**MATH4994 — Capstone Projects in Mathematics and Economics**

**Topic 3 – Voting methods and social choice theory**

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## 3.1 Social choice procedures

A "*social choice procedure*" is a function where a typical input is a sequence of individual preference rankings of the alternatives and an output is a single alternative, or a single set of alternatives if we allow ties.

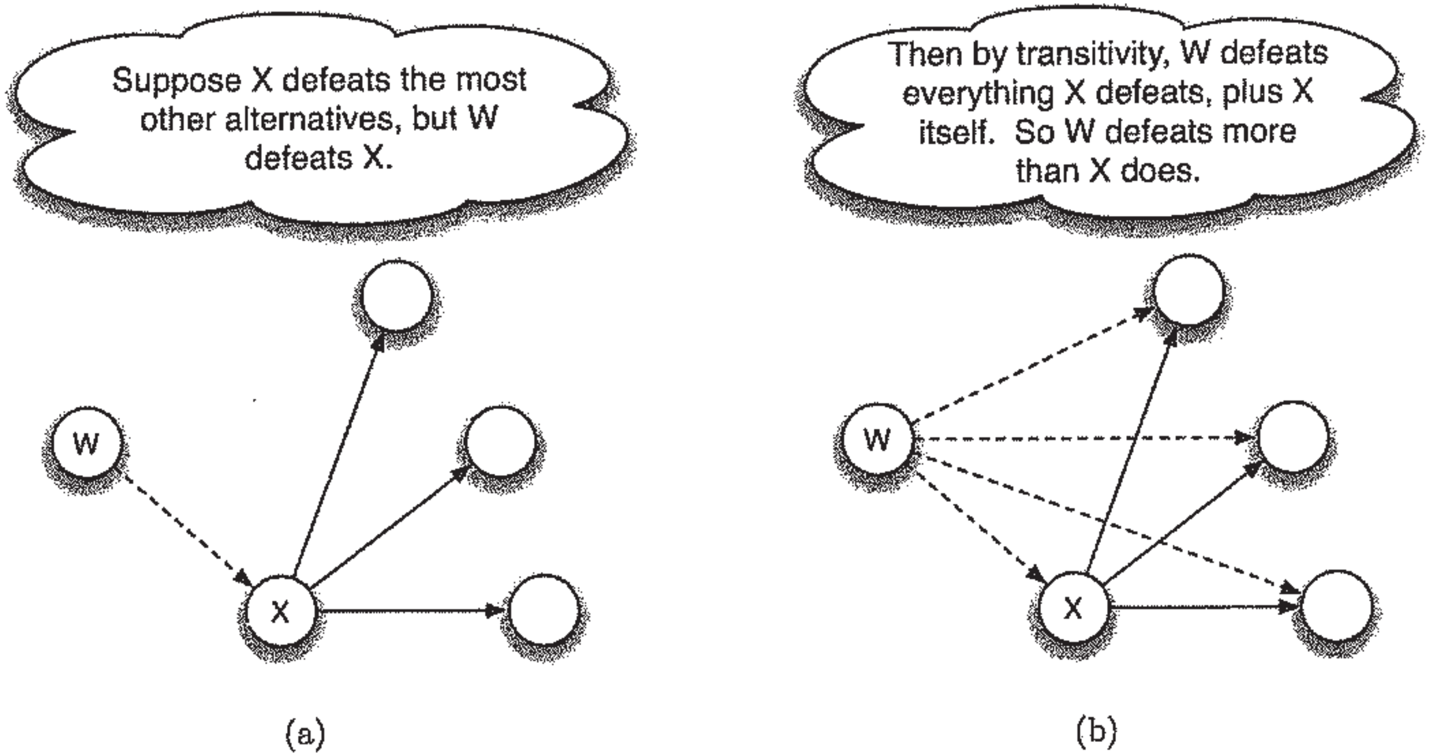
- A sequence of individual preference lists is called a 'profile'.
- The output is called the "social choice" or winner if there is no tie, or the "social choice set" or "those tied for winner" if there is a tie.
- How to use the information of individual preference rankings among the alternatives in the determination of the winner?
- What are the intuitive criteria to judge whether a social choice is "reasonably" acceptable? Is the winner being the least unpopular, broadly acceptable, winning in all one-for-one contests, etc?

## Individual preferences

A group of people is evaluating a set of possible alternatives. We suppose that for each individual, he is able to determine a preference between any pair of alternatives: $X \succ_i Y$, where $X$ and $Y$ is the pair of alternatives and $i$ is the individual.

*Completeness and transitivity*

Completeness means for each pair of distinct alternatives $X$ and $Y$, either she prefers $X$ to $Y$, or she prefers $Y$ to $X$, but not both. We exclude "ties" and "no preference". Transitivity means if an individual $i$ prefers $X$ to $Y$ and $Y$ to $Z$, then $i$ should also prefer $X$ to $Z$.




With complete and transitive preferences, the alternative $X$ that defeats the most others (number of alternatives defeated by $X$ is largest) in fact defeats all of them. If not, some other alternative $W$ would defeat $X$ (as in (a)), but then by transitivity $W$ would defeat more alternatives than $X$ does (as in (b)).

## Rank list and preference relations

From a ranked list, we could define a preference relation $\succ_i$ very simply: $X \succ_i Y$ if alternative $X$ comes before alternative $Y$ in $i$'s ranked list. That is, the preference relation arises from the ranked list. Also, if a preference arises from a ranked list of the alternatives, then it must be complete and transitive.

How to construct a ranked list arising from $\succ_i$ that is complete and transitive, where each alternative is preferred to all the alternatives that come after it.

*Procedures*

- First, we identify the alternative $X$ that defeats the most other alternatives in pairwise comparisons. That is, $X \succ_i Z$ for the most other choices of $Z$. Actually, this $X$ would defeat all the other alternatives: $X \succ_i Z$ for all other $Z$.

- Next, we remove $X$ from the set of alternatives; and repeat exactly the same process on the remaining alternatives. The preferences defined by $\succ_i$ are still complete and transitive on the remaining alternatives. Call $Y$ to be the alternative that defeats the most others in this reduced set.

- Now, $Y$ defeats every alternative in the original set except for $X$, so we put $Y$ second in the list. Remove it too from the set of alternatives, and continue in this way until we exhaust the finite set of alternatives.

## Examples of social choice procedures

### 1. Plurality voting

Declare as the social choice(s) to be the alternative(s) with the largest number of first-place rankings in the individual preference lists.

*1980 US Presidential election*: Democrat Jimmy Carter, Republican Ronald Reagan and Independent John Anderson

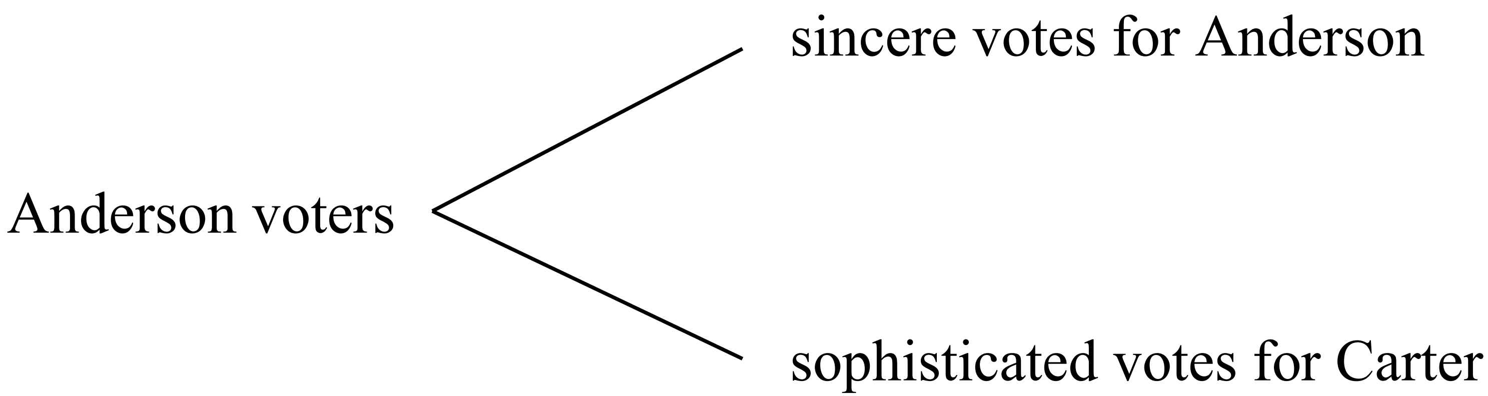
| *Reagan voters* (45%) | *Anderson voters* (20%) | *Carter voters* (35%) |
| --- | --- | --- |
| $R$ | $A$ | $C$ |
| $A$ | $C$ | $A$ |
| $C$ | $R$ | $R$ |

If voters can cast only one vote for their best choice, then Reagan would win with 45% of the vote.

- Reagan was perceived as much more conservative than Anderson who in turn was more conservative than Carter.

Since the chance of Anderson winning is slim, Anderson voters may cast their votes *strategically* to Carter so that their second choice could win.

- Anderson's voter's *sincere* strategy is to vote for her first choice.
- Reagan voters have a straightforward strategy: to vote sincerely.
- Adopting an admissible strategy that is not *sincere* is called *sophisticated* voting.



## Example

| 3 voters | 2 voters | 4 voters | "$c$" wins with first-choice votes; |
|---|---|---|---|
| $a$ | $b$ | $c$ | but 5-to-4 majority of |
| $b$ | $a$ | $b$ | voters rank $c$ last. |
| $c$ | $c$ | $a$ | |

Consider pairwise one-for-one contests:-

$b$ beats $a$ by 6 to 3; $b$ beats $c$ by 5 to 4; $a$ beats $c$ by 5 to 4.

Note that $b$ beats the other two in pairwise contests but $b$ is not the winner. Also, $c$ loses to the other two in pairwise contests but $c$ is the winner. This is like Chen in 2000 Taiwan election.

- Condorcet Winner criterion: The one who wins in all one-for-one contests should be the social choice.
- Condorcet Loser criterion: The one who loses in all one-for-one contests should not be the social choice.

## Plurality voting with run-off

Second-step election between the top two vote-getters in plurality election if no candidate receives a majority. This is used in the French presidential election.

**Example**

| *6 voters* | *5 voters* | *4 voters* | *2 voters* |
|---|---|---|---|
| $a$ | $c$ | $b$ | $b$ |
| $b$ | $a$ | $c$ | $a$ |
| $c$ | $b$ | $a$ | $c$ |

"a" with 11 votes beats "$b$" with 6 votes in the run-off

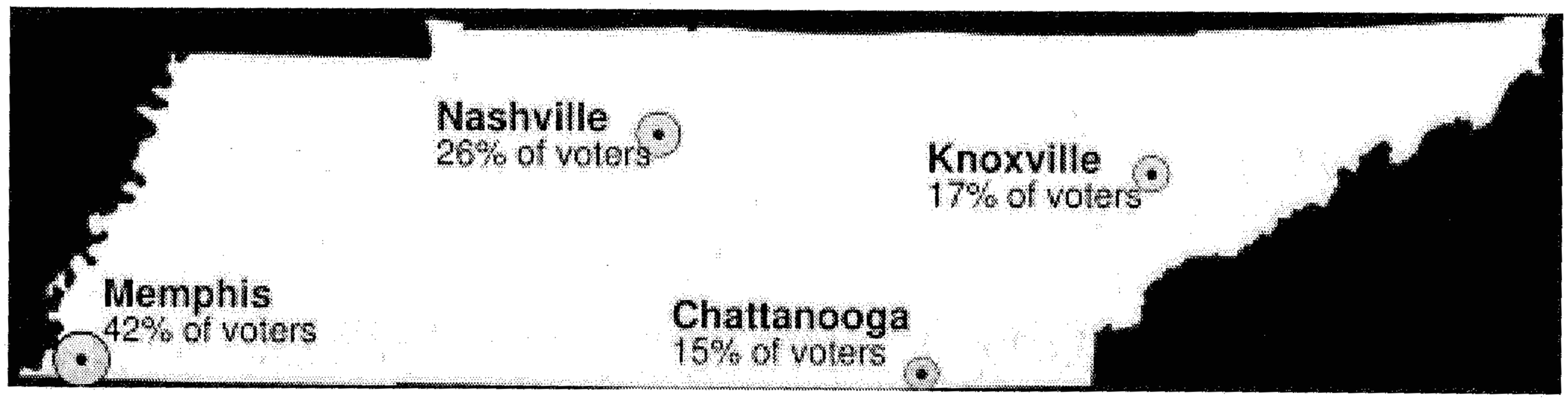
Now, suppose the last 2 voters change their preferences to $abc$, then "$c$" beats "$a$" in the run-off by a vote count of 9 to 8. The moving up of "$a$" in the last 2 voters indeed hurts "$a$". This example demonstrates failure of monotonicity.

## 2. Borda count

One uses each preference list to award "points" to each of $n$ alternatives: bottom of the list gets zero, next to the bottom gets one point, the top alternative gets $n-1$ points.

The alternative(s) with the highest "scores" is the social choice.

- It sometimes elects broadly acceptable candidates, rather than those preferred by the majority, the Borda count is considered as a consensus-based electoral system, rather than a majoritarian one.



The candidates for the capital of the State of Tennessee are:

- Memphis, the state's largest city, with 42% of the voters, but located far from the other cities
- Nashville, with 26% of the voters, almost at the center of the state and close to Memphis
- Knoxville, with 17% of the voters
- Chattanooga, with 15% of the voters

| **42% of votors (close to Memphis)** | **26% of voters (close to Nashville)** | **15% of voters (close to Chattanooga)** | **17% of voters (close to Knoxville)** |
|---|---|---|---|
| **1. Memphis** | **1. Nashville** | **1. Chattanooga** | **1. Knoxville** |
| 2. Nashville | 2. Chattanooga | 2. Knoxvilla | 2. Chattanooga |
| 3. Chattanooga | 3. Knoxville | 3. Nashville | 3. Nashville |
| 4. Knoxvilla | 4. Memphis | 4. Memphis | 4. Memphis |

| City | First | Second | Third | Fourth | Total points |
|---|---|---|---|---|---|
| Memphis | $42 \times 3$ | 0 | 0 | 0 | 126 |
| Nashville | $26 \times 3$ | $42 \times 2$ | $32 \times 1$ | 0 | 194 |
| Chattanooga | $15 \times 3$ | $43 \times 2$ | $42 \times 1$ | 0 | 173 |
| Knoxville | $17 \times 3$ | $15 \times 2$ | $26 \times 1$ | 0 | 107 |

- The winner is Nashville with 194 points.

*Modification*: Voters can be permitted to rank only a subset of the total number of candidates with all unranked candidates being given zero point.

## 3. Hare's elimination procedure

If no alternative is ranked first by a *majority* of the voters, the alternative(s) with the smallest number of first place votes is (are) crossed out from all reference orderings, and the first place votes are counted again.

**Example**

| *5 voters* | *2 voters* | *3 voters* | *3 voters* | *4 voters* |
|---|---|---|---|---|
| $a$ | $b$ | $c$ | $d$ | $e$ |
| $b$ | $c$ | $b$ | $b$ | $b$ |
| $c$ | $d$ | $d$ | $c$ | $c$ |
| $d$ | $e$ | $e$ | $e$ | $d$ |
| $e$ | $a$ | $a$ | $a$ | $a$ |

"$b$" is eliminated first.

| *5 voters* | *2 voters* | *3 voters* | *3 voters* | *4 voters* |
|---|---|---|---|---|
| $a$ | $c$ | $c$ | $d$ | $e$ |
| $c$ | $d$ | $d$ | $c$ | $c$ |
| $d$ | $e$ | $e$ | $e$ | $d$ |
| $e$ | $a$ | $a$ | $a$ | $a$ |

Next, "$d$" is eliminated.

| *5 voters* | *2 voters* | *3 voters* | *3 voters* | *4 voters* |
|---|---|---|---|---|
| $a$ | $c$ | $c$ | $c$ | $e$ |
| $c$ | $e$ | $e$ | $e$ | $c$ |
| $e$ | $a$ | $a$ | $a$ | $a$ |

There is still no majority winner, so "$e$" is crossed off. Lastly, "$c$" is then declared the winner.

- Under plurality with run-off, $a$ and $e$ are the two top vote-getters, ending $e$ as the social choice.

## 4. Coombs elimination procedure

Eliminate the alternative with the *largest* number of *last place* votes, until when one alternative commands the majority support.

Reconsider the example, the steps of elimination are

| *5 voters* | *2 voters* | *3 voters* | *3 voters* | *4 voters* |
|---|---|---|---|---|
| $b$ | $b$ | $c$ | $d$ | $e$ |
| $c$ | $c$ | $b$ | $b$ | $b$ |
| $d$ | $d$ | $d$ | $c$ | $c$ |
| $e$ | $e$ | $e$ | $e$ | $d$ |

"$e$" is eliminated, leaving

| *5 voters* | *2 voters* | *3 voters* | *3 voters* | *4 voters* |
|---|---|---|---|---|
| $b$ | $b$ | $c$ | $d$ | $b$ |
| $c$ | $c$ | $b$ | $b$ | $c$ |
| $d$ | $d$ | $d$ | $c$ | $d$ |

"$b$", with 11 first place votes, is now the winner.

## Example

| *5 voters* | *2 voters* | *4 voters* | *2 voters* |
|---|---|---|---|
| $a$ | $b$ | $c$ | $c$ |
| $b$ | $c$ | $a$ | $b$ |
| $c$ | $a$ | $b$ | $a$ |

- Coombs procedure eliminates "$c$" and chooses "$a$".
- If the last two voters change to favor "$a$" over "$b$", then "$b$" will be eliminated and "$c$" will win. （又是幫他變成害他）

## 5. Dictatorship

Choose one of the voters and call her the dictator. The alternative on top of her list is the social choice.

## 6. Sequential pairwise voting (more than 2 alternatives)

- Two alternatives are voted on first; the majority winner is then paired against the third alternative, etc. The order in which alternatives are paired is called the *agenda* of the voting.

**Example**

$A$: Reagan administration – supported bill to provide arms to the Contra rebels.

$H$: Democratic leadership bill to provide humanitarian aid but not arms.

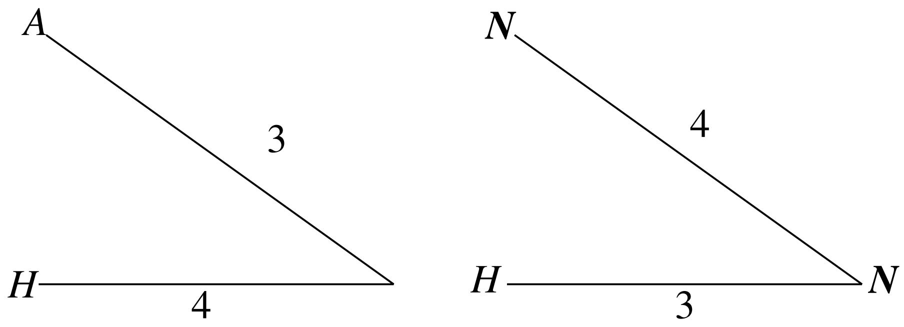
$N$: giving no aid to the rebels.

First, the form of aid is voted, then decide on whether aid or no aid is given to the rebels. In the Congress agenda, the first vote was between $A$ and $H$, with the winner to be paired against $N$.

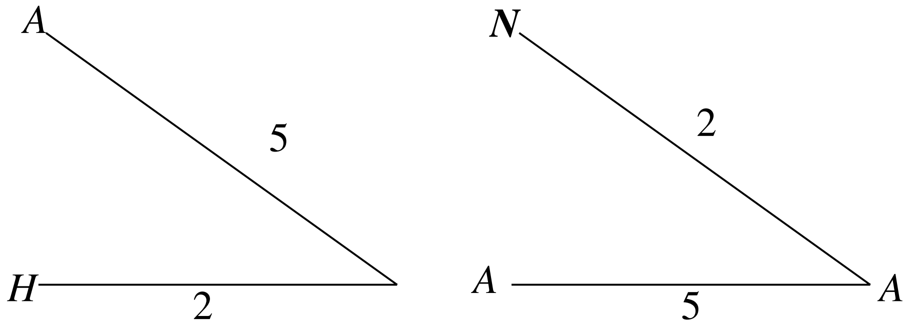
Suppose the preferences of the voters are:

| *Conservative Republicans* | *Moderate Republicans* | *Moderate Democrats* | Liberal Democrats |
| --- | --- | --- | --- |
| $A$ | $A$ | $H$ | $N$ |
| $N$ | $H$ | $A$ | $H$ |
| $H$ | $N$ | $N$ | $A$ |
| (2 voters) | (1 voter) | (2 voters) | (2 voters) |

- The Conservative Republicans may think that humanitarian aid is non-effective, either no arms or no aid at all. Moderate Democrats and Republicans may think that some form of aid is at least useful, so they put "no aid" at the bottom.




