scores lower after allocating the Townhouse and Tickets.
- Since the sum of points is 100, as a result, the value of Tickets is increased to be slightly below that of Ben.

Intuitively, Annie might do better under this scenario. By indicating that she values the townhouse only slightly more than Ben, she hopes to win the townhouse but at a lower cost, thereby winning a higher percentage of the painting as well.

In the first step of the process, Annie still gets the townhouse and the painting, and Ben gets the Red Sox tickets. Annie has 52 points (according to her false point allocations), and Ben has 50. The painting is split since the ratio of values is lowest. Solving for x gives the fraction of the painting that Annie keeps:

$$32 + 20x = 50 + 20(1 - x) = 70 - 20x \text{ giving } x = \frac{19}{20}.$$

By lowering the point to the Townhouse as much as possible, Annie has $18 = 50 - 32$ points of difference. Furthermore, she gains more by lowering the point of painting from 30 to 20. This is achieved by increasing the point to tickets.

Knowledge on both sides

With this kind of knowledge on both sides, it becomes much riskier to submit false preferences. While it may be to someone's advantage to be dishonest (Annie might still get lucky if Ben chooses to submit his true point allocations even with knowledge of Annie's preferences), this strategy can also backfire, resulting in an outcome that is worse than the honest outcome.

For example, if Ben thinks that Annie will be honest, he may submit the following point allocations:

Annie's sincere valuations	*Item*	*Ben's fake valuations*
50	Townhouse	45
20	Red Sox Tickets	25
30	painting	30
100	Total	100

If Annie were honest, then Ben and Annie would each get 52.5 points based on Ben's fake valuations, though this would really constitute over 68 points for Ben according to his true valuations. The resulting point is higher than 62 if both are honest.

But if Annie and Ben both submit these false preferences, the result is not good for either.

In the first step of the process, Annie receives the Red Sox tickets, and Ben gets the townhouse and painting. The ratio for the painting is 1.5 while the ratio for the townhouse is $\frac{45}{32}$, strictly less.

The following calculation gives the fraction of the townhouse to be given to Annie:

$$48 + 22x = 30 + 45(1 - x) = 75 - 45x \text{ giving } x = \frac{27}{77}.$$

So Annie gets just over a third of the townhouse and the Red Sox tickets, while Ben gets just under $\frac{2}{3}$ of the townhouse and the painting. Although this appears to be just over 59 points for each with the false point allocations, both Annie and Ben do much worse according to their true preferences.

Annie's share give her $50 \times \frac{27}{77} + 20$, roughly 37.5 points and Ben's share gives him $30 \times \frac{50}{77} + 20$, just under 39.5 points. Both Annie and Ben would have fared much better had they been honest!

Final remark
In addition to guaranteeing an allocation that is envy-free, equitable, and efficient, the adjusted winner procedure also promotes honesty. This is true at least when knowledge of the other party's preference is not strictly one-sided.

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