1. Consider the function

\[ f(S, \tau) = \left( \frac{S}{B} \right)^{\lambda} c_E \left( \frac{B^2}{S}, \tau \right), \]

where \( c_E(S, \tau) \) is the price of a vanilla European call option, \( \lambda \) is a constant parameter. Show that \( f(S, \tau) \) satisfies the Black-Scholes equation

\[ \frac{\partial f}{\partial \tau} = \frac{\sigma^2}{2} S^2 \frac{\partial^2 f}{\partial S^2} + rS \frac{\partial f}{\partial S} - rf \]

when \( \lambda \) is chosen to be \( -\frac{2r}{\sigma^2} + 1 \).

*Hint:* Substitution of \( f(S, \tau) \) into the Black-Scholes equation gives

\[ \frac{\partial f}{\partial \tau} \left[ \frac{\sigma^2}{2} S^2 \frac{\partial^2 f}{\partial S^2} + rS \frac{\partial f}{\partial S} - rf \right] = \left( \frac{S}{B} \right)^{\lambda} \left[ \frac{\partial c_E}{\partial \tau} - \frac{\sigma^2}{2} \xi^2 \frac{\partial^2 c_E}{\partial \xi^2} \right. \]

\[ + (\lambda - 1) \sigma^2 \xi \frac{\partial c_E}{\partial \xi} - \lambda (\lambda - 1) \frac{\sigma^2}{2} c_E - r\lambda c_E + r \xi \frac{\partial c_E}{\partial \xi} + rc_E \],

where \( c_E = c_E(\xi, \tau), \xi = \frac{B^2}{S} \).

2. Let the price process \( S_t \) be governed by

\[ \frac{dS_t}{S_t} = r \, dt + \sigma \, dW^Q_t, \]

where \( W^Q_t \) is a Brownian motion under the equivalent martingale measure \( Q \). We write the arbitrage-free time-\( t \) value of a “\( g \)-claim” as

\[ \pi^g(t) = e^{-r(T-t)} E^Q_t[g(S_T)] = e^{-r(T-t)} f(S_t, t). \]

Let \( p = 1 - \frac{2r}{\sigma^2} \) and \( H > 0 \) be a constant. We define a new function \( \hat{g} \) by

\[ \hat{g}(x) = \left( \frac{x}{H} \right)^p \left( \frac{H^2}{x} \right). \]

We call \( \hat{g} \) to be \( g \)'s reflected claim. Show that the arbitrage-free time-\( t \) value of this \( \hat{g} \)-claim is given by

\[ \pi^{\hat{g}}(t) = e^{-r(T-t)} \left( \frac{S_t}{H} \right)^p \left( \frac{H^2}{S_t} \right) f \left( \frac{H^2}{S_t}, t \right). \]

Use the above result to deduce the price of an up-and-out put option with upstream barrier \( H \) and strike price \( X \). Distinguish the two cases: (i) \( H > X \), and (ii) \( H \leq X \).
**Hint:** Define the process
\[ Z_t = \left( \frac{S_t}{H} \right)^p, \]
so that \( dZ_t = p\sigma Z_t \, dW_t^Q \). The Radon-Nikodym derivative
\[ \frac{dQ^Z}{dQ} = \frac{Z(T)}{Z(0)} \]
defines the probability \( Q^Z \sim Q \). Show that
\[ \pi^Z(t) = e^{-r(T-t)} \left( \frac{S_t}{H} \right)^p E^Q \left[ g \left( \frac{H^2}{S_T} \right) \right]. \]
Girsanov’s Theorem tells us that
\[ dW_t^Q = dW_t^Q - p\sigma \, dt. \]
Hence, \( W_t^Q \) as defined above is a \( Q^Z \)-Brownian motion. Setting \( Y_t = \frac{H^2}{S_t} \), show that
\[ dY_t = rY_t \, dt + \sigma Y_t \left( dW_t^Q \right). \]
That is, the law of \( Y \) under \( Q^Z \) is the same as the law of \( S \) under \( Q \).

3. By applying the following transformation on the dependent variable \( c \) in the Black-Scholes equation
\[ c = e^{\alpha y + \beta \tau} w, \]
where \( \alpha = \frac{1}{2} - \frac{r}{\sigma^2}, \beta = -\frac{\alpha^2 \sigma^2}{2} - r \), show that the convective diffusion equation
\[ \frac{\partial c}{\partial \tau} = \frac{\sigma^2}{2} \frac{\partial^2 c}{\partial y^2} + \left( r - \frac{\sigma^2}{2} \right) \frac{\partial c}{\partial y} - rc \]
is reduced to the prototype diffusion equation
\[ \frac{\partial w}{\partial \tau} = \frac{\sigma^2}{2} \frac{\partial^2 w}{\partial y^2}, \]
while the auxiliary conditions are transformed to become
\[ w(0, \tau) = e^{-\beta \tau} R(\tau) \text{ and } w(y, 0) = \max(e^{\alpha y}(e^y - X), 0). \]
Consider the following diffusion equation defined in a semi-infinite domain
\[ \frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2}, \quad x > 0 \text{ and } t > 0, \quad a \text{ is a positive constant,} \]
with initial condition: \( v(x, 0) = f(x) \) and boundary condition: \( v(0, t) = g(t) \), the solution to the diffusion equation is given by
\[ v(x, t) = \frac{1}{2a\sqrt{\pi t}} \int_0^\infty \int_x^\infty f(\xi) [e^{-x-\xi^2/4a^2t} - e^{-(x+\xi^2)(x+\xi^2)/4a^2t}] \, d\xi \]
\[ + \frac{x}{2a\sqrt{\pi t}} \int_0^t \frac{e^{-x^2/4a^2\omega}}{\omega^{3/2}} g(t - \omega) \, d\omega. \]
Using the above form of solution, show that the price of the European down-and-out call option is given by

