MATH6380B - Advanced Topics in Derivative Pricing Models

Homework One

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1. Consider the function
$$f(S,\tau) = \left(\frac{S}{B}\right)^{\lambda} c_E\left(\frac{B^2}{S},\tau\right),$$
where $c_E(S,\tau)$ is the price of a vanilla European call option, λ is a constant parameter. Show that $f(S,\tau)$ satisfies the Black-Scholes equation
$$\frac{\partial f}{\partial \tau} = \frac{\sigma^2}{2}S^2\frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} - rf$$
when λ is chosen to be $-\dfrac{2r}{\sigma^2}+1$.

Hint: Substitution of $f(S,\tau)$ into the Black-Scholes equation gives
$$\begin{aligned}
&\frac{\partial f}{\partial \tau} - \left[\frac{\sigma^2}{2}S^2\frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} - rf\right] \\
&= \left(\frac{S}{B}\right)^{\lambda}\left[\frac{\partial c_E}{\partial \tau} - \frac{\sigma^2}{2}\xi^2\frac{\partial^2 c_E}{\partial \xi^2}\right. \\
&\quad \left. + (\lambda-1)\sigma^2\xi\frac{\partial c_E}{\partial \xi} - \lambda(\lambda-1)\frac{\sigma^2}{2}c_E - r\lambda c_E + r\xi\frac{\partial c_E}{\partial \xi} + rc_E\right],
\end{aligned}$$
where $c_E = c_E(\xi,\tau), \xi = \dfrac{B^2}{S}$.

2. Let the price process S_t be governed by
$$\frac{dS_t}{S_t} = r\ dt + \sigma\ dW_t^Q,$$
where W_t^Q is a Brownian motion under the equivalent martingale measure Q. We write the arbitrage-free time-t value of a "g-claim" as
$$\pi^g(t) = e^{-r(T-t)}E_t^Q[g(S_T)] = e^{-r(T-t)}f(S_t,t).$$
Let $p = 1 - \dfrac{2r}{\sigma^2}$ and $H > 0$ be a constant. We define a new function $\hat{g}$ by
$$\hat{g}(x) = \left(\frac{x}{H}\right)^p g\left(\frac{H^2}{x}\right).$$
We call $\hat{g}$ to be g's reflected claim. Show that the arbitrage-free time-t value of this $\hat{g}$-claim is given by
$$\pi^{\hat{g}}(t) = e^{-r(T-t)}\left(\frac{S_t}{H}\right)^p f\left(\frac{H^2}{S_t},t\right).$$
Use the above result to deduce the price of an up-and-out put option with upstream barrier H and strike price X. Distinguish the two cases: (i) $H > X$, and (ii) $H \le X$.

Hint: Define the process

$$Z_t = \left(\frac{S_t}{H}\right)^p,$$

so that $dZ_t = p\sigma Z_t \, dW_t^Q$. The Radon-Nikodym derivative

$$\frac{dQ^Z}{dQ} = \frac{Z(T)}{Z(0)}$$

defines the probability $Q^Z \sim Q$. Show that

$$\pi^{\hat{g}}(t) = e^{-r(T-t)}\left(\frac{S_t}{H}\right)^p E_t^{Q^Z}\left[g\left(\frac{H^2}{S_T}\right)\right].$$

Girsanov's Theorem tells us that

$$dW_t^{Q^Z} = dW_t^Q - p\sigma \, dt.$$

Hence, $W_t^{Q^Z}$ as defined above is a Q^Z-Brownian motion. Setting $Y_t = \dfrac{H^2}{S_t}$, show that

$$dY_t = rY_t \, dt + \sigma Y_t(-dW_t^{Q^Z}).$$

That is, the law of Y under Q^Z is the same as the law of S under Q.

3. By applying the following transformation on the dependent variable c in the Black-Scholes equation

$$c = e^{\alpha y + \beta\tau} w,$$

where $\alpha = \dfrac{1}{2} - \dfrac{r}{\sigma^2}, \beta = -\dfrac{\alpha^2\sigma^2}{2} - r$, show that the convective diffusion equation

$$\frac{\partial c}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 c}{\partial y^2} + \left(r - \frac{\sigma^2}{2}\right)\frac{\partial c}{\partial y} - rc$$

is reduced to the prototype diffusion equation

$$\frac{\partial w}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 w}{\partial y^2},$$

while the auxiliary conditions are transformed to become

$$w(0,\tau) = e^{-\beta\tau}R(\tau) \text{ and } w(y,0) = \max(e^{\alpha y}(e^y - X), 0).$$

Consider the following diffusion equation defined in a semi-infinite domain

$$\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2}, \quad x > 0 \text{ and } t > 0, \quad a \text{ is a positive constant},$$

with initial condition: $v(x,0) = f(x)$ and boundary condition: $v(0,t) = g(t)$, the solution to the diffusion equation is given by

$$v(x,t) = \frac{1}{2a\sqrt{\pi t}} \int_0^\infty f(\xi)[e^{-x-\xi)^2/4a^2 t} - e^{-(x+\xi)^2/4a^2 t}] \, d\xi$$
$$+ \frac{x}{2a\sqrt{\pi}} \int_0^t \frac{e^{-x^2/4a^2\omega}}{\omega^{3/2}} g(t-\omega) \, d\omega.$$