Sincere voting



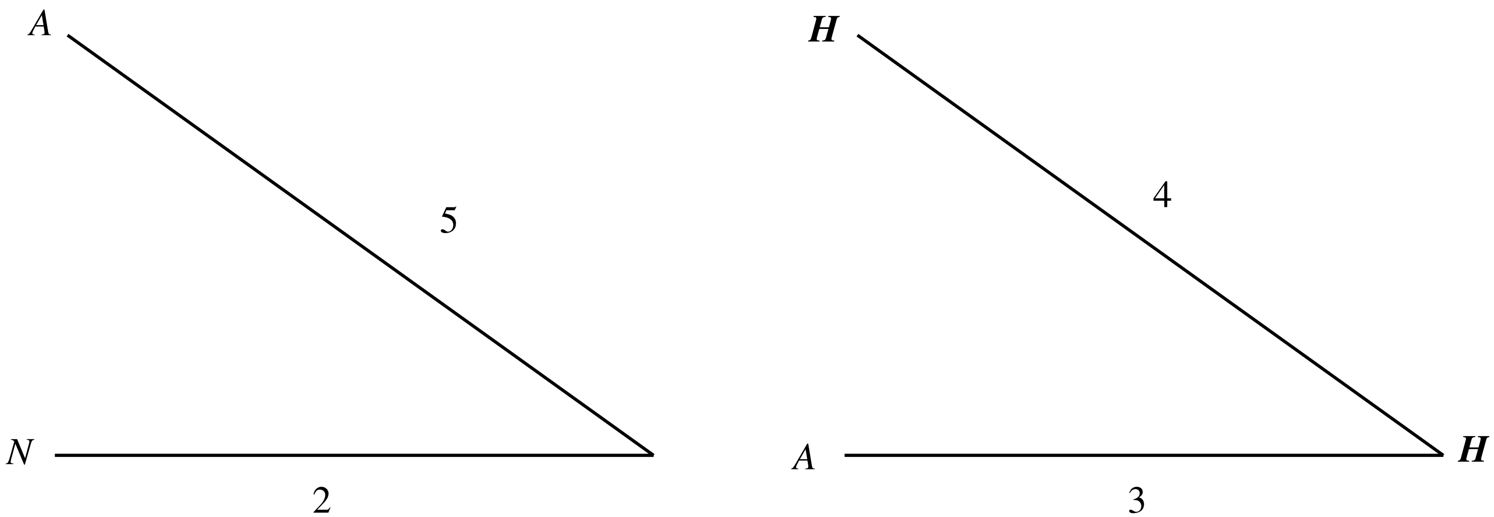

Sophisticated voting

By sophisticated voting, if voters can make $A$ to win first, then $A$ can beat $N$ by 5 to 2.

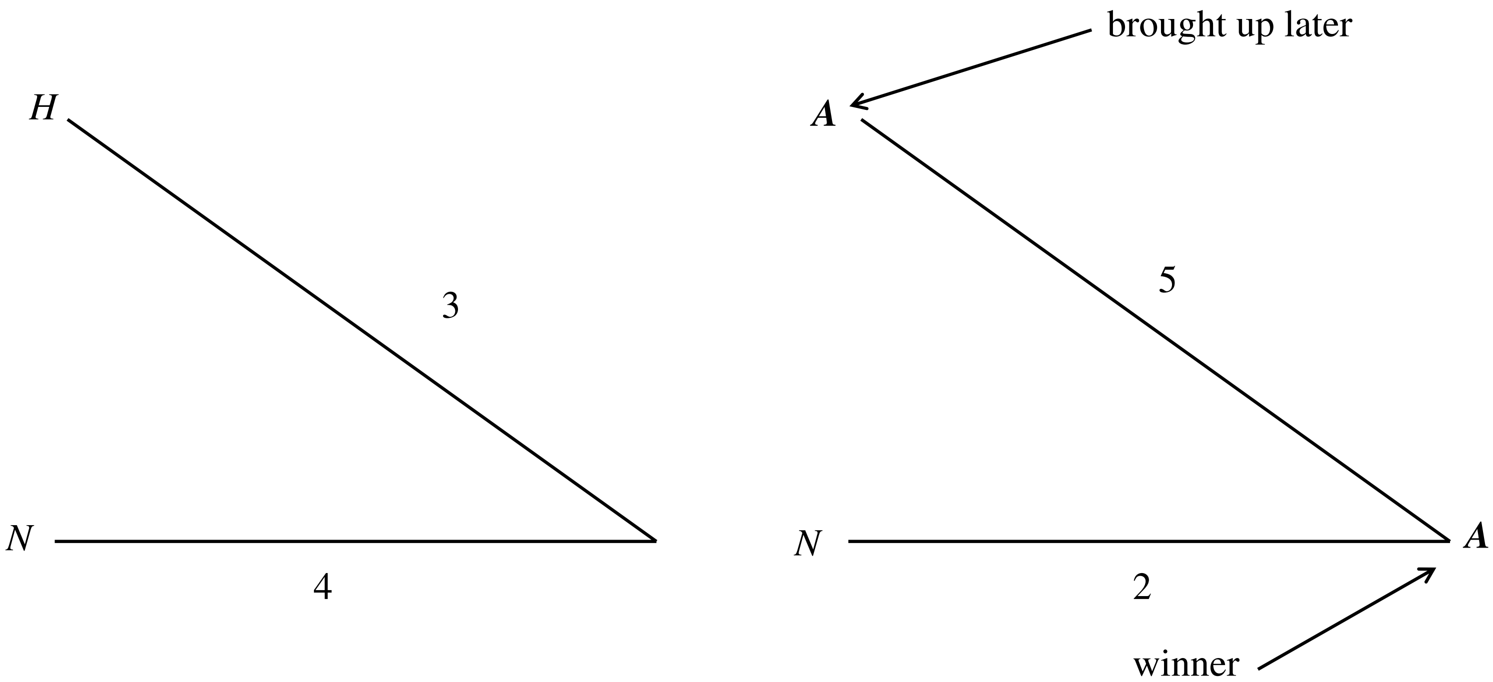
Republicans should vote sincerely for $A$, the liberal Democrats should vote sincerely for $N$, but the moderate Democrats should have voted sophisicatedly for $A$ ($N$ is the last choice for moderate Democrats).

*Alternative agenda*

- produce *any one* of the alternatives as the winner under sincere voting:




Sincere voting

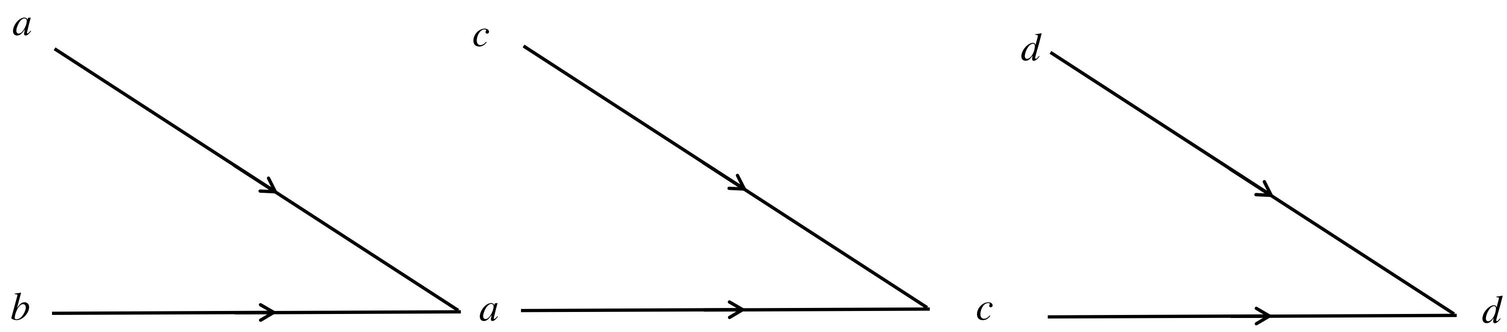



Sincere voting

Remark: The later you bring up your favored alternative, the better chance it has of winning.

**Example (failure of Pareto condition for sequential voting)**

| Voters are unanimous in preferring $b$ to $d$. | 1 *voter* | 1 *voter* | 1 *voter* |
|---|---|---|---|
| | $a$ | $c$ | $b$ |
| | $b$ | $a$ | $d$ |
| | $d$ | $b$ | $c$ |
| | $c$ | $d$ | $a$ |




Note that all voters prefer $b$ to $d$ but $d$ is the winner (violation of the Pareto condition). Note that $b$ is knocked out in the first stage and $d$ enters into the one-for-one contest latest.

## Example (plurality versus pairwise contest)

3 candidates are running for the Senate. By some means, we gather the information on the "preference order" of the voters.

| 22% | 23% | 15% | 29% | 7% | 4% |
|---|---|---|---|---|---|
| $D$ | $D$ | $H$ | $H$ | $J$ | $J$ |
| $H$ | $J$ | $D$ | $J$ | $H$ | $D$ |
| $J$ | $H$ | $J$ | $D$ | $D$ | $H$ |

$\boxed{\text{Top choice only}}$ 45% for D, 44% for $H$ and 11% for $J$; D emerges as the "close"' winner.

$\boxed{\text{One-for-one contest between } H \text{ and } D}$ H scores $(15 + 29 + 7)\% = 51\%$; $D$ scores $(22 + 23 + 4)\% = 49\%$.

## Condorcet paradox

Consider the following 3 preference listings of 3 alternatives, which are obtained by placing the last choice in the earlier list as the top choice in the new list. This is called the Condorcet profile.

| list #1 | list #2 | list #3 |
|---|---|---|
| $a$ | $c$ | $b$ |
| $b$ | $a$ | $c$ |
| $c$ | $b$ | $a$ |

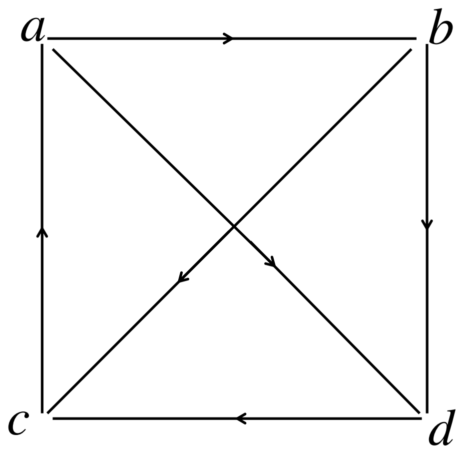
If $a$ is the social choice, then #2 and #3 agree that $c$ is better than $a$.
If $b$ is the social choice, then #1 and #2 agree that $a$ is better than $b$.
If $c$ is the social choice, then #1 and #3 agree that $b$ is better than $c$.

Two-thirds of the people are "constructively unhappy" in the sense of having a single alternative that all agree is superior to the proposed social choice. Generalization to $n$ alternatives and $n$ people, unhappiness of $\frac{n-1}{n}$ of the people is involved.

## Loss of transitivity in pairwise contests

If $a$ is preferred to $b$ and $b$ is preferred to $c$, then we expect $a$ to be preferred to $c$.

| 1 *voter* | 1 *voter* | 1 *voter* |
|---|---|---|
| $a$ | $c$ | $b$ |
| $b$ | $a$ | $d$ |
| $d$ | $b$ | $c$ |
| $c$ | $d$ | $a$ |




$a$ beats $b$ in pairwise contest, $b$ beats $c$ in pairwise contest but $a$ loses to $c$ in pairwise contest.

## Chair paradox

"Apparent power" held by the Chair with tie-breaking power needs not correspond to control over outcomes.

Consider the same example as in the voting paradox of Condorcet:

| $K$ | $G$ | $H$ |
|---|---|---|
| $a$ | $b$ | $c$ |
| $b$ | $c$ | $a$ |
| $c$ | $a$ | $b$ |

The social choice is determined by the *plurality voting procedure* where voter $K$ (Chair) also has a tie-breaking vote. Since only the top choices are considered in plurality voting, the preference lists are not regarded as inputs for the social choice procedure, but only be used for reference that shows the extent to which each of $K$, $G$ and $H$ should be happy with the social choice. Assume complete information in the sense that the preference lists are known to all voters.

*Weakly dominant strategy*

Fix a player $P$ and consider two strategies $V(x)$ and $V(y)$ for $P$. Here, $V(x)$ denotes "vote for alternative $x$". The strategy $V(x)$ is said to be weakly dominating the other strategy $V(y)$ for player $P$ if

1. For every possible scenario (votes of the other players), the social choice resulting from $V(x)$ is at least as good for player $P$ as that resulting from $V(y)$.

2. There is at least one scenario in which the social choice resulting from $V(x)$ is strictly better for player $P$ than that resulting from $V(y)$.

A strategy is said to be weakly dominant for player $P$ if it weakly dominates every other available strategy.

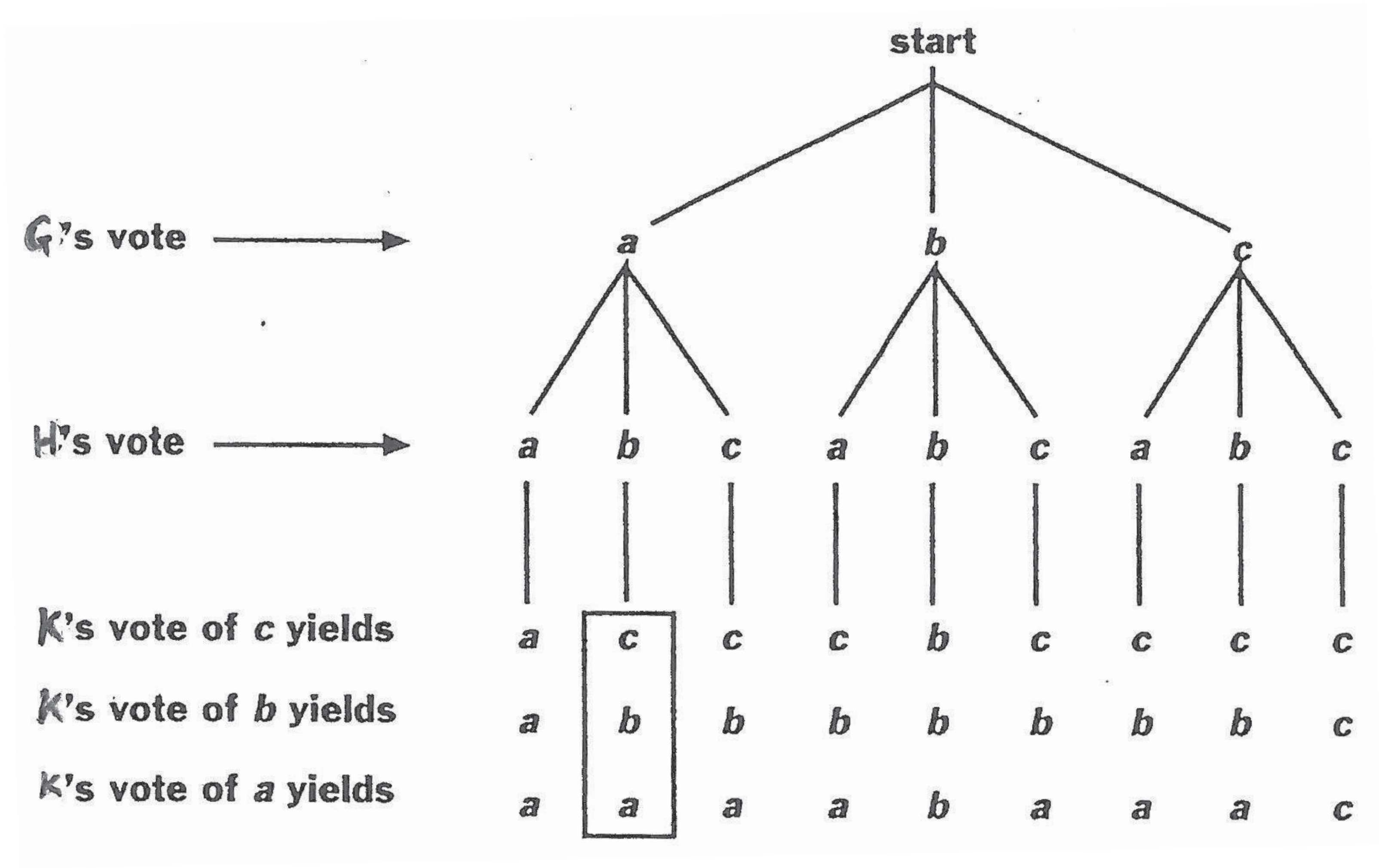
How do we determine whether a strategy is weakly dominant? We list all possible scenarios and compare the result achieved by using this strategy and all other strategies – use a tree to list all scenarios.

*Proposition*

"Vote for alternative $a$" is a weakly dominant strategy for Chair $K$.

*Proof* Consider the 9 possible scenarios for the choices of $G$ and $H$ that are listed in a tree.

- Whenever there is a tie, Chair's choice wins.
- In the first case, $G$'s vote is $a$ and $H$'s vote is $a$, then the outcome is always $a$, independent of the choice of $K$.
- In the sixth case, $G$'s vote is $b$ and $H$'s vote is $c$, then the outcome matches with $K$'s vote since $K$ is the Chair (tie-breaker).




Only when both of the other two players vote for $b$ or $c$, while $K$ votes for $a$, the outcome is not $a$ (fifth and ninth columns).

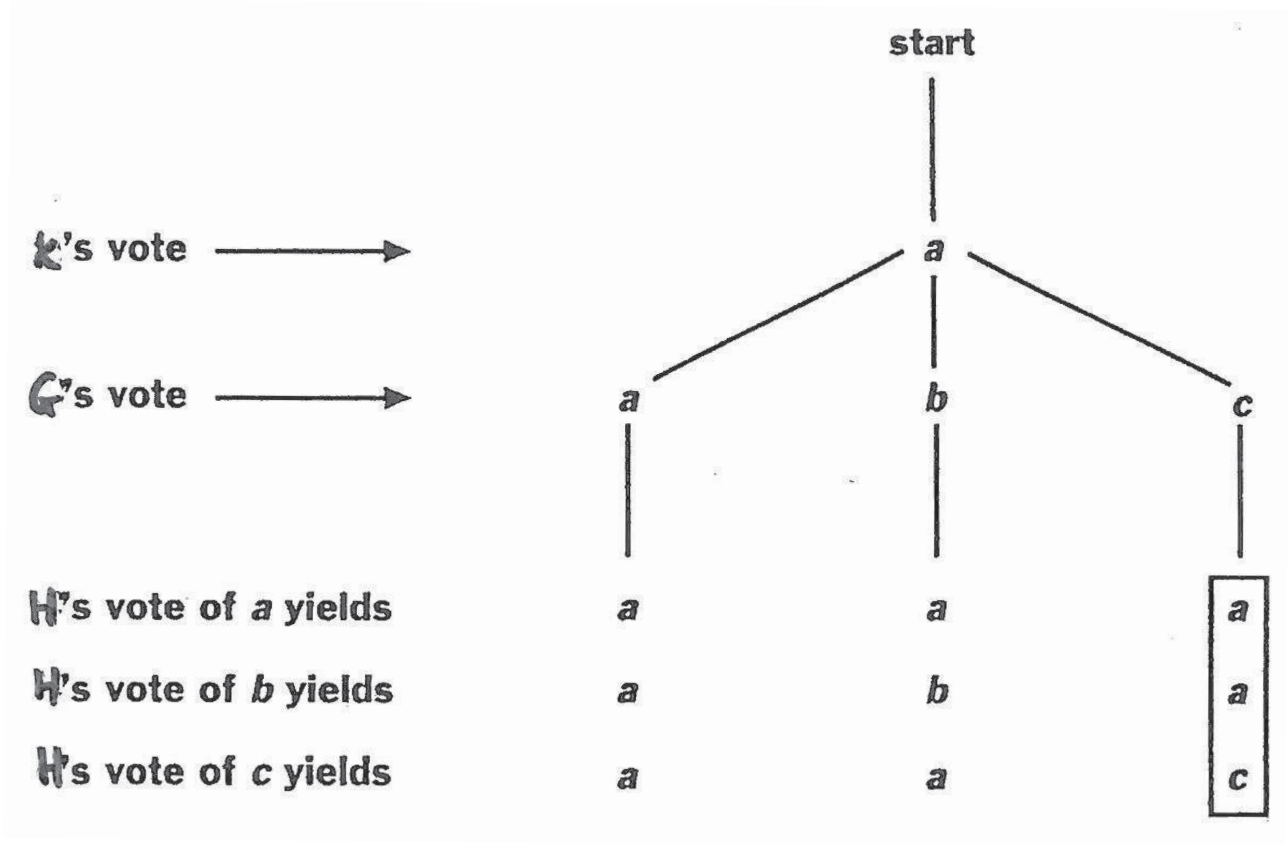
*Player $K$'s strategies*

Recall that $K$'s preference is $(a\ b\ c)$. The outcome at the bottom of each column (corresponding to $K$'s vote of $a$) is never worse for $K$ than either of the outcomes (corresponding to $K$'s vote of either $b$ or $c$) above it, and that it is strictly better than both in at least one case.

- It is not necessary for $K$ to know the preference lists of $G$ and $H$ since the determination of the weakly dominant strategy of $K$ is based on exploring all 9 scenarios of alternatives chosen by $G$ and $H$.
- Player $K$ appears to have no rational justification for voting for anything except $a$.
- If we assume that $K$ will definitely go with his weakly dominant strategy, then we analyze what rational self-interest will dictate for the other 2 players accordingly.

*Player $H$'s strategies*

In the last column, $H$'s vote of $b$ yields $a$ since $K$ is the Chair (tie-breaker).




"Vote for $c$" is a weakly dominant strategy for $H$ since $H$'s preference is ($c \quad a \quad b$). Actually, when $H$ has $a$ as the second choice, the weakly dominant strategy is to vote for his top choice $c$.