\[
c(y, \tau) = e^{\alpha y + \beta \tau} \left\{ \frac{1}{\sqrt{2\pi \tau \sigma}} \int_0^\infty \max(e^{-\alpha \xi}(e^\xi - X), 0) \left[e^{-\frac{(y-\xi)^2}{2\sigma^2 \tau}} - e^{-\frac{(y+\xi)^2}{2\sigma^2 \tau}}\right] d\xi \\ + \frac{y}{\sqrt{2\pi \sigma}} \int_0^\tau \frac{e^{-\beta(\tau-\omega)}e^{-\frac{y^2}{2\sigma^2 \omega}}}{\omega^{3/2}} R(\tau - \omega) d\omega \right\}.
\]

Assuming \( B < X \), show that the price of the European down-and-out call option is given by

\[
c(S, \tau) = c_E(S, \tau) - \left( \frac{B}{S} \right)^{\delta-1} c_E \left( \frac{B^2}{S^2}, \tau \right) \\ + \int_0^\tau e^{-r\omega} \frac{\ln \frac{S}{B}}{\sqrt{2\pi \sigma}} \exp \left( -\frac{\left[ \ln \frac{S}{B} + (r - \frac{\sigma^2}{2})\omega \right]^2}{2\omega} \right) \omega^{3/2} R(\tau - \omega) d\omega.
\]

The last term represents the additional option premium due to the rebate payment.

4. Consider a European down-and-out partial barrier call option where the barrier provision is activated only between option’s starting date (time 0) and \( t_1 \). Here, \( t_1 \) is some time earlier than the expiration date \( T \), where \( 0 < t_1 < T \). Let \( B \) and \( X \) denote the down-barrier and strike, respectively, where \( B < X \). Let the dynamics of \( S_t \) be governed by

\[
dS_t = rdS + \sigma dZ_t
\]

under the risk neutral measure \( Q \). Assuming \( S_0 > B \), show that the down-and-out call price is given by

\[
\text{call price} = e^{-rT} E_Q \left[ (S_T - X)1_{S_T > X} 1_{\{S_{t_1} > B\}} \right] \\ = S_0 \left[ N \left( d_1, e_1; \sqrt{\frac{t_1}{T}} \right) - \left( \frac{B}{S} \right)^{\delta+1} N \left( d'_1, e'_1; \sqrt{\frac{t_1}{T}} \right) \right] \\ - e^{-rT} X \left[ N \left( d_2, e_2; \sqrt{\frac{t_1}{T}} \right) - \left( \frac{B}{S} \right)^{\delta-1} N \left( d'_2, e'_2; \sqrt{\frac{t_1}{T}} \right) \right],
\]

where

\[
d_1 = \frac{\ln \frac{S_0}{X} + (r + \frac{\sigma^2}{2}) \frac{T}{\sqrt{T}}}{\sigma \sqrt{T}}, \quad d_2 = d_1 - \sigma \sqrt{T},
\]

\[
d'_1 = d_1 + \frac{2 \ln \frac{B}{S_0}}{\sigma \sqrt{T}}, \quad d'_2 = d'_1 - \sigma \sqrt{T},
\]

\[
e_1 = \frac{\ln \frac{S_0}{B} + (r + \frac{\sigma^2}{2}) \frac{t_1}{\sqrt{t_1}}}{\sigma \sqrt{t_1}}, \quad e_2 = e_1 - \sigma \sqrt{t_1},
\]

\[
e'_1 = e_1 + \frac{2 \ln \frac{B}{S_0}}{\sigma \sqrt{t_1}}, \quad e'_2 = e'_1 - \sigma \sqrt{t_1},
\]

\[
\delta = \frac{2r}{\sigma^2}.
\]

Find the corresponding price function when \( t_1 \) is set equal to \( T \).
5. The density function \( \phi_n \) of the \( n \)-variate unit variance Brownian motion with constant drifts and one-sided barrier satisfies the following forward Fokker-Planck equation with a semi-infinite domain in the first independent variable \( x_1 \) and infinite domain in the remaining independent variables

\[
\frac{\partial \phi_n}{\partial t} = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} \frac{\partial^2 \phi_n}{\partial x_i \partial x_j} - \sum_{j=1}^{n} \mu_j \frac{\partial \phi_n}{\partial x_j}, \quad t > 0, \ b_1 < x_1 < \infty, \ -\infty < x_j < \infty, \ j = 2, \ldots, n.
\]

