MATH6380B - Advanced Topics in Derivative Pricing Models

Homework Two

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1. As an alternative approach to derive the value of a European floating strike lookback call, we consider

$$\begin{aligned} c_{f\ell}(S, m, \tau) &= e^{-r\tau} E_Q[S_T - \min(m, m_t^T)] \\ &= S - e^{-r\tau} E_Q[\min(m, m_t^T)], \end{aligned}$$

where $S_t = S, m_{T_0}^t = m$ and $\tau = T - t$. We may decompose the above expectation calculation into two terms:

$$E_Q[\min(m, m_t^T)] = mP(m \leq m_t^T) + E_Q[m_t^T \mathbf{1}_{\{m > m_t^T\}}].$$

Show that the first term is given by

$$\begin{aligned} & mP\left(\ln \frac{m_t^T}{S} \geq \ln \frac{m}{S}\right) \\ &= m\left[N\left(-\frac{\ln \frac{m}{S} + \mu\tau}{\sigma\sqrt{\tau}}\right) - \left(\frac{S}{m}\right)^{1-\frac{2r}{\sigma^2}} N\left(\frac{\ln \frac{m}{S} + \mu\tau}{\sigma\sqrt{\tau}}\right)\right]. \end{aligned}$$

The second term can be expressed as

$$E_Q[m_t^T \mathbf{1}_{\{m_t^T > m\}}] = \int_{-\infty}^{\ln \frac{m}{S}} Se^y f_{\min}(y)\, dy.$$

(a) Show that the density function of $y_T = \ln \frac{m_T^t}{s}$ is given by

$$\begin{aligned} f_{\min}(y) = &\frac{1}{\sigma\sqrt{\tau}} n\left(\frac{-y + \mu\tau}{\sigma\sqrt{\tau}}\right) + \frac{2\mu}{\sigma^2} e^{\frac{2\mu y}{\sigma^2}} N\left(\frac{y + \mu\tau}{\sigma\sqrt{\tau}}\right) \\ &+ e^{\frac{2\mu y}{\sigma^2}} \frac{1}{\sigma\sqrt{\tau}} n\left(\frac{y + \mu\tau}{\sigma\sqrt{\tau}}\right). \end{aligned}$$

(b) Find the price function $c_{f\ell}(S, m, \tau)$ of the European floating strike lookback call option.

2. Using the following form of the distribution function of m_t^T

$$P(m \leq m_t^T) = N\left(\frac{-\ln \frac{m}{S} + \mu\tau}{\sigma\sqrt{\tau}}\right) - \left(\frac{S}{m}\right)^{1-\frac{2r}{\sigma^2}} N\left(\frac{\ln \frac{m}{S} + \mu\tau}{\sigma\sqrt{\tau}}\right),$$

show that $P(m \leq m_t^T)$ becomes zero when $S = m$.

3. Suppose we use a straddle (combination of a call and a put with the same strike m) in the rollover strategy for hedging the floating strike lookback call and write

$$c_{f\ell}(S, m, \tau) = c_E(S, \tau; m) + p_E(S, \tau; m) + \text{ strike bonus premium}.$$

Find an integral representation of the strike bonus premium in terms of the distribution functions of S_T and m_t^T. Compare the strike bonus premium as given by the following alternative representation:

$$\text{strike bonus premium } = e^{-r\tau} \int_0^m P(m_t^T \leq \xi \leq S_T)\, d\xi.$$

4. Prove the following put-call parity relation between the prices of the fixed strike lookback call and floating strike lookback put:

$$c_{fix}(S, M, \tau; X) = p_{f\ell}(S, \max(M, X), \tau) + S - Xe^{-r\tau}.$$

Deduce that

$$\frac{\partial c_{fix}}{\partial M} = 0 \quad \text{for} \quad M < X.$$

Give a financial interpretation why c_{fix} is insensitive to M when $M < X$.

5. Derive the following partial differential equation for the floating strike lookback put option

$$\frac{\partial V}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial \xi^2} - \left(r + \frac{\sigma^2}{2}\right)\frac{\partial V}{\partial \xi}, \quad 0 < \xi < \infty, \tau > 0,$$

where $V(\xi, \tau) = p_{f\ell}(S, M, t)/S$ and $\tau = T - t$, $\xi = \ln \dfrac{M}{S}$. The auxiliary conditions are

$$V(\xi, 0) = e^{\xi} - 1 \quad \text{and} \quad \frac{\partial V}{\partial \xi}(0, \tau) = 0.$$

Solve the above Neumann boundary value problem to obtain the put price formula.

Hint: Define $W = \dfrac{\partial V}{\partial \xi}$ so that W satisfies the same governing differential equation but the boundary condition becomes $W(0, \tau) = 0$. Solve for $W(\xi, \tau)$, then integrate W with respect to ξ to obtain V. Be aware that an arbitrary function $\phi(t)$ is generated upon integration with respect to ξ. Obtain an ordinary differential equation for $\phi(t)$ by substituting the solution for V into the original differential equation.