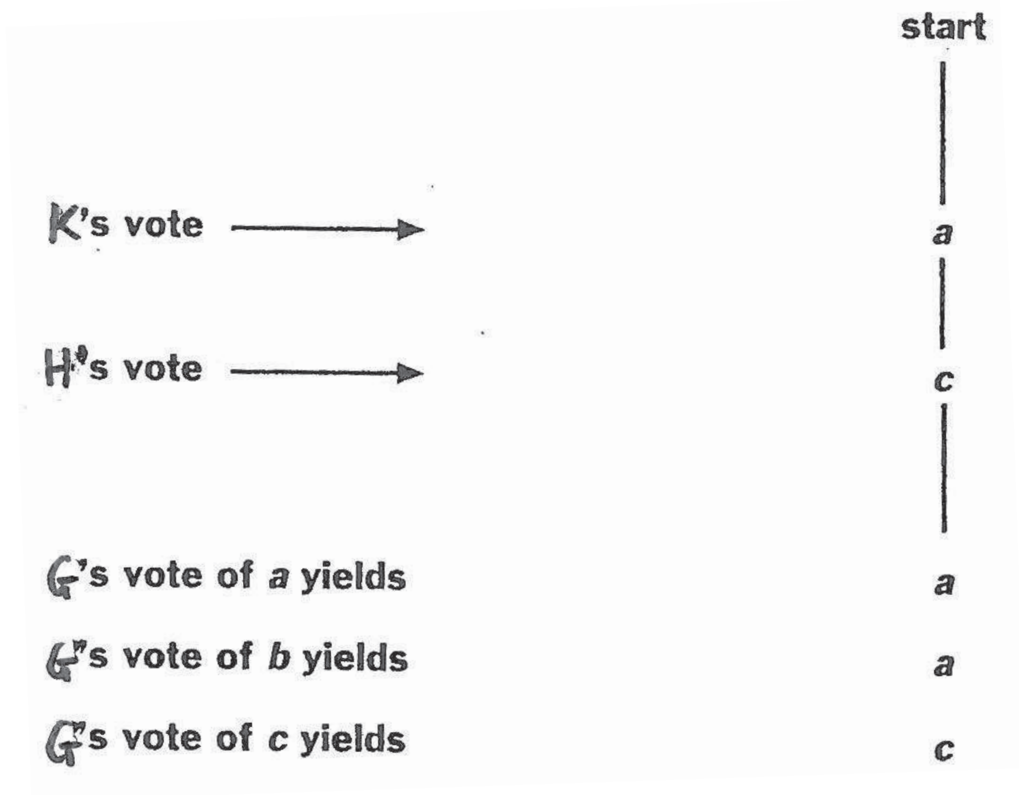
*Player $G$'s strategies*

| | | start | |
|---|---|---|---|
| $K$'s vote | | $a$ | |
| $H$'s vote | $a$ | $b$ | $c$ |
| $G$'s vote for $a$ yields | $a$ | $a$ | $a$ |
| $G$'s vote for $b$ yields | $a$ | $b$ | $a$ |
| $G$'s vote for $c$ yields | $a$ | $a$ | $c$ |

$G$'s preference: $(b \quad c \quad a)$

Note that $G$ has $a$ as his last choice. "Vote for $b$" is not a weakly dominant strategy for $G$ since in the third column, $G$ is worse off by voting for $b$. If $G$ is able to acquire $H$'s preference list, and suppose $G$ knows $H$'s second choice is $a$, then $G$ can deduce his weakly dominant strategy. If $G$ knows $H$'s third choice is $a$, then $G$ has no weakly dominant strategy.

Since Player $K$ definitely votes for $a$ and Player $H$ definitely votes for $c$, the strategy "vote for $c$" is a weakly dominant strategy for Player $G$ since Player $G$ cannot get $b$ even she votes for $b$.




$K$ votes for $a$, $G$ votes for $c$ and $H$ votes for $c$ yield $c$. Alternative $c$ is $K$'s least preferred alternative even though $K$ had the additional "tie-breaking" power. The additional power as Chair forces the other two voters to vote sophisticatedly.

*Remarks*

1. Since Chair $K$ is endowed with the tie-breaking power, his weakly dominant strategy is definitely "*vote for* $a$", independent of his second and third choice and the preference lists of $G$ and $H$.

2. For the other two players $G$ and $H$, if a player (either $G$ or $H$) has $a$ as the second choice, then the weakly dominant strategy is "*vote for his top choice*". Suppose both $G$ and $H$ have $a$ as the second choice, then all players vote for their top choice. This is uninteresting and does not reveal the Chair paradox. In this case, the Chair can realize his top choice $a$ as the social choice due to the tie-breaking power.

3. In the present example, $G$ has $a$ as the third choice while $H$ has $a$ as the second choice. In order that $G$ can be sure that "*vote for* $c$" is weakly dominant, $G$ has to acquire the knowledge that $H$ has $a$ as the second choice. This is the key point to show how the Chair paradox is revealed.

4. Suppose $G$ knows that $H$ has $a$ as the third choice, there will be no weakly dominant strategy for $G$. Under the scenario that both $G$ and $H$ put $a$ as the last choice, they can hardly come into term to have one player to vote for the top choice of the other player in order to beat the Chair. If both $G$ and $H$ choose to vote for their own top choice, both will be worse off since the outcome will be $a$.

## 3.2 Desirable properties of voting methods

Some properties on the social choice that are, at least intuitively, desirable. If ties were not allowed, then we could have said "the" social choice instead of "a" social choice.

*Pareto condition*

Whenever every voter puts $x$ strictly above $y$, the social preference list puts alternative $x$ strictly above $y$. In the context of social choice procedure, if everyone prefers $x$ to $y$, then $y$ cannot be a social choice.

*Condorcet Winner Criterion* (Condorcet winner may not exist)

If there is an alternative $x$ which could obtain a majority of votes in pairwise contests against *every* other alternative, a voting rule should choose $x$ as the winner.

*Condorcet Loser Criterion* (Condorcet loser may not exist)

If an alternative $y$ would lose in pairwise majority contests against every other alternative, a voting rule should *not* choose $y$ as a winner.

*Monotonicity Criterion*(幫他不會導致害他)

If $x$ is a winner under a voting rule, and one or more voters change their preferences in a way favorable to $x$ (without changing the order in which they prefer any other alternatives), then $x$ should still be a winner.

*Independence of irrelevant alternatives (IIA)*

For any pair of alternatives $x$ and $y$, if a preference list is changed but the relative position of $x$ and $y$ to each other is not changed, then the new list can be described as arising from upward and downward shifts of alternatives other than $x$ and $y$. Changing preferences toward these other alternatives should be irrelevant to the social preference of $x$ to $y$.

IIA requires that whenever a pair of alternatives is ranked the same way in two preference profiles over the same sets of alternatives, then the voting rule must order these two alternatives identically.

In the context of social choice procedure, suppose we start with $x$ as a winner while $y$ is a non-winner, people move some other alternative $z$ around, then we cannot guarantee that $x$ is still a winner. However, the independence of irrelevant alternatives at least claims that $y$ should remain a non-winner since $x$ remains to be ahead of $y$ under the modified preference profile.

## Glimpse of Impossibility

There is no *social choice procedure* for three or more alternatives that satisfies both independence of irrelevant alternatives and the Condorcet winner criterion.

*Proof by contradiction*

Suppose we have a social choice procedure that satisfies both independence of irrelevant alternatives and the Condorcet winner criterion. We then show that if this procedure is applied to the profile that constitutes Condorcet's voting paradox, then it produces no winner.

- Recall that the sequential pairwise voting method and Nanson method satisfy the Condorcet winner criterion. Based on this Impossibility Lemma, both methods cannot satisfy IIA.

Assume that we have a social choice procedure that satisfies both independence of irrelevant alternatives and the Condorcet winner criterion. Consider the following profile from the voting paradox of Condorcet:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{pmatrix} c \\ a \\ b \end{pmatrix} \begin{pmatrix} b \\ c \\ a \end{pmatrix} .$$

We would like to show that every alternative is a non-winner.

*Claim 1* The alternative $a$ is a non-winner.

Consider the following profile (obtained by moving alternative $b$ down in the third preference list from the voting paradox profile):

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{pmatrix} c \\ a \\ b \end{pmatrix} \begin{pmatrix} c \\ b \\ a \end{pmatrix} .$$

We focus on $c$ and $a$ ($b$ is considered as the irrelevant alternative) and show that $c$ is always the winner.

- Notice that $c$ *is a Condorcet winner* (defeating both other alternatives by a margin of 2 to 1). Thus, the social choice procedure that satisfies the condorcet winner criterion must produce $c$ as the only winner. Thus, $c$ is a winner and $a$ is a non-winner for this profile.

- Suppose now that the *third voter* moves the irrelevant alternative $b$ up on his or her preference list. The profile then becomes that of the voting paradox. But no one changed his or her mind about whether $c$ is preferred to $a$ or $a$ is preferred to $c$. By "independence of irrelevant alternatives", and because we had $c$ as a winner and $a$ as a non-winner in the profile with which we began the proof of the claim, we can conclude that $a$ is still a non-winner when the procedure is applied to the voting paradox profile.

*Claim 2* The alternative $b$ is a non-winner.

- Consider the following profile (obtained by moving alternative $c$ down in the second preference list from the voting paradox profile):

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{pmatrix} a \\ c \\ b \end{pmatrix} \begin{pmatrix} b \\ c \\ a \end{pmatrix} .$$

  Notice that $a$ *is a Condorcet winner* (defeating both other alternatives by a margin of 2 to 1). Thus, our social choice procedure (which we are assuming satisfies the Condorcet winner criterion) must produce $a$ as the only winner. Thus, $a$ is a winner and $b$ is a non-winner for this profile.

- Suppose now that the *second voter* moves $c$ up on his or her preference list. The profile then becomes that of the voting paradox. But no one changed his or her mind about whether $a$ is preferred to $b$ or $b$ is preferred to $a$. By "independence of irrelevant alternatives", and because we had $a$ as a winner and $b$ as a non-winner in the profile with which we began the proof of the claim, we can conclude that $b$ is still a non-winner when the procedure is applied to the voting paradox profile.

*Claim 3* It can be shown similarly that the alternative $c$ is a non-winner.

- The above three claims show that when our procedure produces *no* winner for the Condorcet profile. But a social choice procedure must always produce at least one winner. Thus, we have a contradiction and the proof is complete.

**Positive results**

1. The plurality procedure satisfies the Pareto condition.

   *Proof*: If everyone prefers $x$ to $y$, then $y$ is not on the top of any list (let alone a plurality), and thus $y$ is certainly not a social choice.

2. The Borda count satisfies the Pareto condition.

   *Proof*: If everyone prefers $x$ to $y$, then $x$ receives more points from each list than $y$. Thus, $x$ receives a higher total than $y$ and so $y$ cannot be a winner.

3. The Hare system satisfies the Pareto condition.

   *Proof*: If everyone prefers $x$ to $y$, then $y$ is not on the top of any list. Thus, either we have immediate winner and $y$ is not among them or the procedure moves on and $y$ is eliminated before $x$. Hence, $y$ is not a winner.

4. Sequential pairwise voting satisfies the Condorcet winner criterion.

   *Proof*: A Condoret winner (if exists) always wins the kind of one-on-one contest that is used to produce the winner in sequential pairwise voting.

5. The plurality procedure satisfies monotonicity.

   *Proof*: If $x$ is the winner under plurality, then $x$ is on the top of the most lists. Moving $x$ up one spot on some list (and making no other changes) certainly preserves this.

6. The Borda count satisfies monotonicity

   *Proof*: Swapping $x$'s position with the alternative above $x$ on some list adds one point to $x$'s score and subtracts one point from that of the other alternative; the scores of all other alternatives remain the same.

7. Sequential pairwise voting satisfies monotonicity.

   *Proof*: Moving $x$ up on some list only improves $x$'s chances in one-on-one contests.

8. The dictatorship procedure satisfies the Pareto condition.

   *Proof*: If everyone prefers $x$ to $y$, then, in particular, the dictator does. Hence, $y$ is not on top of the dictator's list and so is not a social choice.

9. A dictatorship satisfies monotonicity.

   *Proof*: If $x$ is the social choice then $x$ is already on top of the dictator's list. Hence, the exchange of $x$ with some alternative immediately above $x$ must be taking place on some list other than that of the dictator and have no impact on the decision of the social choice. Thus, $x$ is still the social choice.

10. A dictatorship satisfies independence of irrelevant alternatives.

    *Proof*: We are just required to look at the preference list of the dictator. Changing preferences of other alternatives in others' lists has no impact on the social preference of $x$ to $y$ in the dictator's list.

## Negative results

1. Sequential pairwise voting with a fixed agenda does not satisfy the Pareto condition.

   *Proof:*

   | **Voter 1** | **Voter 2** | **Voter 3** |
   |---|---|---|
   | $a$ | $c$ | $b$ |
   | $b$ | $a$ | $d$ |
   | $d$ | $b$ | $c$ |
   | $c$ | $d$ | $a$ |

   Everyone prefers $b$ to $d$. But with the agenda $a$ $b$ $c$ $d$, $a$ first defeats $b$ by a score of 2 to 1, and then $a$ loses to $c$ by this same score. Alternative $c$ now goes on to face $d$, but $d$ defeats $c$ again by a 2 to 1 score. Thus, alternative $d$ is the social choice even though everyone prefers $b$ to $d$. Alternative $d$ has the advantage that it is bought up later.

2. The plurality procedure fails to satisfy the Condorcet winner criterion.

*Proof*: Consider the three alternatives $a, b$, and $c$ and the following sequence of nine preference lists grouped into voting blocs of size four, three, and two.

| **Voters 1–4** | **Voters 5–7** | **Voters 8–9** |
|---|---|---|
| $a$ | $b$ | $c$ |
| $b$ | $c$ | $b$ |
| $c$ | $a$ | $a$ |

- With the plurality procedure, alternative $a$ is clearly the social choice since it has four first-place votes to three $b$ and two for $c$.
- $b$ is a Condorcet winner, $b$ would defeat $a$ by a score of 5 to 4 in one-on-one competition, and $b$ would defeat $c$ by a score of 7 to 2 in one-on-one competition.

3. Borda count does not satisfy the Condorcet winner criterion and violates "Independence of Irrelevant Alternatives".

| *3 voters* | *2 voters* | *Borda count:* |
|---|---|---|
| $a$ | $b$ | "$a$" is 6 |
| $b$ | $c$ | "$b$" is 7 |
| $c$ | $a$ | "$c$" is 2. |

"$b$" is the Borda winner but "$a$" is the Condorcet winner since 3 out of 5 voters place "$a$" above both "$b$" and "$c$". Worse, "$a$" has an absolute majority of first place votes.

Why "$b$" wins in the Borda count? The presence of "$c$" enables the last 2 voters to weigh their votes for "$b$" over "$a$" more heavily than the first 3 voters' votes for "$a$" over "$b$". If "$c$" is put to the lowest choice, then "$a$" is chosen as the Borda winner. This shows a violation of "Independence of Irrelevant Alternatives" for the Borda count method.

4. A dictatorship does not satisfy the Condorcet winner criterion.

*Proof*: The Condorcet winner may not be the dictator's top choice. Consider the three alternatives $a, b$ and $c$, and the following three preference lists:

| **Voter 1** | **Voter 2** | **Voter 3** |
|---|---|---|
| $a$ | $c$ | $c$ |
| $b$ | $b$ | $b$ |
| $c$ | $a$ | $a$ |

Assume that Voter 1 is the dictator. Then, $a$ is the social choice, although $c$ is clearly the Condorcet winner since it defeats both others by a score of 2 to 1.

5. The Hare procedure does not satisfy the Condorcet winner criterion.

*Proof*:

| **Voters 1–5** | **Voters 6–9** | **Voters 10–12** | **Voters 13–15** | **Voter 16–17** |
|---|---|---|---|---|
| $a$ | $e$ | $d$ | $c$ | $b$ |
| $b$ | $b$ | $b$ | $b$ | $c$ |
| $c$ | $c$ | $c$ | $d$ | $d$ |
| $d$ | $d$ | $e$ | $e$ | $e$ |
| $e$ | $a$ | $a$ | $a$ | $a$ |

- $b$ is the Condorcet winner: $b$ defeats $a$ (12 to 5), $b$ defeats $c$ (14 to 3), $b$ defeats $d$ (14 to 3), $b$ defeats $e$ (13 to 4).
- On the other hand, the social choice according to the Hare procedure is definitely not $b$. That is, no alternative has the nine first place votes required for a majority, and so $b$ is deleted from all the lists since it has only two first place votes.

6. The Hare procedure does not satisfy monotonicity.

*Proof*

| **Voters 1–7** | **Voters 8–12** | **Voters 13-16** | **Voter 17** |
| --- | --- | --- | --- |
| $a$ | $c$ | $b$ | $b$ |
| $b$ | $a$ | $c$ | $a$ |
| $c$ | $b$ | $a$ | $c$ |

Since no alternative has 9 or more of the 17 first place votes, we delete the alternatives with the fewest first place votes. In this case, that would be alternatives $c$ and $b$ with only five first place votes each as compared to seven for $a$. But now $a$ is the only alternative left, and so it is obviously on top of a majority (in fact, all) of the lists. Thus, $a$ is the social choice when the Hare procedure is used.

Favorable-to-$a$-change yields the following sequence of preference lists:

| **Voters 1–7** | **Voters 8–12** | **Voters 13-16** | **Voter 17** |
|---|---|---|---|
| $a$ | $c$ | $b$ | $a$ |
| $b$ | $a$ | $c$ | $b$ |
| $c$ | $b$ | $a$ | $c$ |

If we apply the Hare procedure again, we find that no alternative has a majority of first place votes and so we delete the alternative with the fewest first place votes. In this case, that alternative is $b$ with only four. But the reader can now easily check that with $b$ so eliminated, alternative $c$ is on top of 9 of the 17 lists. This is a majority and so $c$ is the soical choice.

7. The plurality procedure does not satisfy independence of irrelevant alternatives.

| **Voter 1** | **Voter 2** | **Voter 3** | **Voter 4** |
|---|---|---|---|
| $a$ | $a$ | $b$ | $c$ |
| $b$ | $b$ | $c$ | $b$ |
| $c$ | $c$ | $a$ | $a$ |

When the plurality procedure is used, $a$ is a winner and $b$ is a non-winner. Suppose that Voter 4 changes his or her list by moving the alternative $c$ down between $b$ and $a$. The lists then become:

| **Voter 1** | **Voter 2** | **Voter 3** | **Voter 4** |
|---|---|---|---|
| $a$ | $a$ | $b$ | $b$ |
| $b$ | $b$ | $c$ | $c$ |
| $c$ | $c$ | $a$ | $a$ |

Notice that we still have $b$ over $a$ in Voter 4's list. However, plurality voting now has $a$ and $b$ tied for the win with two first place votes each. Thus, although no one changed his or her mind about whether $a$ is preferred to $b$ or $b$ to $a$, the alternative $b$ went from being a non-winner to being a winner.

8. The Hare procedure fails to satisfy independence of irrelevant alternatives.

*Proof:*

| **Voter 1** | **Voter 2** | **Voter 3** | **Voter 4** |
|---|---|---|---|
| $a$ | $a$ | $b$ | $c$ |
| $b$ | $b$ | $c$ | $b$ |
| $c$ | $c$ | $a$ | $a$ |

Alternative $a$ is the social choice when the Hare procedure is used because it has at least half the first place votes, $a$ is a winner and $b$ is a non-winner.

| **Voter 1** | **Voter 2** | **Voter 3** | **Voter 4** |
|---|---|---|---|
| $a$ | $a$ | $b$ | $b$ |
| $b$ | $b$ | $c$ | $c$ |
| $c$ | $c$ | $a$ | $a$ |

Notice that we still have $b$ over $a$ in Voter 4's list. Under the Hare procedure, we now have $a$ and $b$ tied for the win, since each has half the first place votes. Thus, although no one changed his or her mind about whether $a$ is preferred to $b$ or $b$ to $a$, the alternative $b$ went from being a non-winner to being a winner.

9. Sequential pairwise voting with a fixed agenda fails to satisfy independence of irrelevant alternatives.

*Proof:*

Consider the alternative $c, b$ and $a$, and assume this reverse alphabetical ordering is the agenda. Consider the following sequence of three preference lists:

| **Voter 1** | **Voter 2** | **Voter 3** |
|---|---|---|
| $c$ | $a$ | $b$ |
| $b$ | $c$ | $a$ |
| $a$ | $b$ | $c$ |

In sequential pairwise voting, $c$ would defeat $b$ by the score of 2 to 1 and then lose to $a$ by this same score. Thus, $a$ would be the social choice (and thus $a$ is a winner and $b$ is a non-winner).

Suppose that Voter 1 moves $c$ down between $b$ and $a$, yielding the following lists:

| **Voter 1** | **Voter 2** | **Voter 3** |
|---|---|---|
| $b$ | $a$ | $b$ |
| $c$ | $c$ | $a$ |
| $a$ | $b$ | $c$ |

Now, $b$ first defeats $c$ and then $b$ goes on to defeat $a$. Hence, the new social choice is $b$. Thus, although no one changes his or her mind about whether $a$ is preferred to $b$ or $b$ to $a$, the alternative $b$ went from being a non-winner to being a winner. This shows that independence of irrelevant alternatives fails for sequential pairwise voting with a fixed agenda.

- Pareto condition is satisfied for most voting methods except the sequential pairwise voting.
- Independence of Irrelevant Alternatives is the hardest to be satisfied except dictatorship.
- Monotonicity is satisfied for most voting methods except elimination methods.
- Hare method fails in most criteria except the Pareto condition.
- Dictatorship satisfies most criteria except the Condorcet Winner Criterion (since the dictator and condorcet winner may not be the same person).
- Surprisingly, the sequential pairwise voting fails the Pareto condition (easiest) but satisfies the Condorcet Winner Criterion (hardest).

| | **Pareto** | **Condorcet Winner Criterion** | **Monotonicity** | **Independence of Irrelevant Alternatives** |
|---|---|---|---|---|
| Plurality | Yes | No | Yes | No |
| Borda | Yes | No | Yes | No |
| Hare | Yes | No | No | No |
| Seq pairs | No | Yes | Yes | No |
| Dictator | Yes | No | Yes | Yes |

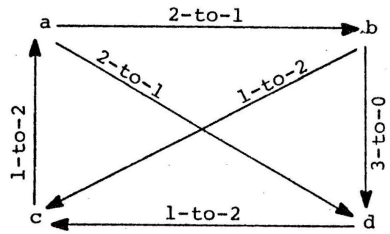
## 3.3 Condorcet voting methods

1. **Black method**

   Value the Condorcet criterion, but also believe that the Borda count has advantages. Try to achieve 3 “yes” among the 4 criteria.