Show that the following linear transformation of the independent variables

\[
z_j = \begin{cases} 
 x_1 & \text{if } j = 1 \\
 \frac{x_j - \rho_{1j} x_1}{\sqrt{1 - \rho_{1j}^2}} & \text{if } j = 2, 3, \ldots, n
\end{cases}
\]

leads to the splitting of \( \phi_n \) in the following sense:

\[
\phi_n(z_1, z_2, \ldots, z_n, t) = \phi_1(z_1, t)\phi_{n-1}(z_2, \ldots, z_n, t).
\]

The reduced density functions \( \phi_1(z_1, t) \) and \( \phi_{n-1}(z_2, \ldots, z_n, t) \) satisfy, respectively, the following equations

\[
\frac{\partial \phi_1}{\partial t} = \frac{1}{2} \frac{\partial^2 \phi_1}{\partial z_1^2} - \mu_1 \frac{\partial \phi_1}{\partial z_1}, \quad t > 0, \ b_1 < z_1 < \infty,
\]

\[
\frac{\partial \phi_{n-1}}{\partial t} = \frac{1}{2} \sum_{i=2}^{n} \sum_{j=2}^{n} \tilde{\rho}_{ij} \frac{\partial^2 \phi_{n-1}}{\partial z_i \partial z_j} - \sum_{j=2}^{n} \tilde{\mu}_j \frac{\partial \phi_{n-1}}{\partial z_j}, \quad t > 0, \ -\infty < z_j < \infty, \ j = 2, \ldots, n,
\]

where

\[
\tilde{\rho}_{ij} = \frac{\rho_{ij} - \rho_{1i}\rho_{1j}}{\sqrt{(1 - \rho_{1i}^2)(1 - \rho_{1j}^2)}} \quad \text{and} \quad \tilde{\mu}_j = \frac{\mu_j - \rho_{1j} \mu_1}{\sqrt{1 - \rho_{1j}^2}}, \quad i, j = 2, 3, \ldots, n.
\]

Note that both \( \phi_1(z_1, t) \) and \( \phi_{n-1}(z_2, \ldots, z_n, t) \) share the same homogeneous Dirichlet condition at \( z_1 = b_1 \).

**Hint** Consider the \( n \)-dimensional standard Brownian motion \((X_1, X_2, \ldots, X_n)\) with correlation matrix \( R \) whose entries are \( \rho_{ij}, i, j = 1, 2, \ldots, n \). Suppose we define

\[
Z_j = \begin{cases} 
 X_1 & \text{for } j = 1 \\
 \frac{X_j - \rho_{1j} X_1}{\sqrt{1 - \rho_{1j}^2}} & \text{for } j = 2, 3, \ldots, n
\end{cases}
\]

then the joint process \((Z_1, Z_2, \ldots, Z_n)\) is also a \( n \)-dimensional standard Brownian motion.

6. Let the exit time density \( q^+(t; x_0, t_0) \) to the upper barrier \( \ell \) have dependence on the initial state \( X(t_0) = x_0, 0 < x_0 < \ell \). We write \( \tau = t - t_0 \) so that \( q^+(t; x_0, t_0) \) is visualized as \( q^+(x_0, \tau) \). Show that the partial differential equation formulation is given by

\[
\frac{\partial q^+}{\partial \tau} = \mu \frac{\partial q^+}{\partial x_0} + \frac{\sigma^2}{2} \frac{\partial^2 q^+}{\partial x_0^2}, \quad 0 < x_0 < \ell, \ \tau > 0,
\]

with auxiliary conditions:

\[
q^+(0, \tau) = 0, \quad q^+(\ell, \tau) = \delta(\tau) \quad \text{and} \quad q^+(x_0, 0) = \delta(\ell - x_0).
\]
By solving the above partial differential equation, show that

\[ q^+(t; x_0, t_0) = e^{\frac{\mu}{\sigma^2}(t-x_0)} \frac{\sigma^2}{\ell^2} \sum_{k=1}^{\infty} e^{-\lambda_k(t-t_0)} k \pi \sin \frac{k \pi (\ell - x_0)}{\ell}, \]

where

\[ \lambda_k = \frac{1}{2} \left( \frac{\mu^2}{\sigma^2} + \frac{k^2 \pi^2 \sigma^2}{\ell^2} \right), \quad k = 1, 2, \ldots. \]

7. Suppose the dynamics of the logarithm of the stock price \( S_t \) is governed by

\[ d \ln S_t = \left( r - \frac{\sigma^2}{2} \right) dt + \sigma dW_t, \]

the density function of \( \ln S_T \) conditional on \( \ln S_0 \) at time 0 is given by

\[ f(S_T; S_0) = \frac{1}{\sqrt{2\pi \sigma^2 T}} \exp \left( - \frac{(\ln \frac{S_T}{S_0} - \lambda T)^2}{2\sigma^2 T} \right), \quad \lambda = r - \frac{\sigma^2}{2}. \]

(a) Let \( \tau_L \) denote the first passage time of the stock price to the lower barrier \( L \), where \( L < S_0 \).