Using the above form of solution, show that the price of the European down-and-out call option is given by

$$
\begin{aligned}
c(y,\tau) = e^{\alpha y+\beta\tau}\Bigg\{ & \frac{1}{\sqrt{2\pi\tau}\sigma}\int_0^\infty \max(e^{-\alpha\xi}(e^\xi - X), 0) \\
& \left[e^{-(y-\xi)^2/2\sigma^2\tau} - e^{-(y+\xi)^2/2\sigma^2\tau}\right] d\xi \\
& + \frac{y}{\sqrt{2\pi}\sigma}\int_0^\tau \frac{e^{-\beta(\tau-\omega)}e^{-y^2/2\sigma^2\omega}}{\omega^{3/2}} R(\tau-\omega)\, d\omega \Bigg\}.
\end{aligned}
$$

Assuming $B < X$, show that the price of the European down-and-out call option is given by

$$
\begin{aligned}
c(S,\tau) = {} & c_E(S,\tau) - \left(\frac{B}{S}\right)^{\delta-1} c_E\left(\frac{B^2}{S},\tau\right) \\
& + \int_0^\tau e^{-r\omega}\frac{\ln\frac{S}{B}}{\sqrt{2\pi}\sigma}\frac{\exp\left(\frac{-\left[\ln\frac{S}{B}+\left(r-\frac{\sigma^2}{2}\right)\omega\right]^2}{2\sigma^2 w}\right)}{\omega^{3/2}} R(\tau-\omega)\, d\omega.
\end{aligned}
$$

The last term represents the additional option premium due to the rebate payment.

4. Consider a European down-and-out *partial barrier* call option where the barrier provision is activated only between option's starting date (time 0) and t_1. Here, t_1 is some time earlier than the expiration date T, where $0 < t_1 < T$. Let B and X denote the down-barrier and strike, respectively, where $B < X$. Let the dynamics of S_t be governed by

$$
\frac{dS_t}{S_t} = r\,dt + \sigma\,dZ_t
$$

under the risk neutral measure Q. Assuming $S_0 > B$, show that the down-and-out call price is given by

$$
\begin{aligned}
\text{call price } &= e^{-rT}E_Q\left[(S_T - X)\mathbf{1}_{\{S_T>X\}}\mathbf{1}_{\{m_0^{t_1}>B\}}\right] \\
&= S_0\left[N\left(d_1, e_1; \sqrt{\frac{t_1}{T}}\right) - \left(\frac{B}{S}\right)^{\delta+1} N\left(d_1', e_1'; \sqrt{\frac{t_1}{T}}\right)\right] \\
&\quad - e^{-rT}X\left[N\left(d_2, e_2; \sqrt{\frac{t_1}{T}}\right) - \left(\frac{B}{S}\right)^{\delta-1} N\left(d_2', e_2'; \sqrt{\frac{t_1}{T}}\right)\right],
\end{aligned}
$$

where

$$
\begin{aligned}
d_1 &= \frac{\ln\frac{S_0}{X} + \left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, & d_2 &= d_1 - \sigma\sqrt{T}, \\
d_1' &= d_1 + \frac{2\ln\frac{B}{S_0}}{\sigma\sqrt{T}}, & d_2' &= d_1' - \sigma\sqrt{T}, \\
e_1 &= \frac{\ln\frac{S_0}{B} + \left(r+\frac{\sigma^2}{2}\right)t_1}{\sigma\sqrt{t_1}}, & e_2 &= e_1 - \sigma\sqrt{t_1}, \\
e_1' &= e_1 + \frac{2\ln\frac{B}{S_0}}{\sigma\sqrt{t_1}}, & e_2' &= e_1' - \sigma\sqrt{t_1}, \\
\delta &= \frac{2r}{\sigma^2}.
\end{aligned}
$$

Find the corresponding price function when t_1 is set equal to T.

5. The density function ϕ_n of the n-variate unit variance Brownian motion with constant drifts and one-sided barrier satisfies the following forward Fokker-Planck equation with a semi-infinite domain in the first independent variable x_1 and infinite domain in the remaining independent variables

$$\frac{\partial \phi_n}{\partial t} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij}\frac{\partial^2 \phi_n}{\partial x_i \partial x_j} - \sum_{j=1}^{n}\mu_j \frac{\partial \phi_n}{\partial x_j}, \quad t > 0,\, b_1 < x_1 < \infty,\, -\infty < x_j < \infty,\, j = 2, \cdots, n.$$