   - In cases where there is a Condorcet winner, choose it; otherwise, choose the Borda winner.

| voter $A$ | voter $B$ | Voter $C$ |
|---|---|---|
| $a$ | $c$ | $b$ |
| $b$ | $a$ | $d$ |
| $d$ | $b$ | $c$ |
| $c$ | $d$ | $a$ |

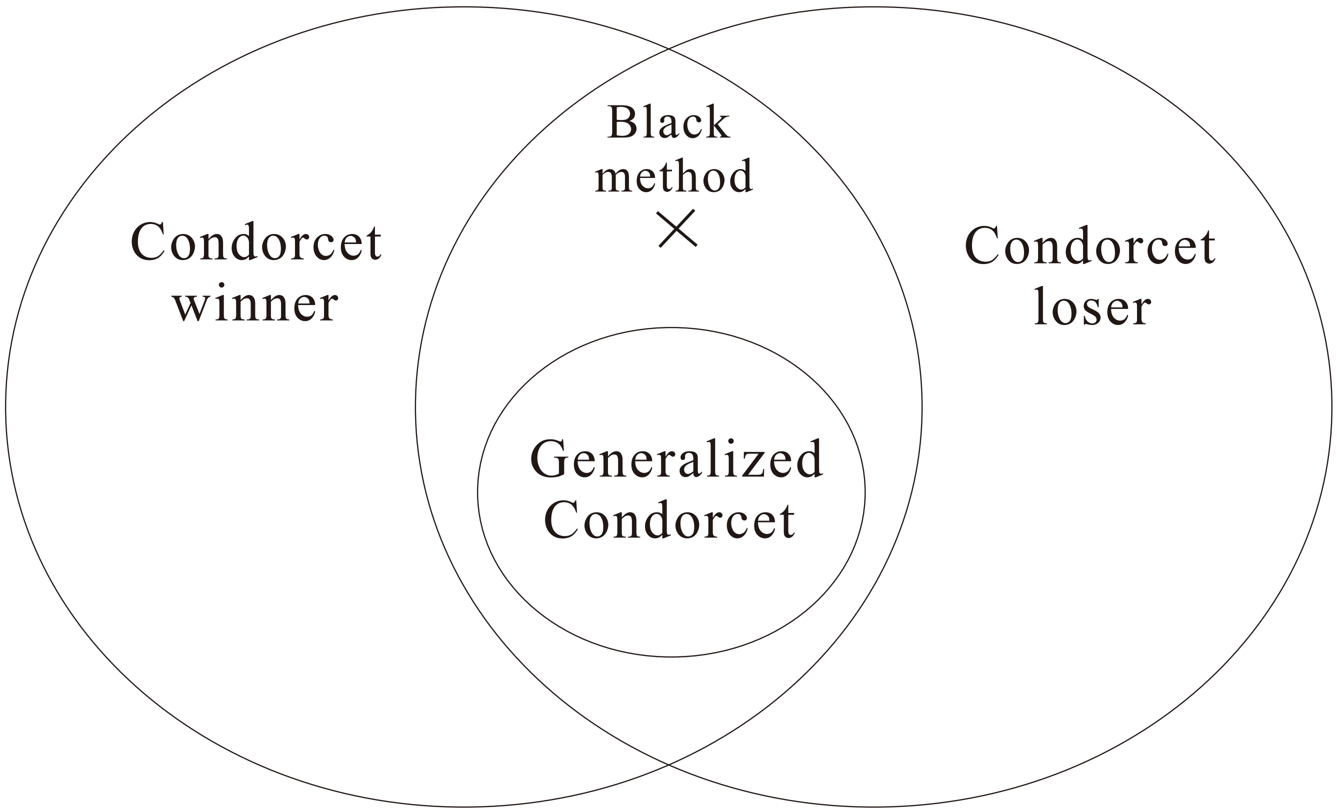


- We check to see if one alternative beats all the other in pairwise contests. If so, that alternative wins. If not, we use the Borda counts to find the Borda winner. Note that a Condorcet loser cannot be a winner under Borda count.

- The Black method satisfies the Pareto, Condorcet loser, Condorcet winner and Monotonicity criteria. However, it does not satisfy the following stronger version of Condorcet criterion.

*Generalized Condorcet criterion*:

If the alternatives can be partitioned into two sets $A$ and $B$ such that every alternative in $A$ beats every alternative in $B$ in pairwise contests, then a voting rule should **not** select an alternative in $B$. This criterion implies both the Condorcet winner and Condorcet loser criteria (take $A$ to be the set which consists of only the Condorcet winner, or $B$ to be the set which consists of only the Condorcet loser).




Generalized Condorcet $\Rightarrow$ Condorcet winner;

Generalized Condorcet $\Rightarrow$ Condorcet loser.

Black method satisfies both the Condorcet winner and Condorcet loser but fails the Generalized Condorcet.

Suppose $x$ is the Condorcet winner, then we choose $A$ to be singleton that contains $x$ only. All other alternatives are put into $B$. We observe that every alternative in $A$ beats every alternative in $B$ in pairwise contest. By the Generalized Condorcet criterion, the single alternative $x$ in $A$ must be the winner. Therefore, the Condorcet winner criterion is observed.

The following example shows that Black's rule violates the Generalized Condorcet criterion:

| *1 Voter* | *1 Voter* | *1 Voter* |
|---|---|---|
| $a$ | $b$ | $c$ |
| $b$ | $c$ | $a$ |
| $x$ | $x$ | $x$ |
| $y$ | $y$ | $y$ |
| $z$ | $z$ | $z$ |
| $w$ | $w$ | $w$ |
| $c$ | $a$ | $b$ |

- If we partition the alternatives as $A = [a, b, c]$ and $B = [x, y, z, w]$, then every alternative in $A$ beats every alternative in $B$ by a 2-to-1 vote.

- Furthermore, there is no Condorcet winner, since alternatives $a$ and $b$ and $c$ beat each other cyclically.

- When we compute Borda counts, we get:

| $a$ | $b$ | $c$ | $x$ | $y$ | $z$ | $w$ |
|---|---|---|---|---|---|---|
| 11 | 11 | 11 | 12 | 9 | 6 | 3 |

  By the Black rule, $x$ is the winner.

  For $a$, $b$ and $c$, they are either at the top or bottom in the lists, so their Borda counts are lower than that of $x$ since $x$ is always at the relatively top positions in the lists.

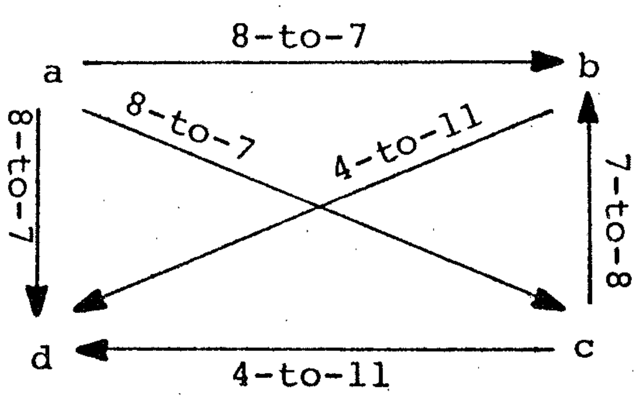
2. **Nanson method**

- It is a Borda elimination scheme which sequentially eliminates the alternative with the lowest Borda count until only one alternative or a collection of tied alternatives remain.
- This procedure always selects the Condorcet winner, if there is one. Note that the Condorcet winner must gather more than half the votes in its pairwise contests with the other alternatives, it must always have a *higher than average* Borda count. It would never have the *lowest* Borda count and can never be eliminated in all steps.
- Nanson's procedure so cleverly reconciles the Borda count with the Condorcet criterion. It is a shame, but perhaps not surprising, to find that it shares the defect of other elimination schemes: failure of monotonicity.

| 3 Voters | 4 Voters | 4 Voters | 4 Voters |
|---|---|---|---|
| $b$ | $b$ | $c$ | $d$ |
| $c$ | $a$ | $a$ | $a$ |
| $d$ | $c$ | $b$ | $c$ |
| $a$ | $d$ | $d$ | $b$ |

The sum among all votes of all alternatives that are above $a$ is $3 \times 3 + 4 + 4 + 4 = 21$ while those below $a$ is $2 \times 4 + 2 \times 4 + 2 \times 4 = 24$.

The pairwise voting diagram is:




so that alternative $a$ is the Condorcet winner. The Borda counts are $a$ : 24, $b$ : 25, $c$ : 26 and $d$ : 15. Hence, alternative $c$ would be the Borda winner, and alternative $a$ would come in next-to-last.

Under Nanson's procedure, alternative $d$ is eliminated and new Borda counts are computed:

| *3 Voters* | *4 Voters* | *4 Voters* | *4 Voters* | |
|---|---|---|---|---|
| $b$ | $b$ | $c$ | $a$ | Borda $a$ : 16 |
| $c$ | $a$ | $a$ | $c$ | counts $b$ : 14 |
| $a$ | $c$ | $b$ | $b$ | $c$ : 15 |

Alternative $b$ is now eliminated, and in the final round alternative $a$ beats $c$ by 8-to-7.

*Failure of monotonicity*

| *8 Voters* | *5 Voters* | *5 Voters* | *2 Voters* |
|---|---|---|---|
| $a$ | $c$ | $b$ | $c$ |
| $b$ | $a$ | $c$ | $b$ |
| $c$ | $b$ | $a$ | $a$ |

- The Borda counts are $a : 21, b : 20$, and $c : 19$. Hence $c$ is eliminated, and then alternative $a$ beats $b$ by 13-to-7.

- If the last two voters change their minds in favor of alternative $a$ over $b$, so that their preference ordering is *cab*, the new Borda counts will be $a : 23, b : 18$ and $c : 19$. Hence $b$ will be eliminated and then $c$ beats $a$ by 12-to-8. The change in alternative $a$'s favor has produced $c$ as the winner.

3. **Copeland method**

- One looks at the results of pairwise contests between alternatives. For each alternative, compute the number of pairwise wins it has minus the number of pairwise losses it has. Choose the alternative(s) for which this difference is largest.
- It is clear that if there is a Condorcet winner, Copeland's rule will choose it: the Condorcet winner will be the only alternative with all pairwise wins and no pairwise losses.
- This method is more likely than other methods to produce ties.
- It may come into spectacular conflict with the Borda count.

| *1 Voter* | *4 Voters* | *1 Voter* | *3 Voters* |
|---|---|---|---|
| $a$ | $c$ | $e$ | $e$ |
| $b$ | $d$ | $a$ | $a$ |
| $c$ | $b$ | $d$ | $b$ |
| $d$ | $e$ | $b$ | $d$ |
| $e$ | $a$ | $c$ | $c$ |

| | | | |
|---|---|---|---|
| Copeland scores: | $a:2$ | Borda scores: | $a:16$ |
| | $b:0$ | | $b:18$ |
| | $c:0$ | | $c:18$ |
| | $d:0$ | | $d:18$ |
| | $e:-2$ | | $e:20$ |

- Alternative $a$ is the Copeland winner and $e$ comes in last, but $e$ is the Borda winner and $a$ comes in last. The two methods produce diametrically opposite results.

- If we try to ask directly whether $a$ or $e$ is better, we notice that the Borda winner $e$ is preferred to the Copeland winner, alternative $a$, by *eight* of the nine voters!

## Summary

- Sequential pairwise voting is bad because of the agenda effect and the possibility of even choosing a Pareto dominated alternative.
- Plurality voting is bad because of the weak mandate （來自選民的授權） it may give. In particular, it may choose an alternative which would lose to *any* other alternative in a pairwise contest. This is a violation of the Condorcet Loser criterion. For example, Chen lost to the other two candidates in pairwise contests in 2000 Taiwan presidential race.
- Plurality with run-off and the elimination schemes due to Hare, Coombs and Nanson all fail to be monotonic: improvement in an alternative's favor can change it from a winner to a loser.
- The Coombs method and Nanson method are generally avoiding disliked alternatives. The Nanson rule always detects a Condorcet winner when there is one and the Coombs scheme almost always does.

- The Borda count takes positional information into full account and generally chooses a non-disliked alternative. Its major difficulty is that it can directly conflict with the plurality rule. Under the Borda count method, it may be possible that an alternative that is more broadly preferred by a majority not be chosen if it is strongly disliked by a minority.

- The Black rule directly combines the virtues of the Condorcet and Borda approaches to voting. The Copeland rule emphasizes the Condorcet approach.

## 3.4 Social welfare functions

1. The input is a sequence of individual preference lists of some set $A$ (the set of alternatives).

2. The output is a listing (perhaps with ties) of the set $A$. This list is called the *social preference list*.

    - Allow ties in the output but not in the input.

*Universality* (Unrestricted domain) – The social welfare function should account for all preferences among all votes to yield a unique and complete ranking of societal choices.

While a social choice procedure produces a winner (or winners if tied), the output of a social welfare function is a “social preference listing” of the alternatives.

Individual preference lists

$$\begin{array}{cccc} b & c & c & \cdots \\ d & a & f & \\ c & b & a & \\ e & f & g & \\ \vdots & \vdots & \vdots & \cdots \end{array}$$

$\longrightarrow$ Social Welfare Function $\longrightarrow$

Social preference list

$$\begin{array}{c} a \\ d \\ e \\ f \\ \vdots \end{array}$$

A social welfare function aggregates individual preference lists into a social preference list.

A social welfare function produces a listing of all alternatives. We can take alternative (or alternatives if tied) at the top of the list as the social choice.

## Social welfare functions for two alternatives

- In this case of having only two alternatives, we may simply vote for one of the alternatives instead of providing a preference list.
- Majority rule declares the lone winner to be whichever alternative which has more than half the votes.

*Some examples of social welfare functions for two alternatives*

1. Designate one person as the dictator.
2. Alternative $x$ is always the social choice.
3. The social choice is $x$ when the number of votes for $x$ is odd.

*Desirable properties of social welfare functions*

1. Anonymity (identity of the voter is irrelevant)

   anonymous (不具名) if the social welfare function is independent of the voters' identities.

   - Direct votes counting satisfies anonymity.
   - Dictatorship does not satisfy anonymity. That is, anonymity implies non-dictatorship.

2. Neutrality (identity of the alternative is irrelevant)

   neutral if it is independent of the identities of the alternatives

   "Fixing a particular alternative as always the social choice" does not satisfy neutrality.

As another example, a particular alternative is the lone winner if that alternative receives more than 1/3 of votes and tie if otherwise. This fails neutrality.

3. Monotonicity (winning status will not be altered when more votes are received by the alternative)

   If outcome is $L$, and one or more votes are changed from $H$ to $L$, then the outcome is still $L$.

For example, taking $x$ to be the social choice when the number of votes for $x$ is odd does not satisfy monotonicity.

## Quota system

$n$ voters and 2 alternatives; fix a number $q$ that satisfies

$$\frac{n}{2} < q \leq n + 1.$$

If one of the alternatives has $q$ or more votes, then it *alone* is the social choice. If otherwise, then both alternatives have less than $q$ votes and the outcome is a tie.

1. If $n$ is odd and $q = \dfrac{n+1}{2}$, then the quota system is just the majority rule.

2. What would happen when $n$ is even and $q = \frac{n}{2} + 2$? One alternative may receive $\frac{n}{2} + 1$ while the other receives $\frac{n}{2} - 1$. It leads to a tie since none of the alternatives has $q$ or more votes. In this case, the majority rule is not observed since one of the alternatives receiving more than half of the votes is not declared to be the winner.

3. If $q = n + 1$ and there are only $n$ people, then the outcome is always a tie. This corresponds to the procedure that declares the social choice to be a tie between the two alternatives regardless of how the people vote.

4. If we do not impose $q > \frac{n}{2}$, then it is possible that both alternatives achieve quota. For example, take $n = 11$ and $q = 5$. It is possible that one alternative has 5 votes and the other alternative has 6 votes. Both are declared winner and this violates the condition for "lone winner".

All quota systems satisfy anonymity, neutrality, and monotonicity. The first two properties are seen to be automatically satisfied by any quota system since the procedure performs the direct votes counting. The last property is also obvious since adding more votes should not move the status from winner to "non-winner".

**Theorem**

Suppose we have a social welfare function for two alternatives that is anonymous, neutral, and monotone, then the procedure must be a quota system.

*Proof*

According to the definition of a quota system, it suffices to prove the following 2 conditions:

1. The alternative $L$ alone is the social choice precisely when $q$ or more people vote for $L$.
2. $\frac{n}{2} < q \leq n + 1$.

- Since the social welfare function is anonymous, so the outcome depends on the number of people who vote for, say, $L$.

- Let $G$ denote the set of all numbers $k$ such that $L$ is the lone-winner (獨贏) when exactly $k$ people vote for $L$.

  (a) When $G = \phi$, this implies that $L$ cannot be the lone winner. Also, $H$ cannot be the lone winner by neutrality. In this case, the outcome is always a tie.

  (b) If $G$ is not empty, then we let $q$ be the smallest number in $G$.

  It is easily seen that Monotonicity $\Rightarrow$ Property (1)

*Remark* Case (a) corresponds to $q = n + 1$. It is superfluous to take $q$ to be larger than $n + 1$.

- By neutrality, if $k$ is in $G$, then $n - k$ is definitely not in $G$. Otherwise, $H$ is a lone-winner when exactly $n - k$ people voted for $H$ (occurring automatically as $k$ people voted for $L$). Now, both $H$ and $L$ win. This leads to a contradiction that $L$ wins alone.

For example, take $n = 11$ and $q = 8$. Now, $k = 9$ is in $G$ but $n - k = 2$ cannot be in $G$. Otherwise, if 2 votes are sufficient for $L$ to win, then 2 votes are also sufficient for $H$ to win (neutrality property). However, when $L$ receives 9 votes, then $H$ receives 2 votes automatically. Both $H$ and $L$ win and this is contradicting to $L$ wins alone when it receives 9 votes.

- By invoking monotonicity and neutrality, if $k$ is in $G$, then $n-k$ cannot be as large as $k$. If otherwise, suppose $n - k \geq k$, then $n - k \in G$ due to monotonicity, a contradiction to neutrality. Thus, $n - k < k$ or $n < 2k$. Hence, $n/2 < k$ for any number that is in $G$. Recall that $q$ is the smallest number in $G$. Therefore, we deduce that $q > n/2$.

- Lastly, $q \leq n$ when $G$ is non-empty and it suffices to take $q$ to be $n+1$ when $G = \phi$. Thus,

$$n/2 < q \leq n + 1.$$

*Remark*

When $n$ is odd and we choose $q > \frac{n+1}{2}$, it is possible that the votes of both alternatives cannot achieve the quota. In this case, we have a tie. For example, we take $n = 11$ and $q = 7$, suppose $L$ has 6 votes and $H$ has 5 votes, then a tie is resulted.

**May Theorem**

If the number of voters is odd and *ties are excluded*, then the only social welfare function for two alternatives that satisfies anonymity, neutrality and monotonicity is majority rule.

If ties are excluded, we must have $q \leq \frac{n+1}{2}$. On the other hand, $q > \frac{n}{2}$. When $n$ is odd, the choice of $q$ must be $\frac{n+1}{2}$. This is just the Majority Rule where an alternative receiving more than half of the votes is the lone winner.

**Reasonable social welfare function**

A social welfare function is called *reasonable* if it satisfies the following three conditions:

1. Pareto condition: Society put alternative $x$ strictly above $y$ whenever every individual puts $x$ strictly above $y$. As a consequence, suppose the input consists of a sequence of identical lists, then this single list should also be the social preference list produced as output.

   Therefore, Pareto condition implies the non-imposition property of a social welfare function. Non-imposition means that every societal preference order can be achievable by some profile of individual preference lists. In mathematical term, this is just the surjective property of the social welfare function.

2. Independence of irrelevant alternatives (IIA):

   For example, in the set of 6 voters, the 1st and the 4th voters place $x$ above $y$ while others place $y$ above $x$. If we move other alternatives around to produce a new sequence, the social preference ordering between $x$ and $y$ remains unchanged.

$$
\begin{pmatrix} \cdot \\ \cdot \\ x \\ \cdot \\ \cdot \\ \cdot \\ y \\ \cdot \\ \cdot \end{pmatrix}
\cdots
\begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ x \\ \cdot \\ \cdot \\ y \\ \cdot \\ \cdot \end{pmatrix}
\cdots
\quad \xrightarrow[\text{alternatives around}]{\text{moving other}} \quad
\begin{pmatrix} x \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ y \end{pmatrix}
\cdots
\begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ x \\ y \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}
\cdots
$$

**Interpretation of Independence of Irrelevant Alternatives**

Suppose we have two different sequences of individual preference lists and exactly the same set of people have alternative $x$ over alternative $y$ in their list.

IIA dictates that we either get $x$ over $y$ in both social preference lists, or $y$ over $x$ in both social preference lists. It is not possible to have $x$ over $y$ in one social preference list but $y$ over $x$ in another social preference list. The positioning of alternatives other than $x$ and $y$ in the individual preference lists is irrelevant to the question of whether $x$ is socially preferred to $y$ or $y$ is socially preferred to $x$. In other words, the social relative ranking (higher or lower) of two alternatives $x$ and $y$ depends only on their relative ranking by every individual.

**Non-dictatorship**

There is no individual whose preference always prevails, that is, no individual's preference list is always the social preference list.

In our later proof of the Arrow Impossibility Theorem, it is necessary to have no ties in the output. This does not raise any concern due to the following proposition.

**Proposition**

If $A$ has at least three elements, then any social welfare function for $A$ that satisfies both IIA and the Pareto condition will never produce ties in the output.

*Proof*

- Assume, for contradiction, there exist some sequences of individual preference lists that result in a social preference list in which the alternatives $a$ and $b$ are tied, even though we are not allowing ties in any of the individual preference lists.

- By virtue of IIA, we know that $a$ and $b$ will remain tied as long as we do not change any individual preference list in a way that reverses that voter's ranking of $a$ and $b$.

Let $c$ be any alternative that is distinct from $a$ and $b$. Let $X$ be the set of voters who have $a$ over $b$ in their individual preference lists, and let $Y$ be the rest of the voters (who therefore have $b$ over $a$ in their lists).

We group voters that place $a$ over $b$ in $X$ and voters that place $b$ over $a$ in $Y$.

$$\overbrace{\begin{matrix} a & \ldots & a \\ b & & b \end{matrix}}^{X} \qquad\qquad \overbrace{\begin{matrix} b & \ldots & b \\ a & & a \end{matrix}}^{Y}$$

yields

$$ab \text{ (tied).}$$

We try to show that contradiction arises due to tied outcome for $a$ and $b$, while the social welfare function has to satisfy IIA and the Pareto condition.