Using the reflection principle, show that

\[ f(S_T; S_0|\tau_L < T) = f \left( S_T; \frac{L^2}{S_0} \right) \left( \frac{L}{S_0} \right)^{2\lambda/\sigma^2}. \]

(b) Similarly, let \( \tau_U \) denote the first passage time of the stock price to the upper barrier \( U \), where \( S_0 < U \). Show that

\[ f(S_T; S_0|\tau_U < T) = f \left( S_T; \frac{U^2}{S_0} \right) \left( \frac{U}{S_0} \right)^{2\lambda/\sigma^2}. \]

(c) Let \( \tau_{U/L} (\tau_{L/U}) \) be the first time that the stock price process hits the upper barrier \( U \) (lower barrier \( L \)) after hitting the lower barrier \( L \) (upper barrier \( U \)). That is,

\[ \tau_{U/L} = \inf\{t|S(t) = U, t > \tau_L\} \]

\[ \tau_{L/U} = \inf\{t|S(t) = L, t > \tau_U\}. \]

Show that

\[ f_{U/L}(S_T; S_0) = f \left( S_T; \frac{U^2}{L} \right) \left( \frac{U}{L} \right)^{2\lambda/\sigma^2} \]

\[ f_{L/U}(S_T; S_0) = f \left( S_T; \frac{L^2}{U} \right) \left( \frac{L}{U} \right)^{2\lambda/\sigma^2}. \]
(d) Use the density function \( f_{L/U}(S_T; S_0) \) to find the price formula of the call option with strike price \( X \), where \( L < X < U \), and subject to knock-out upon sequential breaching of up-barrier \( U \) first and down-barrier \( L \) afterwards.

(e) Lastly, deduce that the density function of \( \ln S_T \), conditional on the stock price hitting neither the lower barrier \( L \) nor the upper barrier \( U \) before time \( T \), is given by

\[
f(S_T; S_0 | \min(\tau_L, \tau_U) > T) = \sum_{n=-\infty}^{\infty} f \left( S_T; S_0 \left( \frac{U}{L} \right)^{2n} \right) \left( \frac{U}{L} \right)^{2n\lambda/\sigma^2} - f \left( S_T; \frac{U^2}{S_0} \left( \frac{U}{L} \right)^{2n} \right) \left[ \frac{U}{S_0} \left( \frac{U}{L} \right)^n \right]^{2\lambda/\sigma^2}.
\]

8. Consider the Black-Scholes equation with time-dependent model parameters for a standard European option

\[
\frac{\partial P(S, t)}{\partial t} + \frac{1}{2} \sigma(t)^2 S^2 \frac{\partial^2 P(S, t)}{\partial S^2} + \left[ r(t) - d(t) \right] S \frac{\partial P(S, t)}{\partial S} - r(t) P(S, t) = 0,
\]

where \( P \) is the option value, \( S \) is the underlying asset price, \( t \) is the calendar time, \( \sigma \) is the volatility, \( r \) is the risk-free interest rate and \( d \) is the dividend yield. Introducing the new variable \( x = \ln(S/B) \), where \( B \) is the value of the fixed barrier in a barrier option, the above pricing equation is simplified to

\[
\frac{\partial P(x, t)}{\partial t} + \frac{1}{2} \sigma(t)^2 \frac{\partial^2 P(x, t)}{\partial x^2} + \left[ r(t) - d(t) - \frac{1}{2} \sigma(t)^2 \right] \frac{\partial P(x, t)}{\partial x} - r(t) P(x, t) = 0.
\]

(a) Explain why the solution \( P(x, t) \) can be expressed as

\[
P(x, t) = \exp(c_3(t)) \exp \left( c_1(t) \frac{\partial}{\partial x} \right) \exp \left( c_2(t) \frac{\partial^2}{\partial x^2} \right) P(x, 0),
\]
where
\[
c_1(t) = \int_0^t \left[ r(t') - d(t') - \frac{\sigma(t')^2}{2} \right] dt'
\]
\[
c_2(t) = \int_0^t \frac{\sigma(t')^2}{2} dt'
\]
\[
c_3(t) = -\int_0^t r(t') dt'.
\]

(b) Making use of the well known relations
\[
\exp \left( \eta \frac{\partial}{\partial x} \right) f(x) = f(x + \eta),
\]
\[
\exp \left( \eta \frac{\partial^2}{\partial x^2} \right) f(x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi \eta}} \exp \left[ -\frac{(x - y)^2}{4\eta} \right] f(y) dy,
\]
for some parameter \( \eta \), we can easily show that \( P(x, t) \) can be expressed in the form
\[
P(x, t) = \int_{-\infty}^{\infty} G(x, t; x', 0) P(x', 0) dx',
\]
where
\[
G(x, t; x', 0) = \frac{1}{\sqrt{4\pi c_2(t)}} \exp \left( -\frac{\left| x - x' + c_1(t) \right|^2}{4c_2(t)} + c_3(t) \right)
\]
is the Green function of the pricing equation in eq. (1).