Show that the following linear transformation of the independent variables

$$z_j = \begin{cases} x_1 & \text{if } j = 1 \\ \frac{x_j - \rho_{1j}x_1}{\sqrt{1-\rho_{1j}^2}} & \text{if } j = 2, 3, ..., n \end{cases}$$

leads to the splitting of ϕ_n in the following sense:

$$\phi_n(z_1, z_2, ..., z_n, t) = \phi_1(z_1, t)\phi_{n-1}(z_2, ..., z_n, t).$$

The reduced density functions $\phi_1(z_1, t)$ and $\phi_{n-1}(z_2, ..., z_n, t)$ satisfy, respectively, the following equations

$$\begin{aligned} \frac{\partial \phi_1}{\partial t} &= \frac{1}{2}\frac{\partial^2 \phi_1}{\partial z_1^2} - \mu_1 \frac{\partial \phi_1}{\partial z_1}, \quad t > 0,\, b_1 < z_1 < \infty, \\ \frac{\partial \phi_{n-1}}{\partial t} &= \frac{1}{2}\sum_{i=2}^{n}\sum_{j=2}^{n}\tilde{\rho}_{ij}\frac{\partial^2 \phi_{n-1}}{\partial z_i \partial z_j} - \sum_{j=2}^{n}\tilde{\mu}_j \frac{\partial \phi_{n-1}}{\partial z_j}, \quad t > 0,\, -\infty < z_j < \infty,\, j = 2, ..., n, \end{aligned}$$

where

$$\tilde{\rho}_{ij} = \frac{\rho_{ij} - \rho_{1i}\rho_{1j}}{\sqrt{(1-\rho_{1i}^2(1-\rho_{1j}^2)}} \quad \text{and} \quad \tilde{\mu}_j = \frac{\mu_j - \rho_{1j}\mu_1}{\sqrt{1-\rho_{1j}^2}}, \quad i, j = 2, 3, ..., n.$$

Note that both $\phi_1(z_1, t)$ and $\phi_n(z_1, ..., z_n, t)$ share the same homogeneous Dirichlet condition at $z_1 = b_1$.

Hint Consider the n-dimensional standard Brownian motion $(X_1 \quad X_2 \quad \cdots \quad X_n)$ with correlation matrix R whose entries are ρ_{ij}, $i, j = 1, 2, ..., n$. Suppose we define

$$Z_j = \begin{cases} X_1 & \text{for } j = 1 \\ \frac{X_j - \rho_{1j}X_1}{\sqrt{1-\rho_{1j}^2}} & \text{for } j = 2, 3, ..., n \end{cases},$$

then the joint process $(Z_1 \quad Z_2 \quad \cdots \quad Z_n)$ is also a n-dimensional standard Brownian motion.

6. Let the exit time density $q^+(t; x_0, t_0)$ to the upper barrier ℓ have dependence on the initial state $X(t_0) = x_0$, $0 < x_0 < \ell$. We write $\tau = t - t_0$ so that $q^+(t; x_0, t_0)$ is visualized as $q^+(x_0, \tau)$. Show that the partial differential equation formulation is given by

$$\frac{\partial q^+}{\partial \tau} = \mu \frac{\partial q^+}{\partial x_0} + \frac{\sigma^2}{2}\frac{\partial^2 q^+}{\partial x_0^2}, \quad 0 < x_0 < \ell, \quad \tau > 0,$$

with auxiliary conditions:

$$q^+(0, \tau) = 0, \quad q^+(\ell, \tau) = \delta(\tau) \quad \text{and} \quad q^+(x_0, 0) = \delta(\ell - x_0).$$

By solving the above partial differential equation, show that

$$q^+(t; x_0, t_0) = e^{\frac{\mu}{\sigma^2}(\ell - x_0)} \frac{\sigma^2}{\ell^2} \sum_{k=1}^{\infty} e^{-\lambda_k (t-t_0)} k\pi \sin \frac{k\pi(\ell - x_0)}{\ell},$$

where

$$\lambda_k = \frac{1}{2}\left(\frac{\mu^2}{\sigma^2} + \frac{k^2\pi^2\sigma^2}{\ell^2}\right), \quad k = 1, 2, \ldots .$$

7. Suppose the dynamics of the logarithm of the stock price S_t is governed by

$$d\ln S_t = \left(r - \frac{\sigma^2}{2}\right) dt + \sigma\, dW_t,$$

the density function of $\ln S_T$ conditional on $\ln S_0$ at time 0 is given by

$$f(S_T; S_0) = \frac{1}{\sqrt{2\pi\sigma^2 T}} \exp\left(-\frac{\left(\ln \frac{S_T}{S_0} - \lambda T\right)^2}{2\sigma^2 T}\right), \quad \lambda = r - \frac{\sigma^2}{2}.$$

(a) Let τ_L denote the first passage time of the stock price to the lower barrier L, where $L < S_0$. Using the reflection principle, show that

$$f(S_T; S_0 | \tau_L < T) = f\left(S_T; \frac{L^2}{S_0}\right)\left(\frac{L}{S_0}\right)^{2\lambda/\sigma^2}.$$

(b) Similarly, let τ_U denote the first passage time of the stock price to the upper barrier U, where $S_0 < U$. Show that

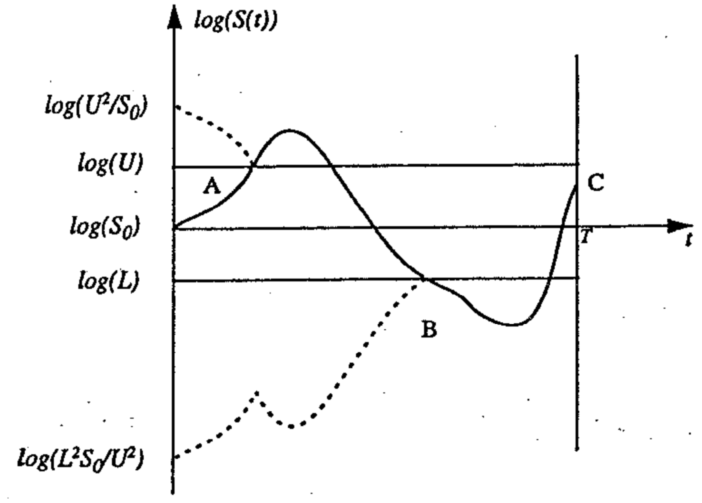
$$f(S_T; S_0 | \tau_U < T) = f\left(S_T; \frac{U^2}{S_0}\right)\left(\frac{U}{S_0}\right)^{2\lambda/\sigma^2}.$$