- Suppose we now insert $c$ between $a$ and $b$ in the lists of the voters in $X$, and we insert $c$ above $a$ and $b$ in the lists of the voters in $Y$. Then we will still get $a$ and $b$ tied in the social preference list (by independence of irrelevant alternatives), and we will get $c$ over $b$ by Pareto, since $c$ is over $b$ in every individual preference list. Thus, we have:

$$\overbrace{\begin{matrix} a \\ c \\ b \end{matrix} \quad \cdots \quad \begin{matrix} a \\ c \\ b \end{matrix}}^{X} \qquad\qquad \overbrace{\begin{matrix} c \\ b \\ a \end{matrix} \quad \cdots \quad \begin{matrix} c \\ b \\ a \end{matrix}}^{Y}$$

yields

$$\begin{matrix} c \\ ab. \end{matrix}$$

- IIA guarantees us that, as for as $a$ versus $c$ goes, we can ignore $b$. Thus, we can conclude that if everyone in $X$ has $a$ over $c$ and everyone in $Y$ has $c$ over $a$, then we get $c$ over $a$ in the social preference list.

- To get our desired contradiction, we insert $c$ differently from what we did before. We insert $c$ under $a$ and $b$ for the voters in $X$, and between $a$ and $b$ for the voters in $Y$. Now, $c$ is below $b$ in both $X$ and $Y$. In the social preference list, we maintain $ab$ to be tied while $c$ is below $b$ due to the Pareto condition.

$$\overbrace{\begin{array}{ccc} a & & a \\ b & \cdots & b \\ c & & c \end{array}}^{X} \qquad \overbrace{\begin{array}{ccc} b & & b \\ c & \cdots & c \\ a & & a \end{array}}^{Y}$$

  yields

$$\begin{array}{c} ab \\ c. \end{array}$$

- IIA guarantees us that, as far as $a$ versus $c$ goes, we can ignore $b$. Thus, we can now conclude that if everyone in $X$ has $a$ over $c$ and everyone in $Y$ has $c$ over $a$, then we get $a$ over $c$ in the social preference list. This is the opposite of what we concluded above, and thus we have the desired contradiction.

## 3.5 Arrow Impossibility Theorem

Dictatorship satisfies Pareto condition (if $x$ is preferred to $y$ by all, including the dictator, then $x$ is socially preferred to $y$) and IIA (moving other alternatives would not change the social ranking of $x$ and $y$).

**Theorem** (Arrow, 1950). If the set of alternatives $A$ has at least three elements and the set $P$ of individuals is finite, then there is no social welfare function for $A$ and $P$ satisfying the Pareto condition, independence of irrelevant alternatives and non-dictatorship.

Under the assumption of Pareto and IIA, there always exists a particular singleton voter where the social preference list is the same as the preference of this singleton voter – a dictator. The only social welfare function satisfying the Pareto condition and IIA is a dictatorship.

## Proof of the Arrow Impossibility Theorem

The proof consists of 3 steps:

1. An alternative $X$ is said to be polarizing with respect to $a$ profile $P$ if $X$ is ranked first or last by every voter in $P$. Pareto and IIA conditions together will enforce $X$ to be placed either the first or last place in the group ranking (social preference list).

2. Identify a potential dictator $j$ who has the power to move $X$ from the last place to the first place in the group ranking.

3. Establish $j$ to be the dictator who can force the ordering of any pair of alternatives.

**First step: Polarizing alternatives**

Let $F$ be a social welfare function satisfying the Pareto and IIA conditions. We will use $P$ to denote a profile of individual rankings and use $F(P)$ to denote the group ranking that $F$ produces. We will work toward identifying an individual $j$ such that $F$ is dictatorship by $j$.

First, let $P$ be a profile in which $X$ is a polarizing alternative. Assume the contrary that $F$ does not place $X$ in either the first or last place in the group ranking $F(P)$. This means that there are other alternatives $Y$ and $Z$ so that $Y \succ X \succ Z$ in the group ranking $F(P)$.

For any individual ranking that puts $Y$ ahead of $Z$, let's change it by sliding $Z$ to the position just ahead of $Y$. This produces a new profile $P'$.

Profile $P$:

| Individual | Ranking |
|---|---|
| 1 | $X \succ \cdots \succ Y \succ \cdots \succ Z \succ \cdots$ |
| 2 | $X \succ \cdots \succ Z \succ \cdots \succ Y \succ \cdots$ |
| 3 | $\cdots \succ Y \succ \cdots \succ Z \succ \cdots \succ X$ |

Profile $P'$:

| Individual | Ranking |
|---|---|
| 1 | $X \succ \cdots \succ Z \succ Y \succ \cdots$ |
| 2 | $X \succ \cdots \succ Z \succ \cdots \succ Y \succ \cdots$ |
| 3 | $\cdots \succ Z \succ Y \succ \cdots \succ X$ |

The polarizing alternative $X$ appears at the beginning or end of every individual ranking. In Profile $P$, $Y$ can be ahead or behind $Z$. To obtain $P'$, we slide $Z$ to be ahead of $Y$ in those individual preference lists which put $Y$ ahead of $Z$ in $P$.

- Since $X$ is a polarizing alternative, the relative order of $X$ and $Z$ does not change in any individual ranking when we move from $P$ to $P'$, nor does the relative order of $X$ and $Y$. By IIA, $F(P)$ and $F(P')$ have the same ordering of $X$ and $Z$, so does $X$ and $Y$. That is, we still have $Y \succ X \succ Z$ in the group ranking $F(P')$.

- However, in $P'$, alternative $Z$ is ahead of alternative $Y$ in every individual ranking, and so by the Pareto condition, we have $Z \succ Y$ in the group ranking $F(P')$. This leads to a contradiction.

**Second step: Identifying a potential dictator**

We create a sequence of profiles with the property that each differs from the next by very little, and we watch how the group ranking under $F$ changes as we move through this sequence. A natural candidate for the dictator will emerge.

We pick one of the alternatives $X$ and start with any profile $P_0$ that has $X$ at the end of each individual ranking. Now, one individual ranking at a time, we move $X$ from the last place to the first place while leaving all other parts of the individual rankings the same. Under such arrangement, $X$ remains to be a polarizing alternative (either at the top or bottom) in all these profiles.

Let $k$ be the total number of voters. Starting from $P_0$, this produces a sequence of profile $P_1$, $P_2$, ..., $P_k$, where $P_i$

(i) has $X$ at the front of the individual rankings of $1, 2, \ldots, i$;

(ii) has $X$ at the end of the individual rankings of $i+1, i+2, \ldots, k$; and

(iii) has the same order as $P_0$ on all other alternatives.

In other words, $P_{i-1}$ and $P_i$ differ only in that individual $i$ ranks $X$ last in $P_{i-1}$, and he ranks $X$ first in $P_i$.

Profile $P_0$:

| Individual | Ranking |
|---|---|
| 1 | $\cdots \succ Y \succ \cdots \succ Z \succ \cdots \succ X$ |
| 2 | $\cdots \succ Z \succ \cdots \succ Y \succ \cdots \succ X$ |
| 3 | $\cdots \succ Y \succ \cdots \succ Z \succ \cdots \succ X$ |

Profile $P_1$:

| Individual | Ranking |
|---|---|
| 1 | $X \succ \cdots \succ Y \succ \cdots \succ Z \succ \cdots$ |
| 2 | $\cdots \succ Z \succ \cdots \succ Y \succ \cdots \succ X$ |
| 3 | $\cdots \succ Y \succ \cdots \succ Z \succ \cdots \succ X$ |

Profile $P_2$:

| Individual | Ranking |
|---|---|
| 1 | $X \succ \cdots \succ Y \succ \cdots \succ Z \succ \cdots$ |
| 2 | $X \succ \cdots \succ Z \succ \cdots \succ Y \succ \cdots$ |
| 3 | $\cdots \succ Y \succ \cdots \succ Z \succ \cdots \succ X$ |

Profile $P_3$:

| Individual | Ranking |
|---|---|
| 1 | $X \succ \cdots \succ Y \succ \cdots \succ Z \succ \cdots$ |
| 2 | $X \succ \cdots \succ Z \succ \cdots \succ Y \succ \cdots$ |
| 3 | $X \succ \cdots \succ Y \succ \cdots \succ Z \succ \cdots$ |

By the Pareto condition, $X$ must be the last in the group ranking $F(P_0)$, and it must be the first in the group ranking $F(P_k)$. Along this sequence, we argue that there must exist the **first** occurrence in which $X$ is not in the last place in the group ranking. If there is no such occurrence, then there will be no change of $X$ from the last place in $F(P_0)$ to the first place in $F(P_k)$, a contradiction. Suppose this first profile is $P_j$. Since $X$ is a polarizing alternative in $P_j$, as it is not in the last place in $F(P_j)$, then it must be in the first place.

The individual $j$ has the power over the outcome for alternative $X$, at least in this sequence of profiles: by switching her own ranking of $X$ from the last to the first, she causes $X$ to move from the last to the first in the group ranking.

In the final step of the proof, we will show that $j$ is in fact a dictator.

**Third step: Establishing that $j$ is a dictator**

1. For any profile $Q$, and any alternatives $Y$ and $Z$ that are different from $X$, the ordering of $Y$ and $Z$ in the group ranking $F(Q)$ is the same as the ordering of $Y$ and $Z$ in $j$'s individual ranking in $Q$.

2. The above result also holds for pairs of alternatives in which one of the alternatives is $X$.

Let $Q$ be any profile, and let $Y$ and $Z$ be alternatives not equal to $X$. Suppose $j$ ranks $Y$ ahead of $Z$ in $Q$. We will show that $F(Q)$ puts $Y$ ahead of $Z$ as well.

Starting from $Q$, we move $X$ to the front of the individual rankings of $1, 2, \ldots, j$, and move $X$ to the end of the individual rankings of $j+1, j+2, \ldots, k$. Then, we move $Y$ to the front of $j$'s individual ranking (just ahead of $X$). We call the resulting profile $Q'$. This guarantees that $Q'$ and $P_{j-1}$ are the same when restricted to $X$ and $Y$, while $Q'$ and $P_j$ are the same when restricting to $X$ and $Z$ (since $Y$ is ahead of $Z$).

- We know that $X$ comes first in the group tanking $F(P_j)$. Since $Q'$ and $P_j$ are the same when restricted to $X$ and $Z$, it follows from IIA that $\mathbf{X \succ Z}$ **in** $\mathbf{F(P_j)}$ as well as $F(Q')$.

- We know that $X$ comes last in the group ranking $F(P_{j-1})$. Since $Q'$ and $P_{j-1}$ are the same when restricted to $X$ and $Y$, it follows from IIA that $\mathbf{Y \succ X}$ **in** $\mathbf{F(P_{j-1})}$ as well as $F(Q')$. By transitivity, we conclude that $\mathbf{Y \succ Z}$ **in** $\mathbf{F(Q')}$.

- $Q$ and $Q'$ are the same when restricted to $Y$ and $Z$, since we produced $Q'$ from $Q$ without ever swapping the order of $Y$ and $Z$ in any individual ranking. By IIA, it follows that $\mathbf{Y \succ Z}$ **in** $\mathbf{F(Q)}$.

- Since $Q$ is any profile, and $Y$ and $Z$ are any pair of alternatives (other than $X$) subject only to the condition that $j$ ranks $Y$ ahead of $Z$, it follows that the ordering of $Y$ and $Z$ in the group ranking is always the same as $j$'s.

We have shown that $j$ is a dictator over all pairs of alternatives that do not involve $X$. The remaining step is to show that $j$ is also a dictator over all pairs involving $X$ as well. We pick any other alternative $W$ different from $X$. We run a similar argument and establish that there is also an individual $\ell$ who is a dictator over all pairs involving $X$ as well, but not involving $W$. It suffices to show that $\ell$ must be $j$.

Suppose that $\ell$ is not equal to $j$. Now, for $X$ and some third alternative $Y$ different from $X$ and $W$, we know that the profiles $P_{j-1}$ and $P_j$ differ **only** in $j$'s individual ranking ($j$ ranks $X$ last in $P_{j-1}$ and first in $P_j$), yet the ordering of $X$ and $Y$ is different between the group rankings $F(P_{j-1})$ and $F(P_j)$. For $\ell \neq j$, the orderings of $X$ and $Y$ in $\ell$'s individual ranking are the same in $P_{j-1}$ and $P_j$.

In one of these two group rankings $F(P_{j-1})$ and $F(P_j)$, where one has $X$ above $Y$ while the other has $Y$ above $X$, the ordering of $X$ and $Y$ must therefore differ from the ordering of $X$ and $Y$ in $\ell$'s individual ranking. This contracts the fact that $\ell$ is a dictator for the pair of $X$ and $Y$. Hence our assumption that $\ell$ is different from $j$ must be false. We then conclude that $j$ is in fact a dictator over all pairs involving $X$.

**Remark**

The original version of the Arrow Impossibility Theorem involves two other criteria: Monotonicity and Non-imposition, rather than Pareto condition.

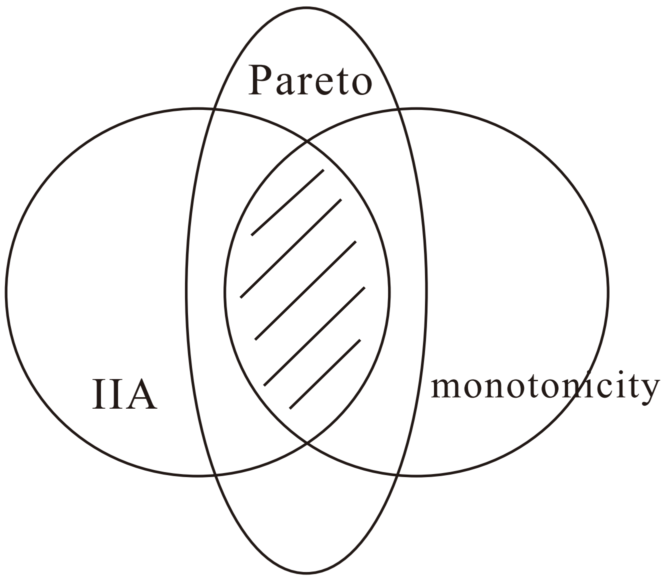
*Monotonicity* states that if one or more voters change their preference lists by putting one alternative higher, then the social preference list should either be changed by ranking that alternative higher or else be unchanged. An alternative cannot be made less popular (lower ranking) by having one rating improved.

We have argued earlier that Pareto condition $\Rightarrow$ non-imposition. Next, we establish that

$$\text{IIA} + \text{monotonicity} \Rightarrow \text{Pareto condition.}$$

Therefore, the version proven earlier represents a stronger version (with less restrictive assumptions required) than the original version of the Theorem.

## Proof of IIA + monotonicity ⇒ Pareto condition




We prove by contradiction. Suppose a social welfare function is not Pareto efficiency but satisfies IIA and monotonicity. Consider a preference profile in which every voter prefers $A$ to $B$ but the outputs prefers $B$ to $A$. This is possible due to failure of Pareto condition.

$$\begin{pmatrix} \vdots \\ A \\ \vdots \\ B \\ \vdots \end{pmatrix} \cdots \begin{pmatrix} \vdots \\ A \\ \vdots \\ B \\ \vdots \end{pmatrix} \longrightarrow \begin{pmatrix} \vdots \\ B \\ \vdots \\ A \\ \vdots \end{pmatrix}$$

Change the profile by moving $B$ up in every voter's list until $B$ is just above $A$. By monotonicity, $B$ remains to be above $A$ in the output.

$$\begin{pmatrix} \vdots \\ B \\ A \\ \vdots \end{pmatrix} \cdots \begin{pmatrix} \vdots \\ B \\ A \\ \vdots \end{pmatrix} \longrightarrow \begin{pmatrix} \vdots \\ B \\ \vdots \\ A \\ \vdots \end{pmatrix}$$

Change every list by moving $A$ down to the position originally occupied by $B$. Since we maintain the relative position of $A$ and $B$ in every list, by IIA, $B$ is still preferred to $A$. Essentially, we swap positions of $A$ and $B$ in every voter's list.

$$\begin{pmatrix} \vdots \\ B \\ \vdots \\ A \\ \vdots \end{pmatrix} \cdots \begin{pmatrix} \vdots \\ B \\ \vdots \\ A \\ \vdots \end{pmatrix} \longrightarrow \begin{pmatrix} \vdots \\ B \\ \vdots \\ A \\ \vdots \end{pmatrix}$$

Since we interchange $A$ with $B$ in every voter's list, we should swap $A$ and $B$ in the outcome so that $A$ should be above $B$. This leads to a contradiction.

We obtain this contradictory result since we start with the assumption of failure of Pareto condition.

Conclusion: IIA $+$ monotonicity $\Rightarrow$ Pareto condition

## Alternative proof

**Theorem** (Arrow, 1950). If the set of alternatives $A$ has at least three elements and the set $P$ of individuals is finite, then there is no social welfare function for $A$ and $P$ satisfying the Pareto condition, independence of irrelevant alternatives, monotonicity, and non-dictatorship.

Alternative statement: The only social welfare function satisfying the Pareto condition, IIA and monotonicity is a dictatorship.

**Definition** 某組人能足夠保證把$a$放在$b$之上

With reference to a social welfare function satisfying Pareto, IIA and monotonicity, suppose $X$ is a set of people, $a$ and $b$ are a pair of distinct alternatives. "**$\mathbf{X}$ can force $\mathbf{a}$ over $\mathbf{b}$**" means

"We get $a$ over $b$ in the social preference list whenever everyone in $X$ places $a$ over $b$ in their individual preference lists."

$X$ always exists by virtue of the Pareto condition: setting $X = P$.

- Under the assumption of IIA and monotonicity, in order to show that $X$ can force $a$ over $b$, it suffices to produce a *single sequence* of individual preference lists for which the following properties all hold.

1. Everyone in $X$ has $a$ over $b$ in their lists.

2. Everyone not in $X$ has $b$ over $a$ in their lists.

3. The resulting social preference list has $a$ over $b$.

- Without monotonicity, "$X$ forces $a$ over $b$" may not be well defined. This is because failure of monotonicity may lead to the scenario where all voters in $X$ places $a$ over $b$ while there is no guarantee that the social preference list places $a$ over $b$ when more voters outside $X$ place $a$ over $b$.

- By virtue of IIA, in showing that $X$ forces $a$ over $b$, it suffices to consider a single sequence of individual preference lists. Other sequences with the same property that everyone in $X$ places $a$ over $b$ would also give $a$ over $b$ in the social preference list.

- By virtue of monotonicity, we allow the "worst scenario" where those not in $X$ place $b$ above $a$. If some of them change their preferences by swapping $b$ over $a$ to $a$ over $b$, $a$ remains to be above $b$ in the social preference list.

- An **empty set** cannot force $a$ above $b$. Why? By property (2), suppose $X$ is the empty set and every one not in $X$ has $b$ over $a$. By virtue of the Pareto condition, the resulting social preference list *cannot* have $a$ over $b$.

## Definition of a "dictating set"

Given a social welfare function, a set $X$ is called a dictating set if $X$ can force $a$ over $b$ for any two distinct pair of alternatives $a$ and $b$ in $A$.

1. If $X$ is the set of all individuals, then $X$ is a dictating set. This follows directly from the Pareto condition. It is guaranteed to have a dictating set once the Pareto condition is satisfied.

2. Let $p$ be one of the individuals. $X$ is a dictating set with single individual $p$ if and only if $p$ is a dictator.

It is obvious that $p$ as dictator satisfies: "force $a$ over $b$" for any pair of alternatives. The set with the single element $p$ is a dictating set. On the other hand, if $p$ as the only single individual in the dictating set that can always force $a$ over $b$ for any pair of alternatives, the social preference list must coincide with this single individual's own preference list, then $p$ is a dictator.

*Remarks*

- A dictator may not exist even a dictating set exists. Even though the whole set $P$ is a dictating set (by virtue of the Pareto condition), it does not suffice to deduce that there exists a dictator.

- *Suppose a dictator exists*, a set is a dictating set if and only if it contains the dictator. If a set contains the dictator, then it is a dictating set. If the set does not contain the dictator, then the set cannot force $a$ over $b$ for any pair of alternatives since the true dictator may choose $b$ over $a$. Therefore, the set cannot be a dictating set.

- If an individual who **can** force one alternative over the other for any pair of alternatives, then he can force his preference list to be the social preference list. This can be done by forcing sequential pairs from his top choice down to the bottom choice to generate the social preference list. Therefore, he is a dictator.

*Key ideas behind the proof*

The strategy for passing from the initial large dictating set $P$ to the very small dictating set $\{p\}$ involves the following: Show that if $X$ is a dictating set, and if we split $X$ into any two sets $Y$ and $Z$ of disjoint partitions (so that everyone in $X$ is in exactly one of the two sets), then one of the split sets is a dictating set.

We start from $P$, then split $P$ into two disjoint sets where one of them is a dictating set. We continue the splitting procedure down to a singleton set. The dictator is identified.

## Five lemmas yielding Arrow's Theorem

### Lemma 1

For notational simplicity, we write "$X$ forces $a$ over $b$" meaning "$X$ has the ability to force $a$ over $b$ in the social preference list under any profile where $X$ places $a$ over $b$". Suppose $X$ forces $a$ over $b$ and $c$ is any alternative distinct from $a$ and $b$. Suppose now that $X$ is split into two disjoint sets $Y$ and $Z$ (either of which may be the empty set) so that each element of $X$ is in exactly one of the two sets. Then either $Y$ forces $a$ over $c$ or $Z$ forces $c$ over $b$.

*Remark*: If $X$ has the power to force $a$ high and $b$ low, then either $Y$ inherits the power to force $a$ high or $Z$ inherits the power to force $b$ low. This is valid for any choice of splitting of $X$ into two disjoint sets.