(c) By the method of images, we can also incorporate an absorbing time dependent barrier along the \( x \)-axis, and the barrier \( S^*(t) \) has time dependence of the form
\[
x^*(t) = \ln(S^*(t)/B) = -c_1(t) - \beta c_2(t),
\]
where \( \beta \) is an adjustable parameter controlling the movement of the barrier. Show that the price of an up-and-out call option is then given by
\[
P(x, t) = \int_{-\infty}^{0} \{ G(x, t; x', 0) - G(x, t; -x', 0) \exp(-\beta x') \} P(x', 0) dx'. \tag{2}
\]

(d) Given the final payoff condition: \( P(x, 0) = \max(S - K, 0) \), where \( K \) is the strike price, perform the integration in eq. (2) to obtain the price function:

\[
P(x, t) = \exp(c_3(t) + c_2(t) + c_1(t) + x) B
\]
\[
\left[ N \left( \frac{x + c_1(t) + 2c_2(t)}{\sqrt{2c_2(t)}} \right) - N \left( \frac{x + c_1(t) + 2c_2(t) - \ln(K/B)}{\sqrt{2c_2(t)}} \right) \right]
\]
\[
- K \exp(c_3(t)) \left[ N \left( \frac{x + c_1(t)}{\sqrt{2c_2(t)}} \right) - N \left( \frac{x + c_1(t) - \ln(K/B)}{\sqrt{2c_2(t)}} \right) \right]
\]
\[
- \exp(c_3(t)) \left[ N \left( \frac{x + c_1(t) + (\beta - 1)[x + c_1(t)] + (\beta - 1)^2 c_2(t)}{\sqrt{2c_2(t)}} \right) B
\]
\[
\left[ N \left( \frac{x + c_1(t) + 2(\beta - 1)c_2(t)}{\sqrt{2c_2(t)}} \right) - N \left( \frac{x + c_1(t) + 2(\beta - 1)c_2(t) + \ln(K/B)}{\sqrt{2c_2(t)}} \right) \right]
\]
\[
+ K \exp(c_3(t) + \beta[x + c_1(t)] + \beta^2 c_2(t))
\]
\[
\left[ N \left( \frac{x + c_1(t) + 2\beta c_2(t)}{\sqrt{2c_2(t)}} \right) - N \left( \frac{x + c_1(t) + 2\beta c_2(t) - \ln(K/B)}{\sqrt{2c_2(t)}} \right) \right].
\]

7
To simulate a fixed barrier, we shall choose an optimal value of the adjustable parameter $\beta$ in such a way that the integral

$$\int_0^t [x^*(t)]^2 dt$$

is minimized. In other words, we try to minimize the deviation from the fixed barrier by varying the parameter $\beta$. Here, $t^*$ denotes the time at which the option price is evaluated. Show that the optimal value of $\beta$ is given by

$$\beta_{opt} = -\frac{\int_0^t c_1(t)c_2(t) dt}{\int_0^t [c_2(t)]^2 dt}.$$ 

9. Let $p_\tau(t)$ be the density function of the stopping time

$$\tau = \inf\{t \geq 0 : W_t \geq g(t)\},$$

where $g(t)$ is a continuous boundary with $g(0) > 0$. We define

$$\phi(x, \Delta) = \exp\left(-\frac{x^2}{2\Delta}\right).$$

Show that

$$P[\tau < t, W_t > y] = \int_y^\infty \int_0^t p_\tau(u)\phi(x - g(u), t - u) du dx.$$ 

Suppose $g$ is a linear boundary on the time interval $[t_i, t_{i+1}]$, show that

$$P[W_s < g(x), t_i \leq s \leq t_{i+1} | W_{t_i} = x_i, W_{t_{i+1}} = x_{i+1}] = 1_{\{g(t_i) > x, g(t_{i+1}) > x_{i+1}\}} \left[1 - \exp\left(-\frac{2[g(t_{i+1}) - x_{i+1}][g(t_i) - x_i]}{t_{i+1} - t_i}\right)\right].$$

Next, we consider the approximation of the following probability:

$$P(a, b) = P[a(t) < W_t < b(t), 0 \leq t \leq T],$$

where $T > 0$ is fixed, the functions $a(t)$ and $b(t)$ are continuous deterministic functions and satisfy $a(t) < b(t), 0 \leq t \leq T$, with $a(0) < 0 < b(0)$. Suppose we discretize the time interval $[0, T]$ by time points

$$t_0 = 0 < t_1 < t_2 < \ldots < t_{n-1} < t_n = T.$$ 

We write $\Delta t_i = t_i - t_{i-1}, \beta_i = b(t_i), \alpha_i = a(t_i)$ and $d_i = \beta_i - \alpha_i$. Suppose we approximate $a(t)$ and $b(t)$ by piecewise linear functions on $[0, T]$ with common nodes $\{t_i\}_{i=1}^n$, show that $P(a, b)$ can be approximated by