(c) Let $\tau_{U/L}$ ($\tau_{L/U}$) be the first time that the stock price process hits the upper barrier U (lower barrier L) after hitting the lower barrier L (upper barrier U). That is,

$$\tau_{U/L} = \inf\{t | S(t) = U,\, t > \tau_L\}$$
$$\tau_{L/U} = \inf\{t | S(t) = L,\, t > \tau_U\}.$$

Show that

$$f_{U/L}(S_T; S_0) = f\left(S_T; S_0\left(\frac{U}{L}\right)^2\right)\left(\frac{U}{L}\right)^{2\lambda/\sigma^2}$$
$$f_{L/U}(S_T; S_0) = f\left(S_T; S_0\left(\frac{L}{U}\right)^2\right)\left(\frac{L}{U}\right)^{2\lambda/\sigma^2}.$$



Double reflection of a sample stock price path

(d) Use the density function $f_{L/U}(S_T; S_0)$ to find the price formula of the call option with strike price X, where $L < X < U$, and subject to knock-out upon sequential breaching of up-barrier U first and down-barrier L afterwards.

(e) Lastly, deduce that the density function of $\ln S_T$, conditional on the stock price hitting neither the lower barrier L nor the upper barrier U before time T, is given by

$$\begin{aligned}&f(S_T; S_0|\min(\tau_L, \tau_U) > T)\\ &= \sum_{n=-\infty}^{\infty} f\left(S_T; S_0\left(\frac{U}{L}\right)^{2n}\right)\left(\frac{U}{L}\right)^{2n\lambda/\sigma^2} - f\left(S_T; \frac{U^2}{S_0}\left(\frac{U}{L}\right)^{2n}\right)\left[\left(\frac{U}{S_0}\right)\left(\frac{U}{L}\right)^{n}\right]^{2\lambda/\sigma^2}.\end{aligned}$$

8. Consider the Black-Scholes equation with time-dependent model parameters for a standard European option

$$\frac{\partial P(S,t)}{\partial t} + \frac{1}{2}\sigma(t)^2 S^2 \frac{\partial^2 P(S,t)}{\partial S^2} + [r(t) - d(t)]S\frac{\partial P(S,t)}{\partial S} - r(t)P(S,t) = 0,$$

where P is the option value, S is the underlying asset price, t is the calendar time, σ is the volatility, r is the risk-free interest rate and d is the dividend yield. Introducing the new variable $x = \ln(S/B)$, where B is the value of the fixed barrier in a barrier option, the above pricing equation is simplified to

$$\frac{\partial P(x,t)}{\partial t} + \frac{1}{2}\sigma(t)^2 \frac{\partial^2 P(x,t)}{\partial x^2} + \left[r(t) - d(t) - \frac{1}{2}\sigma(t)^2\right]\frac{\partial P(x,t)}{\partial x} - r(t)P(x,t) = 0. \tag{1}$$

(a) Explain why the solution $P(x,t)$ can be expressed as

$$P(x,t) = \exp(c_3(t))\exp\left(c_1(t)\frac{\partial}{\partial x}\right)\exp\left(c_2(t)\frac{\partial^2}{\partial x^2}\right)P(x,0),$$

where

$$
\begin{aligned}
c_1(t) &= \int_0^t \left[r(t') - d(t') - \frac{\sigma(t')^2}{2} \right] dt' \\
c_2(t) &= \int_0^t \frac{\sigma(t')^2}{2}\, dt' \\
c_3(t) &= -\int_0^t r(t')\, dt'.
\end{aligned}
$$

(b) Making use of the well known relations

$$
\begin{aligned}
\exp\left(\eta \frac{\partial}{\partial x}\right) f(x) &= f(x+\eta), \\
\exp\left(\eta \frac{\partial^2}{\partial x^2}\right) f(x) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\eta}} \exp\left[-\frac{(x-y)^2}{4\eta}\right] f(y)\, dy,
\end{aligned}
$$

for some parameter η, we can easily show that $P(x,t)$ can be expressed in the form

$$
P(x,t) = \int_{-\infty}^{\infty} G(x,t;x',0) P(x',0)\, dx',
$$

where

$$
G(x,t;x',0) = \frac{1}{\sqrt{4\pi c_2(t)}} \exp\left(-\frac{[x-x'+c_1(t)]^2}{4c_2(t)} + c_3(t)\right)
$$

is the Green function of the pricing equation in eq. (1).