*Proof*

Suppose $X$ forces $a$ over $b$ under a given social welfare function satisfying Pareto, IIA and monotonicity. Consider what happens when the social welfare function under consideration is applied to the following ingeniously chosen profile of individual preference lists as input:

| Everyone in $Y$ | Everyone in $Z$ | Everyone else outside $X$ |
|---|---|---|
| $a$ | $c$ | $b$ |
| $b$ | $a$ | $c$ |
| $c$ | $b$ | $a$ |
| $\vdots$ | $\vdots$ | $\vdots$ |

Here, $a$, $b$ and $c$ are placed among the top three choices in all preference list. By the Pareto condition, the social choice must be either $a$, $b$ or $c$.

Next, we rule out $b$ as a social choice. This is because every voter in $X = Y \cup Z$ ranks $a$ over $b$ in the profile while $X$ forces $a$ over $b$, so $b$ cannot be a social choice.

We consider two separate scenarios under this chosen profile:

(a) The social choice is $a$

Now, every voter in $Y$ places $a$ over $c$ while all voters outside $Y$ places $c$ over $a$. Therefore, we conclude that $Y$ can force $a$ over $c$.

(b) The social choice is $c$

Now, every voter in $Z$ places $c$ over $b$ while all voters outside $Z$ places $b$ over $c$. Therefore, $Z$ can force $c$ over $b$. Since the two scenarios occur separately, we either have $Y$ force $a$ over $c$ or $Z$ forces $c$ over $b$, but not both.

**Lemma 2**

Suppose $X$ forces $a$ over $b$ and $c$ is an alternative distinct from $a$ and $b$. Then $X$ forces $a$ over $c$ and $X$ forces $c$ over $b$.

*Proof*

- Using Lemma 1, as a special case, set $Y = X$ and $Z = \phi$. The conclusion is that either $X$ forces $a$ over $c$ (as desired) or the empty set forces $c$ over $b$. Since an empty set cannot forces $c$ over $b$. Therefore, we must have $X$ forces $a$ over $c$.

- In a completely analogous way, a consideration of the special case of Lemma 1 where $Y$ is the empty set and $Z$ is the whole set $X$. We have $X$ forces $c$ over $b$.

**Lemma 3**

If $X$ forces $a$ over $b$, then $X$ forces $b$ over $a$.

Intuition: The forcing relation is symmetric.

*Proof*

By applying Lemma 2 repeatedly, we deduce that

$$X \text{ forces } \begin{matrix} a \\ b \end{matrix} \Rightarrow X \text{ forces } \begin{matrix} a \\ c \end{matrix} \Rightarrow X \text{ forces } \begin{matrix} b \\ c \end{matrix} \Rightarrow X \text{ forces } \begin{matrix} b \\ a \end{matrix}.$$

**Lemma 4**

Suppose there are two alternatives $a$ and $b$ so that $X$ can force $a$ over $b$, then $X$ can force one alternative over the other in any pair of alternatives; so $X$ is a dictating set.

Intuition: If $X$ has a little local power, then $X$ has complete global power.

*Proof*

(i) For any pair of alternatives $x$ and $y$, where $x$ and $y$ are distinct from $a$ and $b$, we show that $X$ can force $x$ over $y$. Since $a \neq y$, we have

$$X \text{ forces } \begin{matrix} a \\ b \end{matrix} \Rightarrow X \text{ forces } \begin{matrix} a \\ y \end{matrix} \Rightarrow X \text{ forces } \begin{matrix} x \\ y \end{matrix}.$$

(ii) For any pair of alternatives that involve $a$ or $b$ (say $a$), we have

$$X \text{ forces } \begin{matrix} a \\ b \end{matrix} \Rightarrow X \text{ forces } \begin{matrix} a \\ x \end{matrix} \Rightarrow X \text{ forces } \begin{matrix} x \\ a \end{matrix}.$$

In conclusion, $X$ can force any new pair $x$ and $y$, or the pair with one new alternative and one from $a$ or $b$. Hence, $X$ is a dictating set.

## Lemma 5

Suppose that $X$ is a dictating set and suppose that $X$ is split into two disjoint sets $Y$ and $Z$. Then either $Y$ is a dictating set or $Z$ is a dictating set, but not both.

*Proof*

Choose three distinct alternatives $a, b$, and $c$. Since $X$ is a dictating set, we have that $X$ can force $a$ over $b$. Lemma 1 now guarantees that either $Y$ can force $a$ over $c$ (in which case $Y$ is a dictating set by Lemma 4), or $Z$ can force $c$ over $b$ (in which case $Z$ is a dictating set by Lemma 4 again).

To identify whether the split set $Y$ or $Z$ is a dictating set, we check the forcing property by examining whether $a$ over $c$ or $c$ over $b$ in the social preference list under the given social welfare function.

**Final statement**

We start with the whole set $P$, which is a dictating set by virtue of the Pareto condition. We split a given dictating set based on splitting a single element off the set at each step.

If the singleton set is a dictating set, then we are done since the dictator is the singleton voter in that dictating set while the other split set cannot be a dictating set (if a dictator has been identified, then a dictating set must contain the dictator). If otherwise, the other split set is a dictating set since one of the split sets is a dictating set by virtue of Lemma 5. We continue the splitting procedure until a dictator is identified.

**Extensions of the Impossibility Theorem**

- If a social choice function satisfies monotonicity and Pareto condition, then it is a dictatorial social choice function.

- If a social choice function is strategy-proof and satisfies non-imposition, then it satisfies monotonicity and Pareto condition.

As a result, we obtain the *Gibbard-Satterthwaite Theorem*:

If a social choice function is strategy-proof and satisfies non-imposition, then it is a dictatorial social choice function.

## 3.6 Direct democracy - referendum

There has been a shift from "representative democracy" to "direct democracy". Direct democracy through referendums seems to leave no ambiguity on "what did the people want?" Be watchful!

Opponents of referendums are known for their distrust of the abilities of the average citizen to make political decisions. They worry about the susceptibility of the public to well rehearsed advertisements and well financed campaigns by special interests. The problem is "quality of voters".

A referendum is a vote on a set of ordered binary proposals, $1, 2, \ldots, n$, where the outcome of each proposal is decided by majority vote - passage or failure.

## Example

To resolve acute demand of parking spaces on campus, two proposals are put forward.

*Proposal 1* Increase the price of a student parking permit (reduce demand).

*Proposal 2* Build a new parking garage (increase supply).

*Preference lists for the combined proposals in one ballot*

| ordering | Dave | Mike | Pete |
|---|---|---|---|
| 1 | $Y/N$ | $N/Y$ | $N/N$ |
| 2 | $N/Y$ | $Y/Y$ | $N/Y$ |
| 3 | $Y/Y$ | $Y/N$ | $Y/N$ |
| 4 | $N/N$ | $N/N$ | $Y/Y$ |

Each voter votes according to his top choices on the two proposals. Their preference lists are presented as background information for our subsequential theoretical analysis.

Two votes on $N$ for proposal 1, so $N$ wins in proposal 1. Similarly, two votes on $N$ for proposal 2. The final outcome of the referendum is $N/N$, which is the least preferred choice for two voters (Dave and Mike). This is highly undesirable.

Dave: The two proposals are not separate for Dave. He thinks it is appropriate to raise the parking fee if no new parking garage is built, otherwise no raise in parking fee if a new parking garage is built. To him, taking either one of the two measures is better than taking both measures together. The worst scenario is doing nothing.

Mike: Prefers $Y$ in proposal 2, independent on proposal 1. Always agrees to build a new garage.

Pete: Prefers $N$ in proposal 1, independent on proposal 2. No raise in parking fee.

## Observations

1. Pete is *pivotal*. Suppose 100 other students vote like Dave and another 100 other students vote like Mike, Pete's first choice ($N/N$) remains to prevail.

2. Suppose Dave somehow found out that Mike and Pete are going to vote $N/Y$ and $N/N$, respectively, he would vote *insincerely* for his second choice $N/Y$. For Dave, the outcome $N/Y$ is better than $N/N$.

**Separability of preferences**

Let $v$ represent some voter in a referendum election.

- A collection $S$ of proposals in the election (possibly just one) is said to be *separable with respect to* $v$ if $v$'s ranking of the possible combinations of outcomes for all the proposals in $S$ does not depend on the outcome of the election for any of the proposals not in $S$.

- The preferences of $v$ are said to be *separable* if every possible collection of proposals in the election is separable with respect to $v$.

In simple language, a voter has separable preferences over two proposals if foreknowledge of the voting result on one proposal does not affect the voter's preferred choice on the other proposal.

Consider $YY > NN > NY > YN$, learning that the first proposal is going to lose, the voter would choose $N$ on the second proposal.

On the other hand, consider $YY > NY > YN > NN$, whether either proposal wins or loses, the voter has the same preference $Y$ on the other proposal. When the preference order has separable preferences, the voter will never experience regret by voting $Y$ on either proposal

If the first proposal is $Y$, since $YY > YN$, $Y$ is preferred in the second proposal. If the first proposal is $N$, since $NY > NN$, $Y$ is preferred in the second proposal. If the second proposal is $Y$, since $YY > NY$, $Y$ is preferred in the first proposal. If the second proposal is $N$, since $YN > NN$, $Y$ is preferred in the first proposal.

**Example**

Consider the following preference matrix of a voter (0 signifies $N$ and 1 signifies $Y$)

$$R_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix},$$

which is separable with respect to proposal 2 since $Y$ precedes $N$ on proposal 2, independent on $Y$ or $N$ on proposal 1.

However, $R_1$ is not separable with respect to proposal 1 since $Y$ precedes $N$ on proposal 1 if the outcome on proposal 2 is $N$, and vice versa if otherwise.

Next, for the preference matrix of another voter

$$R_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$$

is separable on both proposals. Note that $N$ precedes $Y$ on proposal 1, independent on $Y$ or $N$ on proposal 2.

Lastly, an example of non-separable on both proposals is given by

$$R_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}$$

Note that voter puts $N$ ahead of $Y$ in proposal 2 if outcome of proposal 1 is $Y$, and vice versa if outcome of proposal 1 is different.

For the 3-proposal preference matrix

$$R_4 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

it is seen that $R_4$ is separable on each proposal, separable on the subsets of proposals: $\{p_1, p_2\}$ and $\{p_2, p_3\}$, but not the subset $\{p_1, p_3\}$.

To see the last result, note that

(i) $(0, 1, 1)$ is preferred to $(1, 1, 0)$, (ii) $(1, 0, 0)$ is preferred to $(0, 0, 1)$,

which indicates that $(0, 1)$ is preferred to $(1, 0)$ on $\{p_1, p_3\}$ when $Y$ on proposal 2 and $(1, 0)$ is preferred to $(0, 1)$ when $N$ on proposal 2.

*Separability on proposal 2*

(i) $Y$ precedes $N$ in proposal 2 under $(1,1)$ of proposals 1 and 3 since $(1,1,1) \succ (1,0,1)$.

(ii) $Y$ precedes $N$ in proposal 2 under $(1,0)$ of proposals 1 and 3 since $(1,1,0) \succ (1,0,0)$.

(iii) $Y$ precedes $N$ in proposal 2 under $(0,1)$ of proposals 1 and 3 since $(0,1,1) \succ (0,0,1)$.

(iv) $Y$ precedes $N$ in proposal 2 under $(0,0)$ of proposals 1 and 3 since $(0,1,0) \succ (0,0,0)$.

We conclude that $Y$ precedes $N$ in proposal 2 under all outcomes in proposals 1 and 3.

*Separability on pair of proposals 1 and 2*

The preference order on the two proposals is deduced to be

$$(1,1) \succ (1,0) \succ (0,1) \succ (0,0)$$

based on the following observations. Also, the preference order is independent of the outcome of proposal 3.

(i) $(1,1) \succ (1,0)$ is established since

$$(1,1,1) \succ (1,0,1) \quad \text{and} \quad (1,1,0) \succ (1,0,0).$$

(ii) $(1,0) \succ (0,1)$ is established since

$$(1,0,1) \succ (0,1,1) \quad \text{and} \quad (1,0,0) \succ (0,1,0).$$

(iii) $(0,1) \succ (0,0)$ is established since

$$(0,1,1) \succ (0,0,1) \quad \text{and} \quad (0,1,0) \succ (0,0,0).$$

Separability of the 24 strict preference orders over {YY, YN, NY, NN}

| Orders | Separable? |
|---|---|
| 1. YY>YN>NY>NN | Yes |
| 2. YY>YN>NN>NY | |
| 3. YY>NY>YN>NN | Yes |
| 4. YY>NY>NN>YN | |
| 5. YY>NN>YN>NY | |
| 6. YY>NN>NY>YN | |
| 7. YN>YY>NY>NN | |
| 8. YN>YY>NN>NY | Yes |
| 9. YN>NY>YY>NN | |
| 10. YN>NY>NN>YY | |
| 11. YN>NN>YY>NY | Yes |
| 12. YN>NN>NY>YY | |
| 13. NY>YY>YN>NN | |
| 14. NY>YY>NN>YN | Yes |
| 15. NY>YN>YY>NN | |
| 16. NY>YN>NN>YY | |
| 17. NY>NN>YY>YN | Yes |
| 18. NY>NN>YN>YY | |
| 19. NN>YY>YN>NY | |
| 20. NN>YY>NY>YN | |
| 21. NN>YN>YY>NY | |
| 22. NN>YN>NY>YY | Yes |
| 23. NN>NY>YY>YN | |
| 24. NN>NY>YN>YY | Yes |

Of the $4! = 24$ possible strict preference orders over the four $Y$-$N$ combinations, 8 of them (33.3%) have separable preferences.

When there are 3 proposals, there are $8! = 40,320$ strict preference orders. There are only 384 orders that are separable (less than 1%).

The percentage appears to be negligible for four or more proposals.

Under standard aggregation, $Y$ and $N$ votes are aggregated separately for each proposal to determine a winner. Voters with separable preferences have no regret under standard aggregation. However, voters with nonseparable preferences may find that they might have voted contrary to their interests once all the results are known. Regrets occur.

## Intersection of sets of separability

Suppose $S_1$ and $S_2$ are subsets of proposals that are separable, the example of $R_4$ shows that union of $S_1$ and $S_2$ may not be separable while intersection of $S_1$ and $S_2$ is separable. For example, $R_4$ is separable on $\{p_1, p_2\}$ and $\{p_2, p_3\}$, it is separable on the intersection $\{p_2\}$. However, $R_4$ is separable on $\{p_1\}$ and $\{p_3\}$, but not separable on the union $\{p_1, p_3\}$.

The proof of separability under intersection is quite straightforward. Proposals on $S_1$ are not affected by any proposal outside $S_1$, same for proposals in $S_2$. Recall that $S_1' \cup S_2' = (S_1 \cap S_2)'$, where $S'$ denotes the complement of $S$. Therefore, proposals in $S_1$ and $S_2$ are not affected by any proposal outside both $S_1$ and $S_2$ or outside $S_1 \cap S_2$.

## Problems with nonseparable preferences

If some voters have nonseparable preferences for the outcomes of votes across multiple proposals, if voters do not know each other's preferences, and if the proposals are decided simultaneously by majority rule with binary votes on separate proposals, then a Condorcet winner may not be chosen, and the outcome may be a Condorcet loser.

## Example

| Rank | Voter 1 | Voter 2 | Voter 3 |
|---|---|---|---|
| 1 | YN | NY | NN |
| 2 | YY | YY | YY |
| 3 | NY | YN | NY |
| 4 | NN | NN | YN |

$YY$ is the Condorcet winner among the alternatives $\{YY, YN, NY, NN\}$. The referendum produces the Condorcet loser $NN$ as the outcome.

If some voters have nonseparable preferences for the outcomes of votes across multiple proposals, then the social choice may be Pareto-dominated by all other outcomes.

## Example

A referendum produces an outcome ranked last by every vote (Pareto-dominated since any other alternative represents a Pareto improvement over $YYY$).

| Rank | Voter 1 | Voter 2 | Voter 3 |
|---|---|---|---|
| 1 | YYN | YNY | NYY |
| 2 | YNY | NYY | YYN |
| 3 | NYY | YYN | YNY |
| 4 | NNY | NYN | YNN |
| 5 | YNN | YNN | NYN |
| 6 | NYN | NNY | NNY |
| 7 | NNN | NNN | NNN |
| 8 | YYY | YYY | YYY |

## Positive results under separable preferences

If all voters have separable preferences, then the social choice will not be Pareto-dominated by all other outcomes. To illustrate the result, suppose $YYY$ is Pareto dominated by all other outcomes. Say, every voter prefers $NNY$ to $YYY$. Since all voters have separable preferences, say in the first proposal, $N$ is preferred to $Y$ for any voter. Therefore, $N$ emerges as the winning outcome on the first proposal. As a result, $YYY$ cannot be a winner.

Also a Condorcet winning outcome will be selected wherever one exists. Suppose $YYN$ beats $NNY$, $NYY$, $NNN$, etc in all pairwise contests. This means more voters prefer $YYN$ than $NNY$ (say). On the first proposal, since all voters have separable preferences, the same pattern of choosing $Y$ and $N$ among all voters prevails. Since $YYN$ beats $NNY$, we deduce that $Y$ emerges as the winning outcome in the first proposal. The same argument can be applied to other proposals as well.

## Sequential voting

One way to avoid the problems of nonseparable preferences may be to vote on proposals sequentially. Sequential voting provides information to voters before voting on subsequent proposals.

*Positive result*: Sequential voting prevents the selection of Condorcet losers.

Referring to the earlier example where the Condorcet loser $YYY$ is chosen, once $YY$ in the first two proposals are revealed, all voters would vote for $YYN$ since $YYY$ is the least preferred choice for all voters.

## Measure of desirability

Consider the example

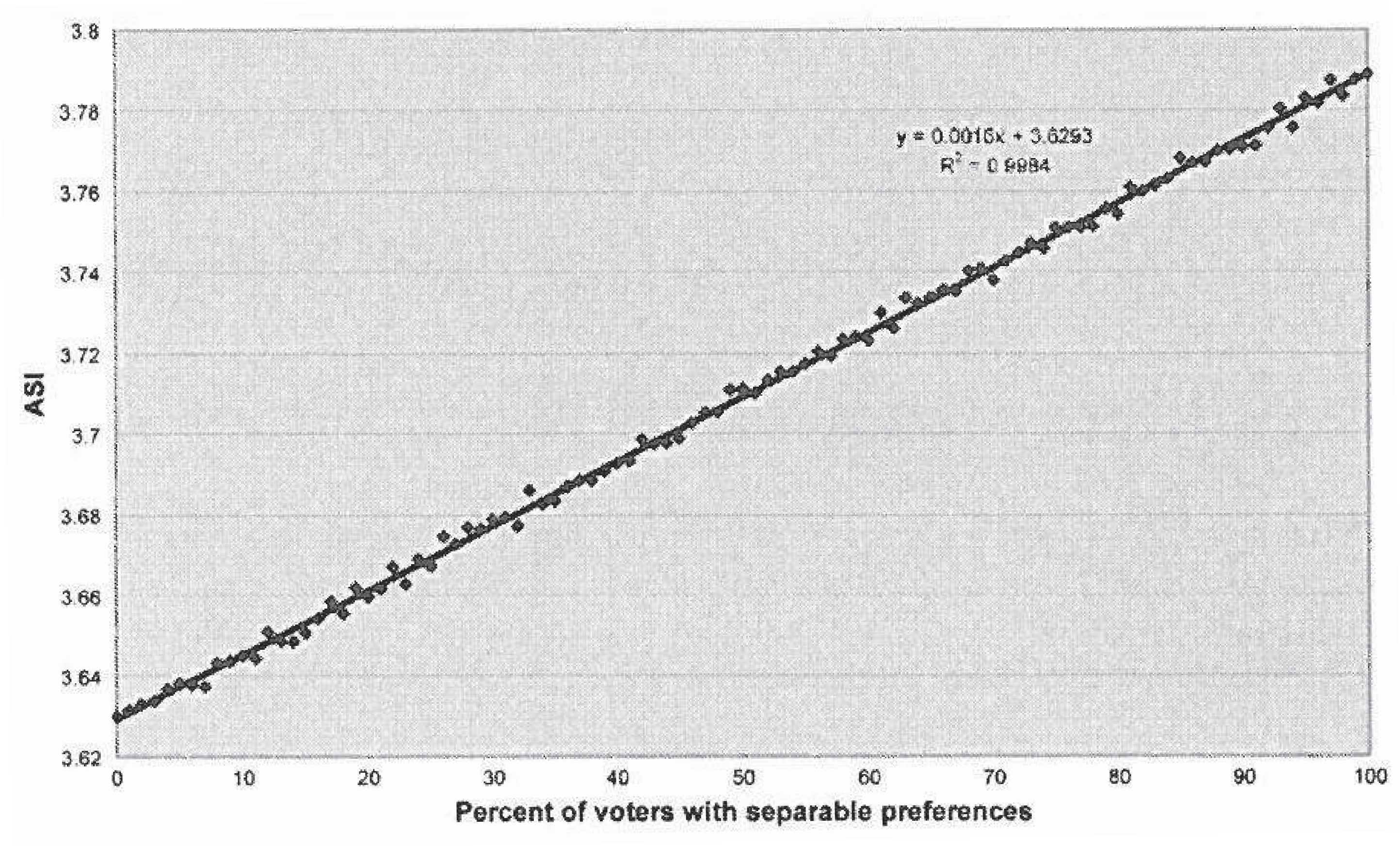
$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix},$$

the outcome is $(1,0)$.

Assign score of 2 points to $(1,0)$ for voter 1 since $(1,0)$ is the second best choice, score of 3 points for voter 2 and score of 1 point for voter 3. The average score is $(2+3+1)/3 = 2$ points. The other outcome $(1,1)$ has an average score of $(3+2+2)/3 = 7/3 > 2$.

The average score refers to the level of desirability of the outcome. Higher score means better desirability of the outcome. When the Pareto-dominated outcome is resulted, the average score is zero since the point assigned to every voter (bottom of his choice) is zero.

Through simulation with different number of proposals and voters, the following plot shows a strong linear relationship between the percentage of voters in an election with separable preferences and the expected desirability score for the election.