$$P(a, b) \approx E[g(W(t_1), W(t_2), \ldots, W(t_n))]$$

where

$$g(x) = \prod_{i=1}^n 1_{\alpha_i < x < \beta_i} \left[1 - \sum_{j=1}^\infty q(i, j)\right],$$
\( \mathbf{x} = (x_1 \ x_2 \ \ldots \ x_n)^T, \Delta x_i = x_i - x_{i-1}, \ x_0 = 0, \) and
\[
q(i, j) = \exp \left( -\frac{2}{\Delta t_i} [j d_{i-1} + (\alpha_{i-1} - x_{i-1})][j d_i + (\alpha_i - x_i)] \right)
- \exp \left( -\frac{2 j}{\Delta t_i} [j d_{i-1} d_i + d_{i-1}(\alpha_i - x_i) - d_i(\alpha_{i-1} - x_{i-1})] \right)
+ \exp \left( -\frac{2}{\Delta t_i} [j d_{i-1} - (\beta_{i-1} - x_{i-1})][j d_i - (\beta_i - x_i)] \right)
- \exp \left( -\frac{2 j}{\Delta t_i} [j d_{i-1} d_i - d_{i-1}(\beta_i - x_i) + d_i(\beta_{i-1} - x_{i-1})] \right).
\]

10. We consider a financial market that is complete, trading can take place continuously without frictions, etc. and there exists a unique risk neutral measure \( Q \) under which all discounted security prices in the market are \( Q \)-martingales. Under the risk neutral measure \( Q \), the riskless interest rate \( r_t \) is assumed to follow the following extended Vasicek process
\[
dr_t = [\phi(t) - \alpha r] dt + \sigma_r dZ_r,
\]
where \( \sigma_r \) is the constant volatility of the interest rate, \( \phi(t) \) and \( \alpha \) are parameters in the mean reversion process. Let \( B(r, t; T) \) denote the price of default free discount bond and the bond price volatility \( \sigma_B(t, T) \) is then given by
\[
\sigma_B(t, T) = \frac{\sigma_r}{\alpha} [1 - e^{-\alpha(T-t)}].
\]
Under the risk neutral measure \( Q \), the firm value process \( A_t \) is assumed to follow the Geometric Brownian motion
\[
\frac{dA_t}{A_t} = r_t dt + \sigma_A dZ_A,
\]
where \( \sigma_A \) is the constant volatility of the firm value process. Further, we assume that the Brownian motions \( Z_r \) and \( Z_A \) are correlated such that \( dZ_r dZ_A = \rho dt \), where \( \rho \) is the constant correlation coefficient.

We assume a simple capital structure of the firm as in the Merton structural debt model, where the firm liabilities consist only a single fixed debt with par value \( F \) and maturity date \( T \). Bondholders are protected by a safety covenant whereby the bondholders can force a reorganization when the firm value \( A_t \) falls to some threshold level \( \nu(t) \). The threshold level is exogenously specified to represent a fraction of the present value of the liabilities so that \( \nu(t) \) takes the form
\[
\nu(t) = \beta FB(r, t; T),
\]
where \( \beta \) is a fractional constant \((0 \leq \beta < 1)\). The special case \( \beta = 0 \) corresponds to nonexistence of intertemporal default. When \( A_t \) reaches the threshold \( \nu(t) \), the bondholders receive only \( f_1 \) \((0 \leq f_1 \leq 1)\) fraction of \( \beta F \). In the case where the strict priority rule is observed, the equity holders would receive nothing, which corresponds to \( f_1 = 1 \). Suppose the firm value has stayed above the default threshold level \( \nu(t) \) throughout the life of the debt, default can occur only at debt maturity. Upon default at maturity, due to possible violation of strict priority rule, the bondholders can receive only \( f_2 \) \((0 \leq f_2 \leq 1)\) fraction of the firm asset.

Let \( T^A_\nu \) denote the first passage time of the firm value process through the barrier \( \nu(t) \), and \( A_T \) denote the asset value at maturity \( T \). Conditional on the occurrence of intertemporal default, corresponding to \( T^A_\nu < T \), the bondholders receive only \( f_1 \) of the liabilities. When \( T^A_\nu \geq T \), the bond survives until maturity. At maturity time \( T \), the bondholders receive the full par value \( F \) if \( A_T \geq F \) and only a fraction \( f_2 \) of the terminal asset value if \( A_T < F \).
(a) Show that the value of the risky debt conditional on \( A_t = A \) and \( r_t = r \) is given by
\[
V(A, r, t) = B(r, t) E_Q \left[ f_1 \beta F_{1(T^* < T)} + F_{1(T^* \geq T, A_T > F)} + f_2 A_T 1_{(T^* \geq T, A_T < F)} \right],
\]
where \( Q \) is the \( T \)-forward measure.