(c) By the method of images, we can also incorporate an absorbing time dependent barrier along the x-axis, and the barrier $S^*(t)$ has time dependence of the form

$$
x^*(t) = \ln(S^*(t)/B) = -c_1(t) - \beta c_2(t),
$$

where β is an adjustable parameter controlling the movement of the barrier. Show that the price of an up-and-out call option is then given by

$$
P(x,t) = \int_{-\infty}^{0} \{G(x,t;x',0) - G(x,t;-x',0)\exp(-\beta x')\} P(x',0)\, dx'. \tag{2}
$$

(d) Given the final payoff condition: $P(x,0) = \max(S-K,0)$, where K is the strike price, perform the integration in eq. (2) to obtain the price function:

$$
\begin{aligned}
P(x,t) = {} & \exp(c_3(t) + c_2(t) + c_1(t) + x) B \\
& \left[N\left(-\frac{x + c_1(t) + 2c_2(t)}{\sqrt{2c_2(t)}}\right) - N\left(-\frac{x + c_1(t) + 2c_2(t) - \ln(K/B)}{\sqrt{2c_2(t)}}\right) \right] \\
& - K \exp(c_3(t)) \left[N\left(-\frac{x + c_1(t)}{\sqrt{2c_2(t)}}\right) - N\left(-\frac{x + c_1(t) - \ln(K/B)}{\sqrt{2c_2(t)}}\right) \right] \\
& - \exp\left(c_3(t) + (\beta-1)[x + c_1(t)] + (\beta-1)^2 c_2(t)\right) B \\
& \left[N\left(\frac{x + c_1(t) + 2(\beta-1)c_2(t)}{\sqrt{2c_2(t)}}\right) - N\left(\frac{x + c_1(t) + 2(\beta-1)c_2(t) + \ln(K/B)}{\sqrt{2c_2(t)}}\right) \right] \\
& + K \exp(c_3(t) + \beta[x + c_1(t)] + \beta^2 c_2(t)) \\
& \left[N\left(\frac{x + c_1(t) + 2\beta c_2(t)}{\sqrt{2c_2(t)}}\right) - N\left(\frac{x + c_1(t) + 2\beta c_2(t) - \ln(K/B)}{\sqrt{2c_2(t)}}\right) \right].
\end{aligned}
$$

(e) To simulate a fixed barrier, we shall choose an optimal value of the adjustable parameter β in such a way that the integral

$$\int_0^{t^*} [x^*(t)]^2 \, dt$$

is minimized. In other words, we try to minimize the deviation from the fixed barrier by varying the parameter β. Here, t^* denotes the time at which the option price is evaluated. Show that the optimal value of β is given by

$$\beta_{opt} = -\frac{-\int_0^{t^*} c_1(t)c_2(t)\, dt}{\int_0^{t^*} [c_2(t)]^2\, dt}.$$

9. Let $p_{\tau(t)}$ be the density function of the stopping time

$$\tau = \inf\{t \geq 0 : W_t \geq g(t)\},$$

where $g(t)$ is a continuous boundary with $g(0) > 0$. We define

$$\phi(x, \Delta) = \frac{\exp\left(-\frac{x^2}{2\Delta}\right)}{\sqrt{2\pi\Delta}}.$$

Show that

$$P[\tau < t, W_t > y] = \int_y^\infty \int_0^t p_{\tau(u)} \phi(x - g(u), t - u)\, du\, dx.$$

Suppose g is a linear boundary on the time interval $[t_i, t_{i+1}]$, show that

$$\begin{aligned} &P[W_s < g(x),\, t_i \leq s \leq t_{i+1} | W_{t_i} = x_i,\, W_{t_{i+1}} = x_{i+1}] \\ =& \mathbf{1}_{\{g(t_i) > x_i,\, g(t_{i+1}) > x_{i+1}\}} \left[1 - \exp\left(-\frac{2[g(t_i) - x_i][g(t_{i+1}) - x_{i+1}]}{t_{i+1} - t_i}\right)\right]. \end{aligned}$$

Next, we consider the approximation of the following probability:

$$P(a, b) = P[a(t) < W_t < b(t),\, 0 \leq t \leq T],$$

where $T > 0$ is fixed, the functions $a(t)$ and $b(t)$ are continuous deterministic functions and satisfy $a(t) < b(t)$, $0 \leq t \leq T$, with $a(0) < 0 < b(0)$. Suppose we discretize the time interval $[0, T]$ by time points

$$t_0 = 0 < t_1 < t_2 < ... < t_{n-1} < t_n = T.$$

We write $\Delta t_i = t_i - t_{i-1}$, $\beta_i = b(t_i)$, $\alpha_i = a(t_i)$ and $d_i = \beta_i - \alpha_i$. Suppose we approximate $a(t)$ and $b(t)$ by piecewise linear functions on $[0, T]$ with common nodes $\{t_i\}_{i=1}^n$, show that $P(a, b)$ can be approximated by

$$P(a, b) \approx E\left[g(W(t_1), W(t_2), ..., W(t_n))\right]$$

where

$$g(\boldsymbol{x}) = \prod_{i=1}^n \mathbf{1}_{\alpha_i < x_i < \beta_i} \left[1 - \sum_{j=1}^\infty q(i, j)\right],$$