*References*

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"A problem with referendums," by Dean Lacy and Emerson Niou.

## 3.7 Majority rule with single-peaked preferences

We try to explain why politicians on both ends of the spectrum tend to gravitate towards the philosophical center. The purpose is to appeal to as many voters as possible.

For alternatives corresponding to linear orderings like a political spectrum from conservative to liberal, we assume that each voter's preferences "fall away" consistently on both sides of their most favorite alternative (when the most favorite alternative is not at the far left or far right).

Let the $k$ alternatives be named $x_1, x_2, \ldots, x_k$, and that voters all perceive them as being arranged in this order (like candidates on a political spectrum).

*Single-peaked preferences*

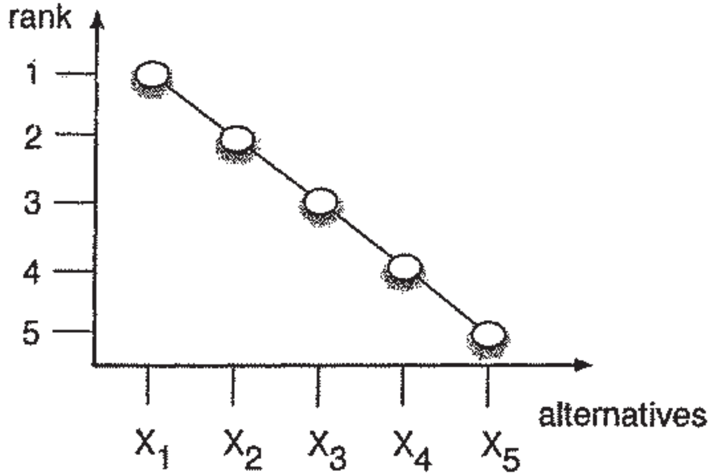
A voter is said to have single-peaked preferences if there are no alternative $X_s$ for which both neighboring alternatives $X_{s-1}$ and $X_{s+1}$ are ranked above $X_s$. We call complete (ties and no preference are excluded) and transitive single-peaked preferences as single-peaked rankings.

Each voter $i$ has a top-ranked alternative $X_t$, and her preferences fall off on both sides of $X_t$:

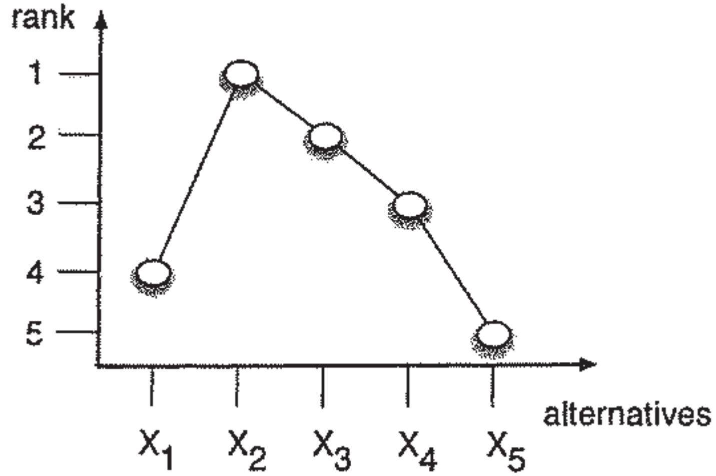
$$X_t \succ_i X_{t+1} \succ_i X_{t+2} \succ_i \cdots$$

and

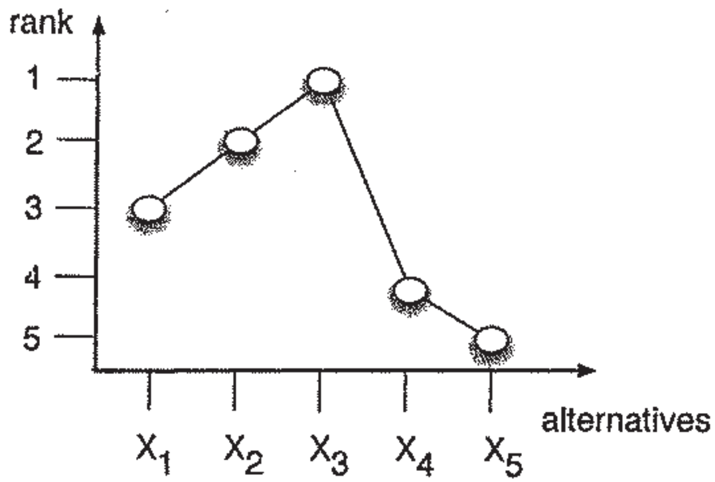
$$X_t \succ_i X_{t-1} \succ_i X_{t-2} \succ_i \cdots$$




(a) *Voter 1's ranking.*




(b) *Voter 2's ranking.*




(c) *Voter 3's ranking.*

With single-peaked preferences, there are no alternative for which both neighboring alternatives are ranked above it.

*Median individual favorite*

We consider the top-ranked alternative for each voter and sort the set of individual favorites from left to right, along the linear order. For example, if the individual favorites of 7 voters were $X_1, X_1, X_2, X_2, X_3, X_4, X_5$, then the median individual favorite would be $X_2$ in the $4^{\text{th}}$ position.

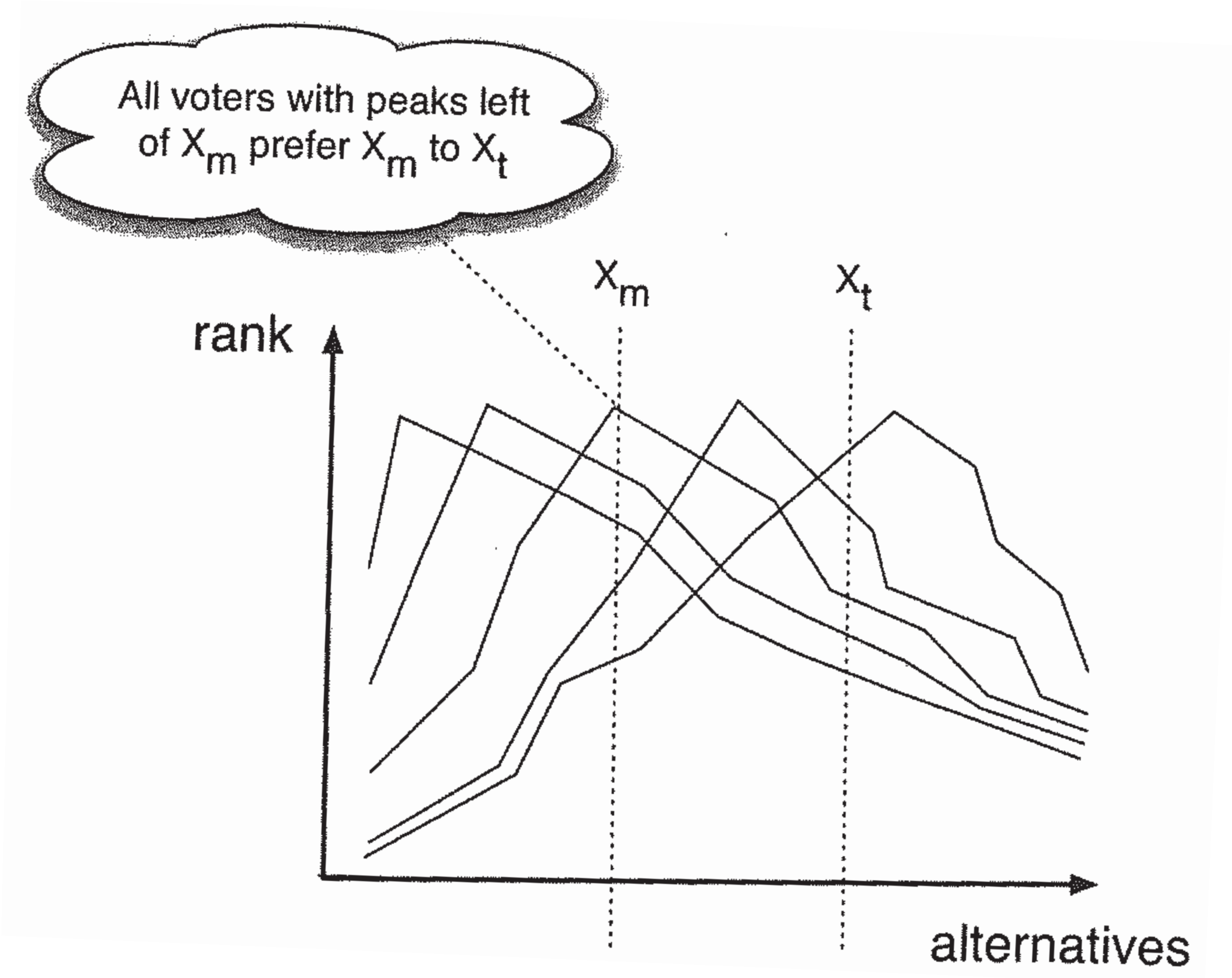
Another daily life example is related to the best location of several grocery stores (alternatives) along a long and straight main road with residential apartments. Residents (voters) choose the store that is closest to their apartments. The store that enjoys being the median individual favorites would benefit best in attracting customers.

## Median Voter Theorem

For simplicity of analysis, we assume the number of voters to be odd so that we do not worry about the possibility of ties. With single-peaked rankings, the median individual favorite defeats every other alternative in a pairwise majority rule.

*Proof*

Let $X_m$ be the median individual favorite, and let $X_t$ be any other alternative. Let's suppose that $X_t$ lies to the right of $X_m$ – that is, $t > m$. The case in which it lies to the left has a completely symmetric argument. We also order the voters in the sorted order of their individual favorites.




The proof that the median individual favorite $X_m$ defeats every other alternative $X_t$ in a pairwise majority vote: if $X_t$ is to the right of $X_m$, then $X_m$ is preferred by all voters whose peak is on $X_m$ or to its left. The symmetric argument applies when $X_t$ is to the left of $X_m$.

The number of voters $k$ is odd, and we know that – since it is the median – $X_m$ is in position $(k+1)/2$ of the sorted list of individual favorites. This means that for everyone in the first $(k+1)/2$ positions, $X_m$ is either their favorite, or their favorite lies to the left of $X_m$. For each voter in this group, $X_m$ and $X_t$ are both on the right-hand "down-slope" of this voter's preferences, but $X_m$ is closer to the peak than $X_t$ is, so $X_m$ is preferred to $X_t$. It follows that everyone in the first $(k+1)/2$ positions prefers $X_m$ to $X_t$. But this is a strict majority of the voters (more than 50%), and so $X_m$ defeats $X_t$ in a pairwise majority vote.

The median individual favorite $X_m$ can always count on gathering a majority of support against any other alternative $X_t$, because for more than half the voters, $X_m$ lies between $X_t$ and each of their respective favorites.

*Majority rule to produce group ranking*

We show how the majority rule among all pairs produces a complete and transitive group ranking: we simply build up the group ranking by identifying group favorites one at a time. That is, we start by finding the median individual favorite and placing it at the top of the group ranking. This is safe to do since the Median Voter Theorem guarantees that it defeats all other alternatives that will come later in the list.

Next, we remove this alternative from each individual ranking. Notice that when we do this, the rankings all remain single-peaked. Essentially, we have simply "decapitated" the peak from each ranking, and the second item in each voter's ranking becomes their new peak. We face with single-peaked rankings on a set of alternatives that is one smaller. So we find the median individual favorite on the remaining alternatives, place it second in the group ranking, and continue in this way until we exhaust the finite set of alternatives.

*Example*

Consider the earlier pictorial example with 3 votes and 5 preferences. We would identify $X_2$ as the median individual favorite, and we would place it first in the group ranking. Once we remove this alternative, we have three single-peaked rankings on the alternatives $X_1$, $X_3$, $X_4$, and $X_5$. The individual favorites in this reduced set are $X_1$, $X_3$ and $X_3$, so $X_3$ is the new median individual favorite, and we place it second in the group ranking. Proceeding in this way, we end up with the group ranking

$$X_2 \succ X_3 \succ X_1 \succ X_4 \succ X_5.$$

Since voter 2 was the original "median voter" in the sense of having the original median individual favorite, the start of the group ranking necessarily agrees with the start of voter 2's individual ranking: they both place $X_2$ first.

However, the full group ranking does not coincide with voter 2's full individual ranking: for example, voters 1 and 3 both prefer $X_1$ to $X_4$, even though voter 2 does not, and the group ranking reflects this.

## 3.8 Cumulative voting

- In *single-winner* plurality voting, each voter is allowed to vote for only one candidate; and the winner of the election is whichever candidate represents a plurality of voters. Voters can elect just one representative from that district, even if another candidate won a substantial percentage of votes.

- In *multi-member* constituencies, with 8,000 voters and 5 to be elected, a coalition of 4001 members can elect 5 candidates of its choice by giving each of the 5 candidates 4,001 votes.

Cumulative voting is a multiple-winner voting system intended to promote proportional representation while also being simple to understand.

| *You may offer up to 3 votes* | | | |
|---|---|---|---|
| 1 | 2 | 3 | |
| ○ | ○ | ○ | Chan |
| ● | ● | ○ | Lee |
| ○ | ○ | ○ | Cheung |
| ○ | ○ | ● | Wong |
| ○ | ○ | ○ | Ho |

2 votes for Lee and 1 vote for Wong

Voters can ‘plump’ their votes, conferring all $n$ votes on a single candidate or distributing their $n$ votes as they please. In cumulative voting, each voter is allotted the same number of votes, while allowing for expression of intensity of candidate preference. For example, 15 voters with 3 votes can cast all votes (almost) equally to Lee and Wong, where 8 voters put 2 votes on Lee and 1 vote on Wong and 7 voters put 1 vote on Lee and 2 votes on Wong.

- Since 1980, Illinois tried "redrawing political districts" in order to guarantee election of political minorities. This takes power away from the people and gives it to politicians and to the courts.
  - There is nothing in the Illinois Constitution or the US Constitution that requires single-member districts.
  - Proportional voting is the system in most European countries. If 7% of the voters support the Green Party, the Green Party gets around 7% of the seats.

- Minority group voters do not have to be made into majorities of voters in deliberately drawn districts in order to elect a candidate. The need to manipulate *district lines* is largely, if not completely, eliminated.

- Cumulative voting is popular in electing the members in the Board of Directors in listed companies. Suppose the stockholder of one share of stock is entitled to cast $n$ votes, where $n$ is the number of seats. Let $n$ be 10, can a major stockholder (visualized as a coalition of voters) of 27% secure 3 seats out of 10? If not, how about 28%?

*Assuring a certain representation*

- Voting literature frequently mentions "thresholds", which designate a fraction of population for which a cohesive group whose population fraction is above the threshold can assure itself a certain level of representation under a method of voting.

- Let $P$ be the total number of voters (population) and $n$ the number of seats to be elected, $P > n$. We want the fraction of population $x/P$ over which the group can elect $k$ of $n$, if the group desires to do so and if they vote strategically. Every voter has $n$ votes.

- In the above example, what is the ratio $x/P$ such that the major stockholder can secure 3 out of 10 seats? Here, $k = 3$ and $n = 10$.

## Theorem

Under cumulative voting, a coalition of $x$ voters can guarantee the election of $k$ candidates if and only if

$$\frac{x}{k} > \frac{P-x}{n-k+1} \quad \Leftrightarrow \quad \frac{x}{P} > \frac{k}{n+1}.$$

*Example*

Let $P = 81$ and $n = 8$. With coalition of size $x = 46$, we have $\frac{46}{81} \approx 0.568$. For $n = 8$, $\frac{1}{n+1} = \frac{1}{9} \approx 0.1111$. Since $0.568 > 5 \times 0.1111$, the election of 5 candidates is guaranteed.

- The commonly cited "threshold of exclusion" for cumulative voting is given by $\frac{1}{n+1}$, above which a minority can assure itself representation. This is just the special case with $k = 1$.

*Proof*

We assume $x$ and $P$ to be very large (like hundred thousand) while $n$ and $k$ are small integers.

(i) $\dfrac{x}{k} > \dfrac{P-x}{n-k+1} \Rightarrow$ election of $k$ candidates.

A coalition of $x$ voters can give each of $k$ candidates $\dfrac{xn}{k}$ votes. For convenience, we assume $x$ to be very large so that $\dfrac{xn}{k}$ is sufficiently close to some integer value. We avoid the nuisance of considering the floor value of $\dfrac{xn}{k}$.

The least popular of $n-k+1$ other candidates could receive no more than $\dfrac{(P-x)n}{n-k+1}$ votes. Thus the coalition of $x$ voters can guarantee the election of $k$ candidates if

$$\frac{xn}{k} > \frac{(P-x)n}{n-k+1} \;\Leftrightarrow\; \frac{x}{k} > \frac{P-x}{n-k+1} \;\Leftrightarrow\; \frac{x}{P} > \frac{k}{n+1}.$$

(ii) election of $k$ candidates $\Rightarrow \dfrac{x}{k} > \dfrac{P-x}{n-k+1}$

By contradiction, suppose $\dfrac{x}{k} \leq \dfrac{P-x}{n-k+1}$, then the other $P-x$ voters can block the election of the $k^{\text{th}}$ candidate of coalition $C$. This is because $\dfrac{(P-x)n}{n-k+1}$ votes is more than $\dfrac{xn}{k}$ votes.

## 3.9 Approval voting

Approval voting (AV) is a voting procedure in which voters can vote for, or approve of, as many candidates as they like in multicandidate elections (those with more than two candidates). Each candidate approved of in a ballot receives one vote, and the candidate with the most votes wins.

- AV is not currently used in any public elections.
- In the late 1980s, AV was used in Eastern Europe and the Soviet Union in the form of "disapproval voting" – permitted to cross off names on ballots (差额选举).

Beginning in 1987, several scientific and engineering societies adopted AV, including the

- Mathematical Association of America (MAA), with about $32,000$ members;
- American Mathematical Society (AMS), with about $30,000$ members;
- Institute for Operations Research and Management Sciences (INFORMS), with about $12,000$ members;
- American Statistical Association (ASA), with about $15,000$ members;
- Institute of Electrical and Electronics Engineers (IEEE), with about $377,000$ members.

**1970 New York Senatorial race**

Ottinger (Democrat), Goodell (Republican-Liberal), Buckley (Conservative)

- Buckley (39%), Ottinger (37%), Goodell (24%)

- Suppose many if not most supporters of Ottinger and Goodell perceived to be liberal, those who voted for Ottinger or Goodell would vote for the other. This would have led to a victory for Ottinger and possibly a 3th-place finish for Buckley.

- Ottinger enjoyed approval by a larger proportion of the electorate and served with a wider mandate.

## Positive aspects

1. *More flexible options*

   - At least, like plurality voting, vote for a single favorite.
   - Vote for all candidates that considered acceptable.
   - If a voter's most preferred choice has little chance of winning, voter can vote for both a first choice and a more viable candidate without worrying about wasting his or her vote on the less popular candidate. Voters would be more likely to come to vote and put their sincere votes.

2. *Likely to elect the Condorcet winner*
   The candidate with the greatest overall support will generally win. Under plurality voting, the Condorcet candidates often lose since they split the vote with one or more other centrist candidates.

3. *Reduce negative campaigning*
   Candidates will have an incentive to broaden their appeals by reaching out for approval to voters who might have a different first choice.

4. *Easy to implement*, unlike more complicated ranking systems.

*Comparison with ranking systems, like Borda count voting*

Borda voting fosters "insincere voting". A voter moves a second choice down to the last place to minimize that candidate's threat to his top choice, so it is highly vulnerable to "irrelevant candidates".

**Actual case studies**

1. *The Institute of Management Science* (TIMS)

    - The use of AV by TIMS in 1987 was preceded by an experiment in which members were sent a nonbinding AV ballot, along with the regular PV ballot, in the 1985 elections.
    - 85% of the members who voted in these elections returned the AV ballot.

*PV and AV vote totals in 1985 TIMS election*

| Candidates | Official PV | Actual AV | Extrapolated AV |
|---|---|---|---|
| A | 166 | 417 | 486 |
| B | 827 | 1,038 | 1,224 |
| C | 835 | 908 | 1,054 |
| Total | 1,828 | 2,363 | 2,764 |
| No. of voters | 1,828 | 1,567 | 1,828 |

- B loses to C by 8 votes under the official PV.
- B wins C by 170 votes under the extrapolated AV.
- Extrapolate the voting pattern of the AV non-respondents (15% of voters) based on sampling.

*C wins more plurality votes while B wins more approval votes*

Approval voting picks a clear winner on the *basis of second choices*. B has a broader acceptance in the electorate than C.

- Eliciting more information from the voters leads to the election of the candidate with the *widest support* (desirable for professional societies).

*Background*

- Management scientists should "*practice what we preach*". Make an informed judgment about the applicability of AV.

- Before undertaking the experiment, inquiries were made of the candidates to ask their *permission to participate* in it. Because of its research potential, all agreed, prefiguring AV's eventual adoption in 1987.

2. *1987 MAA election*

*AV vote totals in 1987 MAA election*

| Candidates | 1-Voters | 2-Voters | 3-Voters | 4-Voters | Total |
|---|---|---|---|---|---|
| A | 848 | 276 | 122 | 21 | 1,267 |
| B | 618 | 275 | 127 | 32 | 1,052 |
| C | 652 | 264 | 134 | 34 | 1,084 |
| D | 660 | 273 | 118 | 31 | 1,082 |
| E | 303 | 132 | 87 | 30 | 552 |
| Total | 3,081 | 1,220 | 588 | 148 | 5,037 |
| No. of voters | 3,081 | 610 | 196 | 37 | 3,924 |

- A wins among narrow (those who cast few votes) but not among wide voters (those who cast many votes).

- A's victory is largely attributable to the substantial margin received from 1-voters (strongest preference intensity), not from the presumably more lukewarm support received from multiple voters.

- Winner (A) received 28% more votes from 1-voters than the 1-voters' runner-up (D) did, just edged out B among 2-voters, but lost to several candidates among 3-voters and among 4-voters.

- A candidate is said to be *AV-dominant* if he wins among all classes of voters - those who cast few votes (narrow voters) and those who cast many votes (wide voters). In this example, the winner A is not AV-dominant.