(b) We define two ratios, the quasi debt-to-asset ratio \( d = F B(r, t; T)/A \) and bankruptcy ratio \( b = \nu/A = \beta d \). Let \( p(A, r, F) \) be the European put price function with strike price \( F \). The analytic expression for the put price \( p(A, r, F) \) under stochastic interest rate is given by
\[
p(A, r, F) = F N(-d_2) - AB(r, \tau; T) N(-d_1),
\]
where
\[
d_2 = -\frac{\ln d}{\sigma_{A,T} \theta} - \frac{\sigma_{A,T} \theta}{2}, \quad d_1 = d_2 + \sigma_{A,T} \theta, \quad \tau = T - t,
\]
and
\[
\sigma_{A,T}^2 = \frac{1}{T-t} \int_t^T \left( \sigma_A^2 - 2 \rho \sigma_A \sigma_B(u, T) + \sigma_B^2(u, T) \right)^2 du.
\]

(c) Find the risk neutral probability of default over the time interval \([0, T]\).

11. Assume that the dynamics of the short rate \( r_t \) and asset value \( V_t \) under the risk neutral probability measure \( P^* \) are governed by
\[
\begin{align*}
    dr_t &= \kappa (\theta - r_t) dt + \sigma_r \, dW^*_r, \\
    d\ln V_t &= \left( r_t - \frac{1}{2} \sigma_V^2 \right) dt + \sigma_V \, dW^*_V,
\end{align*}
\]
where \( W^*_r \) and \( W^*_V \) are standard Brownian motions under the risk neutral probability measure \( P^* \) and \( \theta \) is the long-term mean of short rate adjusted by market price of interest rate risk. To allow for dependence between the firm value and the interest rate, we use the default-free discount bond price as the numeraire. The dynamics of the short rate and asset value under the forward martingale measure \( P_T \) are
\[
\begin{align*}
    dr_t &= \kappa \left( \theta - r_t - \frac{\sigma_r^2}{\kappa} B^{(T-t)}_{(T-t)} \right) dt + \sigma_r \, dW^*_r, \\
    d\ln V_t &= \left( r_t - \frac{1}{2} \sigma_V^2 - \rho \sigma_r \sigma_V B^{(T-t)}_{(T-t)} \right) dt + \sigma_V \, dW^*_V,
\end{align*}
\]
where \( B^{(T-t)}_{(T-t)} = [1 - e^{-\kappa(T-t)}]/\kappa \). The firm’s default time is defined as
\[
\tau = \inf \{t| t \geq 0, V_t \leq K_t\}.
\]
Hence, the risk neutral default probability of the firm during time period \( t \) is
\[
Q(t) = P[\tau < t] = E_0^{P^*}[I_{(\tau < t)}],
\]
where \( E_0^{P^*}[\cdot] \) is the conditional expectation with respect to \( P^* \) at time 0.

We discretize the time interval \([0, T]\) into \( n_t \) equal intervals, and define the time point \( t_m = mT/n_t = m\Delta t, m = 1, 2, \ldots, n_t \). Similarly, we discretize the \( r \)-space into \( n_r \) equal intervals between some chosen minimum \( \underline{r} \) and maximum \( \overline{r} \), and define \( r_k = \underline{r} + k\Delta r, k = 1, 2, \ldots, n_r, \)
where $\Delta r = (\tau - \tau)/n_r$. With deterministic default threshold level $K_t$, show that the default probability of the firm during time period $T$ under the forward martingale measure $P_T$ is

$$Q^F(T) = \sum_{m=1}^{n_s} \sum_{k=1}^{n_r} q(r_k, t_m),$$

where

$$q(r_k, t_1) = \Delta r \Psi(r_k, t_1), \quad k \in \{1, 2, ..., n_r\},$$

$$q(r_k, t_m) = \Delta r \left[ \Psi(r_k, t_m) - \sum_{\nu=1}^{m-1} \sum_{u=1}^{n_r} q(r_u, t_\nu) \psi(r_k, t_m | r_u, t_\nu) \right],$$

$$k \in \{1, 2, ..., n_r\}, \quad m \in \{2, ..., n_t\};$$

$$\Psi(r_t, t) = \pi(r_t, t | r_0, 0) N \left( \frac{\mu_i(r_t, t | 0, r_0, 0)}{\sigma_i(r_t, t | 0, r_0, 0)} \right);$$

$$\psi(r_t, t | r_s, s) = \pi(r_t, t | r_s, s) N \left( \frac{\mu_i(r_t, t | 0, r_s, s)}{\sigma_i(r_t, t | 0, r_s, s)} \right).$$

Here, $\pi(r_t, t | r_s, s)$ is the transition density for the stochastic short rate and $l_t = \ln(K_t/V_t)$ is the log leverage ratio of the firm at time $t$. Also, the conditional mean and variance of $l_t$ are defined by

$$\mu_i(r_t, l_s, r_s) = E^P[l_t | r_i]$$

$$\Sigma_i(r_t, l_s, r_s) = var^P[l_t | r_i].$$

Lastly, show that

$$\mu_i(r_t, l_s, r_s) = E^P[l_t | r_i] = E^P[l_t] + \frac{cov^P[l_t, r_i]}{var^P[l_t]} \left( r_t - E^P[r_i] \right),$$

$$\Sigma_i^2(r_t, l_s, r_s) = var^P[l_t | r_i] = var^P[l_t] - \frac{cov^P[l_t, r_i]^2}{var^P[l_t]},$$