$\boldsymbol{x} = (x_1 \quad x_2 \quad \dots \quad x_n)^T$, $\Delta x_i = x_i - x_{i-1}$, $x_0 = 0$, and

$$
\begin{aligned}
q(i,j) = \exp&\left(-\frac{2}{\Delta t_i}[jd_{i-1} + (\alpha_{i-1} - x_{i-1})][jd_i + (\alpha_i - x_i)]\right) \\
&- \exp\left(-\frac{2j}{\Delta t_i}[jd_{i-1}d_i + d_{i-1}(\alpha_i - x_i) - d_i(\alpha_{i-1} - x_{i-1})]\right) \\
&+ \exp\left(-\frac{2}{\Delta t_i}[jd_{i-1} - (\beta_{i-1} - x_{i-1})][jd_i - (\beta_i - x_i)]\right) \\
&- \exp\left(-\frac{2j}{\Delta t_i}[jd_{i-1}d_i - d_{i-1}(\beta_i - x_i) + d_i(\beta_{i-1} - x_{i-1})]\right).
\end{aligned}
$$

10. We consider a financial market that is complete, trading can take place continuously without frictions, etc. and there exists a unique risk neutral measure Q under which all discounted security prices in the market are Q-martingales. Under the risk neutral measure Q, the riskless interest rate r_t is assumed to follow the following extended Vasicek process

$$dr_t = [\phi(t) - \alpha r]\, dt + \sigma_r\, dZ_r,$$

where σ_r is the constant volatility of the interest rate, $\phi(t)$ and α are parameters in the mean reversion process. Let $B(r,t;T)$ denote the price of default free discount bond and the bond price volatility $\sigma_B(t,T)$ is then given by

$$\sigma_B(t,T) = \frac{\sigma_r}{\alpha}[1 - e^{-\alpha(T-t)}].$$

Under the risk neutral measure Q, the firm value process A_t is assumed to follow the Geometric Brownian motion

$$\frac{dA_t}{A_t} = r_t\, dt + \sigma_A\, dZ_A,$$

where σ_A is the constant volatility of the firm value process. Further, we assume that the Brownian motions Z_r and Z_A are correlated such that $dZ_r\, dZ_A = \rho\, dt$, where ρ is the constant correlation coefficient.

We assume a simple capital structure of the firm as in the Merton structural debt model, where the firm liabilities consist only a single fixed debt with par value F and maturity date T. Bondholders are protected by a safety covenant whereby the bondholders can force a reorganization when the firm value A_t falls to some threshold level $\nu(t)$. The threshold level is exogenously specified to represent a fraction of the present value of the liabilities so that $\nu(t)$ takes the form

$$\nu(t) = \beta F B(r,t;T),$$

where β is a fractional constant $(0 \leq \beta < 1)$. The special case $\beta = 0$ corresponds to non-existence of intertemporal default. When A_t reaches the threshold $\nu(t)$, the bondholders receive only f_1 $(0 \leq f_1 \leq 1)$ fraction of βF. In the case where the strict priority rule is observed, the equity holders would receive nothing, which corresponds to $f_1 = 1$. Suppose the firm value has stayed above the default threshold level $\nu(t)$ throughout the life of the debt, default can occur only at debt maturity. Upon default at maturity, due to possible violation of strict priority rule, the bondholders can receive only f_2 $(0 \leq f_2 \leq 1)$ fraction of the firm asset.

Let T_ν^A denote the first passage time of the firm value process through the barrier $\nu(t)$, and A_T denote the asset value at maturity T. Conditional on the occurrence of intertemporal default, corresponding to $T_\nu^A < T$, the bondholders receive only f_1 of the liabilities. When $T_\nu^A \geq T$, the bond survives until maturity. At maturity time T, the bondholders receive the full par value F if $A_T \geq F$ and only a fraction f_2 of the terminal asset value if $A_T < F$.

(a) Show that the value of the risky debt conditional on $A_t = A$ and $r_t = r$ is given by

$$V(A, r, t) = B(r, t)E_{Q^T}\left[f_1 \beta F \mathbf{1}_{\{T_\nu^A < T\}} + F\mathbf{1}_{\{T_\nu^A \geq T, A_T \geq F\}} + f_2 A_T \mathbf{1}_{\{T_\nu^A \geq T, A_T < F\}}\right],$$

where Q^T is the T-forward measure.

(b) We define two ratios, the quasi debt-to-asset ratio $d = FB(r, t; T)/A$ and bankruptcy ratio $b = \nu/A = \beta d$. Let $p(A, \tau; F)$ be the European put price function with strike price F. The analytic expression for the put price $p(A, \tau; F)$ under stochastic interest rate is given by

$$p(A, \tau; F) = FN(-d_2) - AB(r, \tau; T)N(-d_1),$$

where

$$d_2 = -\frac{\ln d}{\sigma_{A,T}\tau} - \frac{\sigma_{A,T}\tau}{2}, \quad d_1 = d_2 + \sigma_{A,T}\tau, \quad \tau = T - t,$$

and

$$\sigma_{A,T}^2 = \frac{1}{T - t}\int_t^T [\sigma_A^2 - 2\rho\sigma_A\sigma_B(u, T) + \sigma_B^2(u, T)]^2 \, du.$$

(c) Find the risk neutral probability of default over the time interval $[0, T]$.