- In 12 of the 16 multicandidate AV elections in 4 societies, the winners were AV-dominant.

- In the remaining 4 elections in which there was not an AV-dominant winner, the winner won by virtue of receiving greater support among narrow voters than among wide voters (like 1987 MAA election) – more intense and heartfelt.

- A candidate may lose among every possible class of voters – AV-dominated – and still be the AV winner.
  For example, C could emerge as the AV winner if A did badly among wide voters and B did badly among narrow voters.

**How should the rational voter select the subset of candidates from $k$ candidates, $c_1, \ldots, c_k$, to vote for under approval voting?**

- Each voter $v$ defines a real-valued function $f$ on the set of candidates, such that the quantity $f(c_i) - f(c_j)$ is intended to represent the utility or value to voter $v$ of having candidate $c_i$ elected instead of candidate $c_j$.

- Let $S$ denote the set of candidates voted for by $v$, Define the total utility for $v$ by

$$V(S) = \sum [f(c_i) - f(c_j)],$$

  where the summation is over all $i \in S$ and $j \notin S$. We expect $f(c_i) - f(c_j) > 0$ since $c_i$ is favored over $c_j$ by $v$.

Suppose voter $v$ has decided to vote for the candidates in set $S$ and wishes to know if he could improve his total utility by also voting for an additional candidate $c$. We compare the total utility with and without $c$.

Note that $c$ is not originally in $S$ and $f(c_i) - f(c_j)$ without $c$ would be cancelled in the difference of the two total utilities. The impact on the total utility of including $c$ in the set of approved candidates is summarized by

$$\begin{aligned} V(S \cup \{c\}) - V(S) &= \sum_{c_j \notin S} [f(c) - f(c_j)] - \sum_{c'_j \in S} [f(c'_j) - f(c)] \\ &= kf(c) - \sum_{j=1}^{k} f(c_j). \end{aligned}$$

Here, $k$ is the total number of candidates. He can improve total utility by voting for $c$ precisely if

$$f(c) > \frac{1}{k} \sum_{j=1}^{k} f(c_j).$$

- A rational voter should vote for all candidates whom he rates above the average of those running.
  Suppose he rates the 4 candidates with values 10, 8, 7, and 0, he should vote for the top 3 since all three rate above the average (6.25).

## 3.10 Electing committees with diversity requirement

Electing a committee introduces constraints beyond excellence, such as ensuring a balance of gender, tenure, talent, and other characteristics.

A committee of three to be formed for searching the next president in a woman college.

*Goal*: Gender diversification should be respected in the committee.

| **Social sciences** | **Natural sciences** | **Humanities** |
|---|---|---|
| Ann | Carole | Ellen |
| Bob | David | Fred |

Voters have attempted to make their choices of committees that observe *gender diversity*. However, the voters have a preference for the overall composition of the committee that cannot be decomposed into preferences on the individual candidates.

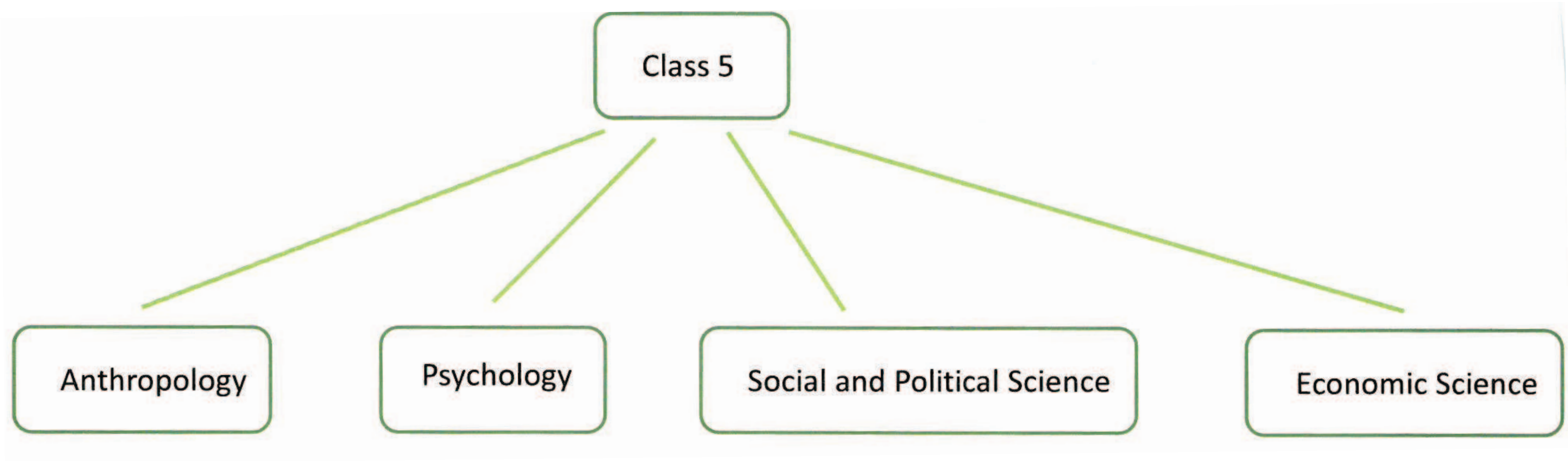
| **Voter 1** | **Voter 2** | **Voter 3** |
|---|---|---|
| Ann, David, Fred | Bob, Carole, Fred | Bob, David, Ellen |

Bob gets 2 votes, David gets 2 votes and Fred gets 2 votes.

Outcome is (Bob, David, Fred), loss of gender diversification in the final outcome.

## National Academy of Sciences

Class 5 consists of 4 sections.

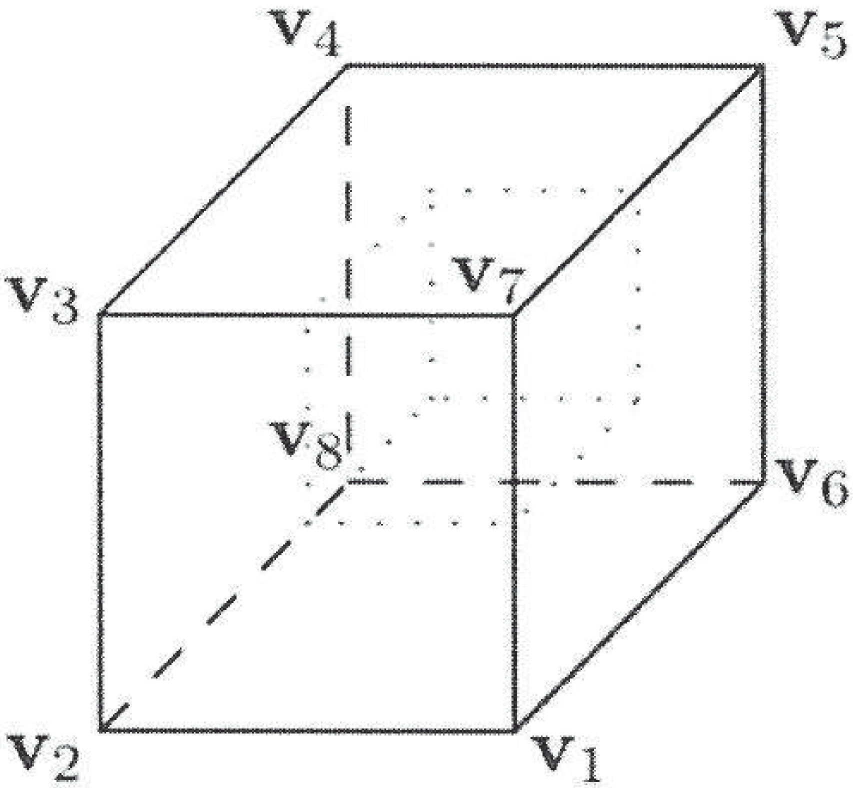



Elect 9 members from around 20 candidates

Besides the consideration of the relative merits of the candidates, there is the need for a balanced representation over the 4 sections. The ideal situation is to elect at least 2 members from each section.

*Strategy*: Vote for the top 2 or 3 candidates from his section, 3 or 4 other top candidates (who represent excellence and will be elected anyway), and then strategically "protect" his section candidates by "wasting" the remaining votes by voting for candidates who probably would not win. Terrible outcome: A weak candidate that fails in excellence may win.

| Vertex | Committee | Vertex | Committee |
|---|---|---|---|
| $\boldsymbol{v}_1$ | $(1, 1, -1)$ or $(m, m, w)$ | $\boldsymbol{v}_4$ | $(-1, -1, 1)$ or $(w, w, m)$ |
| $\boldsymbol{v}_2$ | $(1, -1, -1)$ or $(m, w, w)$ | $\boldsymbol{v}_5$ | $(-1, 1, 1)$ or $(w, m, m)$ |
| $\boldsymbol{v}_3$ | $(1, -1, 1)$ or $(m, w, m)$ | $\boldsymbol{v}_6$ | $(-1, 1, -1)$ or $(w, m, w)$ |
| $\boldsymbol{v}_7$ | $(1, 1, 1)$ or $(m, m, m)$ | $\boldsymbol{v}_8$ | $(-1, -1, -1)$ or $(w, w, w)$ |



If $p_j$ represents the proportion of all voters who prefer committee $j$, $j = 1, \ldots, 6$, then a profile becomes $\boldsymbol{p} = (p_1, \ldots, p_6)$ where $\sum_{j=1}^{6} p_j = 1$. The election outcome $\boldsymbol{V}$ is

$$\boldsymbol{V} = \sum_{j=1}^{6} p_j \boldsymbol{v}_j.$$

| **Voter 1** | **Voter 2** | **Voter 3** |
|---|---|---|
| Ann, David, Fred | Bob, Carole, Fred | Bob, David, Ellen |
| $\boldsymbol{v}_5(-1, 1, 1)$ | $\boldsymbol{v}_3(1, -1, 1)$ | $\boldsymbol{v}_1(1, 1, -1)$ |

With the above preferences, the three listed committees are $\boldsymbol{v}_1$, $\boldsymbol{v}_3$ and $\boldsymbol{v}_5$, corresponding to the preferences of Voters 3, 2, and 1, respectively. We have

$$\boldsymbol{p} = \left(\frac{1}{3}, 0, \frac{1}{3}, 0, \frac{1}{3}, 0\right)$$

and by using the above formula:

$$\boldsymbol{V} = \frac{1}{3}(1, 1, -1) + \frac{1}{3}(1, -1, 1) + \frac{1}{3}(-1, 1, 1) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

Since the components in $\boldsymbol{V}$ are all positive, a committee consisting of all men is elected.

**Theorem**

In electing a three-person committee from among candidates who are slotted in three divisions, suppose each division has two candidates representing two different categories (e.g., tenured and untenured faculty, or men and women, or Nordics and non-Nordics). To reflect a universal intent shared by all voters to elect a committee with representation coming from each of these two categories, an admissible ballot must have at least one candidate from each category (so, two candidates are from one category and the "diversity" candidate is from the second category).

The diversity objective always can be achieved by assigning weights if and only if the weight $w_1$ assigned to the diversity candidate equals the sum of the weights $w_2$ and $w_3$ assigned to the other two candidates.

For example, $(\overset{w_1}{\text{Ann}},\ \overset{w_2}{\text{David}},\ \overset{w_3}{\text{Fred}})$, Ann is the diversity candidate.

Weights assigned are $w_1$, $w_2$ and $w_3$.

Diversity objective in the outcome can be achieved if and only if

$$w_1 = w_2 + w_3.$$

Say, $w_1 = 2$, $w_2 = w_3 = 1$ or $w_1 = 3$, $w_2 = 2$ and $w_1 = 1$.

(1,1,1) method produce (u,u,u) ☹

| Voter \ Division | A | B | C |
|---|---|---|---|
| 1 | t | u | u |
| 2 | t | u | u |
| 3 | t | u | u |
| 4 | u | t | u |
| 5 | u | t | u |
| 6 | u | t | u |
| 7 | u | u | t |
| 8 | u | u | t |
| 9 | u | u | t |
| 10 | u | u | t |
| 11 | u | u | t |
| 12 | t | t | u |
| total | 8u, 4t | 8u, 4t | 7u, 5t |

(2,1,1) method produce (u,u,t) ☺

| Voter \ Division | A | B | C |
|---|---|---|---|
| 1 | t(2x) | u | u |
| 2 | t(2x) | u | u |
| 3 | t(2x) | u | u |
| 4 | u | t(2x) | u |
| 5 | u | t(2x) | u |
| 6 | u | t(2x) | u |
| 7 | u | u | t(2x) |
| 8 | u | u | t(2x) |
| 9 | u | u | t(2x) |
| 10 | u | u | t(2x) |
| 11 | u | u | t(2x) |
| 12 | t | t | u(2x) |
| total | 8u, 7t | 8u, 7t | 8u, 10t |

$$x = \frac{\text{no of votes on } u \text{ in } A - \text{no of votes on } t \text{ in } A}{\text{total no of votes in } A} = \frac{8-4}{12} > 0 \ (u \text{ wins}),$$
$$y = \frac{\text{no of votes on } u \text{ in } B - \text{no of votes on } t \text{ in } B}{\text{total no of votes in } B} = \frac{8-4}{12} > 0 \ (u \text{ wins}),$$
$$z = \frac{\text{no of votes on } u \text{ in } C - \text{no of votes on } t \text{ in } C}{\text{total no of votes in } C} = \frac{7-5}{12} > 0 \ (u \text{ wins}),$$

*Under modified weights*

$$\widetilde{x} = \frac{8-7}{15} > 0 \quad (u \text{ wins}),$$
$$\widetilde{y} = \frac{8-7}{15} > 0 \quad (u \text{ wins}),$$
$$\widetilde{z} = \frac{8-10}{18} < 0 \quad (t \text{ wins}).$$

**Illustration with $(\lambda, 1, 1)$ rule for the 3-member committee**

| | equal weights | modified weights |
|---|---|---|
| $\boldsymbol{v}_1$ | $(1, 1, -1)$ | $(1, 1, -\lambda)$ |
| $\boldsymbol{v}_2$ | $(1, -1, -1)$ | $(\lambda, -1, -1)$ |
| $\boldsymbol{v}_3$ | $(1, -1, 1)$ | $(1, -\lambda, 1)$ |
| $\boldsymbol{v}_4$ | $(-1, -1, 1)$ | $(-1, -1, \lambda)$ |
| $\boldsymbol{v}_5$ | $(-1, 1, 1)$ | $(-\lambda, 1, 1)$ |
| $\boldsymbol{v}_6$ | $(-1, 1, -1)$ | $(-1, \lambda, -1)$ |

Sum of components of each vector under modified weights is always equal to zero when $\lambda = 2$.

(i) $1 \leq \lambda < 2$, take $\boldsymbol{p}_3 = \left(\frac{1}{3}, 0, \frac{1}{3}, 0, \frac{1}{3}, 0\right)$,

outcome $= \frac{1}{3}(2-\lambda,\ 2-\lambda,\ 2-\lambda)$ with all components being positive (winners are all men).

Take $\boldsymbol{p}_2 = \left(0, \frac{1}{3}, 0, \frac{1}{3}, 0, \frac{1}{3}\right)$

outcome $= \frac{1}{3}(\lambda-2,\ \lambda-2,\ \lambda-2)$ with all components being negative (winners are all women).

(ii) $\lambda > 2$, same set of conclusions can be derived.

Gender diversity of the outcome is not guaranteed when $\lambda \neq 2$.

When $\lambda = 2$, for any $\boldsymbol{p} = (p_1, p_2, p_3, \ldots, p_6)$

$$\text{outcome} = p_1(1, 1, -\lambda) + p_2(\lambda, -1, -1) + \cdots + p_6(-1, \lambda, -1)$$

Sum of components in the outcome vector is always zero since sum of components of each vector under modified weights is zero.

Therefore, all components with the same sign in the outcome is ruled out; that is, $(m, m, m)$ or $(w, w, w)$ is ruled out. Gender diversity is guaranteed in the outcome.

## Extension to 5-member committee

There are $2^5 - 2 = 30$ possible choices of committees.

| | *equal weights* | *modified weights* |
|---|---|---|
| $\boldsymbol{v}_1$ | $(1, 1, 1, 1, -1)$ | $(1, 1, 1, 1, \mathbf{-4})$ |
| $\boldsymbol{v}_2$ | $(1, 1, 1, -1, -1)$ | $(2, 2, 2, \mathbf{-3}, \mathbf{-3})$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $\boldsymbol{v}_{29}$ | $(-1, -1, -1, 1, 1)$ | $(-2, -2, -2, \mathbf{3}, \mathbf{3})$ |
| $\boldsymbol{v}_{30}$ | $(-1, -1, -1, -1, 1)$ | $(-1, -1, -1, -1, \mathbf{4})$ |

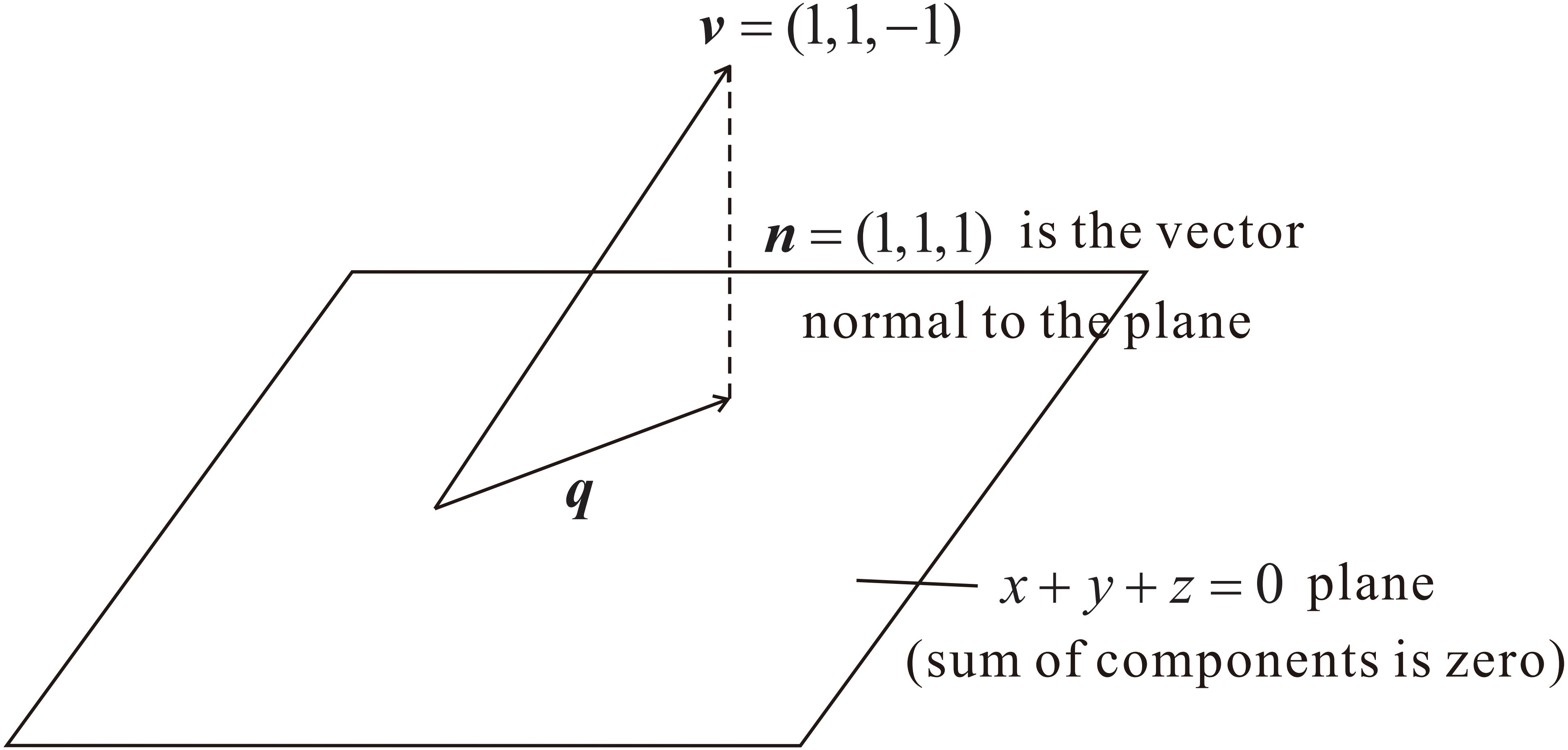
Sum of components of each vector under modified weights is always equal to zero.

**Projection vector onto the plane: $x + y + z = 0$ (minimal deviation)**

Given $\boldsymbol{v} = (1, 1, -1)$, we find its projection $\boldsymbol{q}$ onto the plane: $x+y+z = 0$ so that the projection vector has sum of components to be zero.

The normal to the plane is $\boldsymbol{n} = (1, 1, 1)$.

Note that $\boldsymbol{v} - \boldsymbol{q}$ must be parallel to the normal.



$\boldsymbol{v} = (1,1,-1)$
$\boldsymbol{n} = (1,1,1)$ is the vector
normal to the plane
$\boldsymbol{q}$
$x + y + z = 0$ plane
(sum of components is zero)

We write $\boldsymbol{v} = \alpha\boldsymbol{n}+\boldsymbol{q}$ and determine $\alpha$ such that the sum of components in $\boldsymbol{q}$ is zero. Note that $\boldsymbol{v}-\alpha\boldsymbol{n} = (1-\alpha, 1-\alpha, -1-\alpha)$ with sum of components $= 1 - 3\alpha$.

We set the sum to be zero so that $\alpha = \frac{1}{3}$. We then obtain $\boldsymbol{q} = \left(\frac{2}{3}, \frac{2}{3}, \frac{-4}{3}\right)$. Upon normalization, we obtain the modified vector $(1, 1, -2)$ or the $\lambda = 2$ rule. We assign double weight to the diversity candidate.

The other choice $(1, 19, -20)$ leads to significant deviation from $\boldsymbol{v}$ when compared with the projection vector $\boldsymbol{q}$.