where

$$E^P[l_t] = l_s + \ln \frac{K_t}{K_s} - \left( \frac{\theta}{\kappa} \frac{\sigma_r^2}{\kappa^2} - \frac{\rho \sigma_r \sigma_v}{\kappa} \right) (t - s)$$

$$- \left( \frac{\kappa}{\theta} \frac{\sigma_r^2}{\kappa^2} + \frac{\rho \sigma_r \sigma_v}{\kappa} \right) e^{-\kappa(T-t)} B(t-s)$$

$$- \frac{\sigma_r^2}{2 \kappa^2} e^{-\kappa(T-t)} B(t-s)^2,$$

$$E^P[r_t] = r_s e^{-\kappa(t-s)} + \left( \kappa / \theta \right) \sigma_r^2 \kappa (t-s) + \frac{\sigma_r^2}{\kappa^2} e^{-\kappa(T-t)} B(2t-s),$$

$$var^P[l_t] = \left( \frac{\sigma_r^2}{\kappa^2} + \frac{\rho \sigma_r \sigma_v}{\kappa} \right) (t-s)$$

$$- \left( \rho \sigma_r \sigma_v \sigma_v / \kappa \right) B(t-s) + \frac{\sigma_r^2}{\kappa^2} B(t-s)^2,$$

$$var^P[r_t] = \sigma_r^2 B(2t-s),$$

$$cov^P[l_t, r_t] = \left( \frac{\sigma_r^2}{\kappa^2} + \frac{\rho \sigma_r \sigma_v}{\kappa} \right) B(t-s) - \frac{\sigma_r^2}{\kappa} B(t-s).$$
12. By following the procedure of deriving the price formula of a perpetual down-and-out proportional step call option, find the corresponding price formula of a perpetual up-and-out proportional step put option with upstream barrier $B$, continuous dividend yield $q$ and killing rate $\rho$. Consider the two limits (i) $\rho = 0$, and (ii) $\rho \to \infty$, of the resulting put price formula. Give the financial interpretation of the respective formula under these two limiting cases.

13. The terminal payoff of a delayed barrier call option is $1_{\{\tau_B < \alpha T\}} \max(S_T - K, 0)$, where $B$ is the down-barrier and $K$ is the strike price. Find the time-$t$ price of a seasoned delayed barrier call option during the contract life, $0 < t < T$.

14. Let $X_t$ be the double exponential jump diffusion process whose moment generating function is defined by

$$E[e^{\theta X_t}] = \exp(G(\theta)t),$$

where

$$G(x) = \mu x + \frac{\sigma^2}{2}x^2 + \lambda \left( \frac{pm_1}{\eta_1 - x} + \frac{q\eta_2}{\eta_2 + x} - 1 \right), \quad p + q = 1.$$ 

Let $\tau_b$ denote the stopping time at the barrier $b$ and $X_{\tau_b} - b$ is the overshoot. Show that

(a) $P[X_{\tau_b} - b \geq x] = e^{-t\lambda x}P[X_{\tau_b} - b > 0]$;

(b) $E[e^{-\alpha \tau_b}1_{\{X_{\tau_b} \geq x, t\}}] = e^{-t\lambda x}E[e^{-\alpha \tau_b}1_{\{X_{\tau_b} - b > 0\}}].$

The price of a down-and-out call option is given by

$$c_{\text{down}}(S_0; T; k) = E_Q[e^{-rT} \max(S_T - e^k, 0)1_{\{\tau_b > T\}}].$$

Suppose the underlying asset price process follows the double exponential jump-diffusion process specified above, find the fair price of this down-and-out call option.

15. Consider a discretely monitored down-and-out call option with strike price $X$ and barrier level $B_i$ at discrete time $t_i, i = 1, 2, \cdots, n$. Show that the price of this European barrier call option is given by

$$c_{\text{down}}(S_0; T; X, B_1, B_2, \cdots, B_n) = S_0N_{n+1}(d_1, d_2, \cdots, d_n; \Gamma) - e^{-rT}N_{n+1}(d_1, d_2, \cdots, d_n; \Gamma)$$

where

$$d_1 = \frac{\ln \frac{S_0}{B_1} + \left( r + \frac{\sigma^2}{2} \right) t_i}{\sigma \sqrt{t_i}}, \quad d_2 = d_1 - \sigma \sqrt{t_i}, \quad i = 1, 2, \cdots, n,$$

$$d_{n+1} = \frac{\ln \frac{S_0}{X} + \left( r + \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}}, \quad d_{n+1} = d_{n+1} - \sigma \sqrt{T}.$$

Also, $\Gamma$ is the $(n + 1) \times (n + 1)$ correlation matrix whose entries are given by

$$\rho_{jk} = \frac{\min(t_j, t_k)}{\sqrt{t_j t_k}}, \quad 1 \leq j, k \leq n; \quad \rho_{j, n+1} = \frac{\sqrt{t_j}}{T}, \quad j = 1, 2, \cdots, n.$$

16. Explain how the Broadie-Yamamoto method can be applied to price barrier options under Kou’s double-exponential jump-diffusion model. Give the details of the algorithmic procedures. What would be the operation counts in the computation?