11. Assume that the dynamics of the short rate r_t and asset value V_t under the risk neutral probability measure P^* are governed by

$$dr_t = \kappa(\theta - r_t)\, dt + \sigma_r \; dW_r^*,$$
$$d\ln V_t = \left(r_t - \frac{1}{2}\sigma_V^2\right) dt + \sigma_V \; dW_V^*,$$

where W_r^* and W_V^* are standard Brownian motions under the risk neutral probability measure P^* and θ is the long-term mean of short rate adjusted by market price of interest rate risk. To allow for dependence between the firm value and the interest rate, we use the default-free discount bond price as the numeraire. The dynamics of the short rate and asset value under the forward martingale measure P_T are

$$dr_t = \kappa\left(\theta - r_t - \frac{\sigma_r^2}{\kappa}B_\kappa^{(T-t)}\right) dt + \sigma_r \; dW_r^T,$$
$$d\ln V_t = \left(r_t - \frac{1}{2}\sigma_V^2 - \rho_{rV}\sigma_r\sigma_V B_\kappa^{(T-t)}\right) dt + \sigma_V \; dW_V^T,$$

where $B_\kappa^{(T-t)} = [1 - e^{-\kappa(T-t)}]/\kappa$. The firm's default time is defined as

$$\tau = \inf\{t | t \geq 0, \, V_t \leq K_t\}.$$

Hence, the risk neutral default probability of the firm during time period t is

$$Q(t) = P[\tau < t] = E_0^{P^*}[I_{\{\tau < t\}}],$$

where $E_0^{P^*}[\cdot]$ is the conditional expectation with respect to P^* at time 0.

We discretize the time interval $[0, T]$ into n_t equal intervals, and define the time point $t_m = mT/n_t = m\Delta t$, $m = 1, 2, ..., n_t$. Similarly, we discretize the r-space into n_r equal intervals between some chosen minimum $\underline{r}$ and maximum $\overline{r}$, and define $r_k = \underline{r} + k\Delta r$, $k = 1, 2, ..., n_r$, where $\Delta r = (\overline{r} - \underline{r})/n_r$. With deterministic default threshold level K_t, show that the default probability of the firm during time period T under the forward martingale measure P_T is

$$Q^T(T) = \sum_{m=1}^{n_t}\sum_{k=1}^{n_r} q(r_k, t_m),$$

where

$$\begin{aligned}
q(r_k, t_1) &= \Delta r \Psi(r_k, t_1), \quad k \in \{1, 2, ..., n_r\}, \\
q(r_k, t_m) &= \Delta r \left[\Psi(r_k, t_m) - \sum_{\nu=1}^{m-1}\sum_{u=1}^{n_r} q(r_u, t_\nu)\psi(r_k, t_m | r_u, t_\nu) \right], \\
&\qquad\qquad\qquad k \in \{1, 2, ..., n_r\}, \quad m \in \{2, ..., n_t\}; \\
\Psi(r_t, t) &= \pi(r_t, t | r_0, 0) N\left(\frac{\mu_i(r_t, t | l_0, r_0, 0)}{\Sigma_i(r_t, t | l_0, r_0, 0)} \right); \\
\psi(r_t, t | r_s, s) &= \pi(r_t, t | r_s, s) N\left(\frac{\mu_i(r_t, t | 0, r_s, s)}{\Sigma_i(r_t, t | 0, r_s, s)} \right).
\end{aligned}$$

Here, $\pi(r_t, t | r_s, s)$ is the transition density for the stochastic short rate and $l_t = \ln(K_t/V_t)$ is the log leverage ratio of the firm at time t. Also, the conditional mean and variance of l_t are defined by

$$\begin{aligned}
\mu_i(r_t, l_s, r_s) &= E_s^{P_T}[l_t | r_t] \\
\Sigma_i(r_t, l_s, r_s) &= \mathrm{var}_s^{P_T}[l_t | r_t].
\end{aligned}$$

Lastly, show that

$$\begin{aligned}
\mu_i(r_t, l_s, r_s) &= E_s^{P_t}[l_t | r_t] = E_s^{P_T}[l_t] + \frac{\mathrm{cov}_s^{P_T}[l_t, r_t]}{\mathrm{var}_s^{P_T}[r_t]} \left(r_t - E_s^{P_T}[r_t] \right), \\
\Sigma_i^2(r_t, l_s, r_s) &= \mathrm{var}_s^{P_t}[l_t | r_t] = \mathrm{var}_s^{P_T}[l_t] - \frac{\mathrm{cov}_s^{P_T}[l_t, r_t]^2}{\mathrm{var}_s^{P_T}[r_t]},
\end{aligned}$$

where

$$\begin{aligned}
E_s^{P_T}[l_t] =& \, l_s + \ln\frac{K_t}{K_s} - \left(\frac{\theta}{\kappa} - \frac{\sigma_r^2}{\kappa^2} - \frac{\sigma_V^2}{2} - \frac{\rho_{rV}\sigma_r\sigma_V}{\kappa} \right)(t - s) \\
& - \left(r_s - \frac{\theta}{\kappa} + \frac{\sigma_r^2}{\kappa^2} + \frac{\rho_{rV}\sigma_r\sigma_V}{\kappa} e^{-\kappa(T-t)} \right) B_\kappa^{(t-s)} \\
& - \frac{\sigma_r^2}{2\kappa} e^{-\kappa(T-t)} \left(B_\kappa^{(t-s)} \right)^2, \\
E_s^{P_T}[r_t] =& \, r_s e^{-\kappa(t-s)} + \left(\kappa\theta - \frac{\sigma_r^2}{\kappa} \right) B_\kappa^{(t-s)} + \frac{\sigma_r^2}{\kappa} e^{-\kappa(T-t)} B_{2\kappa}^{(t-s)}, \\
\mathrm{var}_s^{P_T}[l_t] =& \left(\sigma_V^2 + 2\frac{\rho_{rV}\sigma_r\sigma_V}{\kappa} + \frac{\sigma_r^2}{\kappa^2} \right)(t - s) \\
& - 2\left(\frac{\rho_{rV}\sigma_r\sigma_V}{\kappa} + \frac{\sigma_r^2}{\kappa^2} \right) B_\kappa^{(t-s)} + \frac{\sigma_r^2}{\kappa^2} B_{2\kappa}^{(t-s)}, \\
\mathrm{var}_s^{P_T}[r_t] =& \, \sigma_r^2 B_{2\kappa}^{(t-s)}, \\
\mathrm{cov}_s^{P_T}[l_t, r_t] =& \left(\frac{\sigma_r^2}{\kappa} + \rho_{rV}\sigma_r\sigma_V \right) B_\kappa^{(t-s)} - \frac{\sigma_r^2}{\kappa} B_{2\kappa}^{(t-s)}.
\end{aligned}$$

12. By following the procedure of deriving the price formula of a perpetual down-and-out proportional step call option, find the corresponding price formula of a perpetual up-and-out proportional step put option with upstream barrier B, continuous dividend yield q and killing rate ρ. Consider the two limits (i) $\rho = 0$, and (ii) $\rho \to \infty$, of the resulting put price formula. Give the financial interpretation of the respective formula under these two limiting cases.

13. The terminal payoff of a delayed barrier call option is $\mathbf{1}_{\{\tau_B^- < \alpha T\}} \max(S_T - K, 0)$, where B is the down-barrier and K is the strike price. Find the time-t price of a seasoned delayed barrier call option during the contract life, $0 < t < T$.

14. Let X_t be the double exponential jump diffusion process whose moment generating function is defined by

$$E[e^{\theta X_t}] = \exp(G(\theta)t),$$

where

$$G(x) = \mu x + \frac{\sigma^2}{2}x^2 + \lambda\left(\frac{p\eta_1}{\eta_1 - x} + \frac{q\eta_2}{\eta_2 + x} - 1\right), \quad p + q = 1.$$

Let τ_b denote the stopping time at the barrier b and $X_{\tau_b} - b$ is the overshoot. Show that

(a) $P[X_{\tau_b} - b \geq x] = e^{-\eta_1 x} P[X_{\tau_b} - b > 0]$;

(b) $E[e^{-\alpha\tau_b}\mathbf{1}_{\{X_{\tau_b} \geq b + x\}}] = e^{-\eta_1 x} E[e^{-\alpha\tau_b}\mathbf{1}_{\{X_{\tau_b} - b > 0\}}]$.

The price of a down-and-out call option is given by

$$c_{\text{down}}(S_0, T; k) = E_Q[e^{-rT}\max(S_T - e^k, 0)\mathbf{1}_{\{\tau_b > T\}}].$$

Suppose the underlying asset price process follows the double exponential jump diffusion process specified above, find the fair price of this down-and-out call option.

15. Consider a discretely monitored down-and-out call option with strike price X and barrier level B_i at discrete time $t_i, i = 1, 2, \cdots, n$. Show that the price of this European barrier call option is given by

$$\begin{aligned} &c_{d_0}(S_0, T; X, B_1, B_2, \cdots, B_n) \\ &= S_0 N_{n+1}(d_1^1, d_1^2, \cdots, d_1^{n+1}; \Gamma) \; - e^{-rT} N_{n+1}(d_2^1, d_2^2, \cdots, d_2^{n+1}; \Gamma) \end{aligned}$$

where

$$\begin{aligned} d_1^i &= \frac{\ln\frac{S_0}{B_i} + \left(r + \frac{\sigma^2}{2}\right)t_i}{\sigma\sqrt{t_i}}, \quad d_2^i = d_1^i - \sigma\sqrt{t_i}, \quad i = 1, 2, \cdots, n, \\ d_1^{n+1} &= \frac{\ln\frac{S_0}{X} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \quad d_2^{n+1} = d_1^{n+1} - \sigma\sqrt{T}. \end{aligned}$$

Also, Γ is the $(n+1) \times (n+1)$ correlation matrix whose entries are given by

$$\rho_{jk} = \frac{\min(t_j, t_k)}{\sqrt{t_j}\sqrt{t_k}}, \quad 1 \leq j, k \leq n; \quad \rho_{j,n+1} = \sqrt{\frac{t_j}{T}}, \quad j = 1, 2, \cdots, n.$$

16. Explain how the Broadie-Yamamoto method can be applied to price barrier options under Kou's double-exponential jump-diffusion model. Give the details of the algorithmic procedures. What would be the operation counts in the computation?