6. Let $p(S, t; \delta t)$ denote the value of a floating strike lookback put option with discrete monitoring of the realized maximum value of the asset price, where δt is the regular interval between monitoring instants. Suppose we assume the following two-term Taylor expansion of $p(S, t; \delta t)$ in powers of $\sqrt{\delta t}$

$$p(S, t; \delta t) \approx p(S, t; 0) + \alpha\sqrt{\delta t} + \beta \delta t.$$

With $\delta t = 0$, $p(S, t; 0)$ represents the floating strike lookback put value corresponding to continuous monitoring. Let τ denote the time to expiry. By setting $\delta t = \tau$ and $\delta t = \tau/2$, we deduce the following pair of linear equations for α and β:

$$\alpha\sqrt{\frac{\tau}{2}} + \frac{\beta\tau}{2} = p\left(S, t; \frac{\tau}{2}\right) - p(S, t; 0)$$
$$\alpha\sqrt{\tau} + \beta\tau = p(S, t; \tau) - p(S, t; 0).$$

Hence, $p(S, t; \tau)$ is simply the vanilla put value with strike price equal to the current realized maximum asset price M. With only one monitoring instant at the mid-point of the remaining option's life, show that $p\left(S, t; \frac{\tau}{2}\right)$ is given by

$$\begin{aligned}
&p\left(S, t; \frac{\tau}{2}\right) \\
&= -e^{-q\tau}S + e^{-r\tau}\Bigg[MN_2\left(-d_M\left(\frac{\tau}{2}\right) + \sigma\sqrt{\frac{\tau}{2}}, -d_M(\tau) + \sigma\sqrt{\tau}; \frac{1}{\sqrt{2}}\right) \\
&\qquad + e^{(r-q)\tau}SN_2\left(d_M(\tau), d\left(\frac{\tau}{2}\right); \frac{1}{\sqrt{2}}\right) \\
&\qquad + e^{(r-q)\frac{\tau}{2}}SN_2\left(d_M\left(\frac{\tau}{2}\right), -d\left(\frac{\tau}{2}\right) + \sigma\sqrt{\frac{\tau}{2}}; 0\right)\Bigg],
\end{aligned}$$

where

$$d_M(\tau) = \frac{\ln \frac{S}{M} + \left(r - q + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} \text{ and } d(\tau) = \frac{\left(r - q + \frac{\sigma^2}{2}\right)\sqrt{\tau}}{\sigma}.$$

Once α and β are determined, we then obtain an approximate price formula of the discretely monitored floating strike lookback put.

7. The dynamic fund protection feature in an equity-linked fund product guarantees a predetermined protection level K to an investor who owns the underlying fund. Let S_t denote the value of the underlying fund. The dynamic protection replaces the original value of the underlying fund by an upgraded value F_t so that F_t is guaranteed not to fall below K. That is, whenever F_t drops to K, just enough capital will be added by the sponsor so that the upgraded fund value does not fall below K.

   (a) Show that the value of the upgraded fund at maturity time T is given by

   $$F_T = S_T \max\left(1, \max_{0 \le u \le T} \frac{K}{S_u}\right).$$

   (b) Let X_T denote the terminal value of the derivative that provides the dynamic fund protection. Define the lookback state variable

   $$M_t = \max\left(1, \frac{K}{\min\limits_{0 \le u \le t} S_u}\right), \quad 0 \le t \le T,$$

   which is known at the current time t. Show that

   $$\begin{aligned} X_T &= F_T - S_T \\ &= S_T(M_t - 1) + S_T \max\left(\frac{K}{\min\limits_{t \le u \le T} S_u} - M_t, 0\right). \end{aligned}$$

   (c) Under the risk neutral measure Q, let the dynamics of S_t be governed by

   $$\frac{dS_t}{S_t} = r\,dt + \sigma\,dZ_t.$$

   Show that the fair value of the dynamic fund protection is given by (Imai and Boyle, 2001)

   $$\begin{aligned} V(S, M, t) &= E_Q[X_T] \\ &= S[MN(d_1) - 1] + \frac{K}{\alpha}\left(\frac{\widehat{K}}{S}\right)^{\alpha} N(d_2) \\ &\quad + \left(1 - \frac{1}{\alpha}\right) K e^{-r\tau} N(d_3), \quad \tau = T - t, \end{aligned}$$

   where E_Q denotes the expectation under Q conditional on $S_t = S$ and $M_t = M$. The other parameter values are defined by

   $$\begin{aligned} \alpha &= 2r/\sigma^2, \quad \widehat{K} = K/M, \quad k = \ln \frac{S}{\widehat{K}}, \\ d_1 &= \frac{k + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}, \quad d_2 = \frac{-k + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}, \\ d_3 &= \frac{-k - \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}. \end{aligned}$$

8. Let $S_{1,t}$ and $S_{2,t}$ denote the price process of asset 1 and asset 2, respectively. We write $\overline{S}_1[t_1,t_2]$ and $\underline{S}_2[t_1,t_2]$ as the realized maximum of $S_{1,t}$ and realized minimum value of $S_{2,t}$ over the period $[t_1,t_2]$, respectively. The terminal payoff of the European two-asset lookback spread option is given by

$$c_{sp}(S_{1,T}, S_{2,T}, T; K) = \max(\overline{S}_1[T_0,T] - \underline{S}_2[T_0,T] - K, 0),$$

where K is the strike price. Show that

(a) $\overline{S}_1[T_0,t] - \underline{S}_2[T_0,t] - K \geq 0$ (currently in-the-money)

$$\begin{aligned} & c_{sp}(S_1, S_2, t; \overline{S}_1[T_0,t], \underline{S}_2[T_0,T]) \\ =\, & p_{fl}(S_1, t; \overline{S}_1[T_0,t]) + c_{fl}(S_2, t; \underline{S}_2[T_0,t]) + S_1 - S_2 - Ke^{-r(T-t)}, \end{aligned}$$

where p_{fl} and c_{fl} are the price functions of the European floating strike lookback put and lookback call, respectively.

(b) $\overline{S}_1[T_0,t] - \underline{S}_2[T_0,t] - K < 0$ (currently out-of-the-money)

$$\begin{aligned} & c_{sp}(S_1, S_2, t; \overline{S}_1[T_0,t], \underline{S}_2[T_0,T]) \\ =\, & p_{fl}(S_1, t; \overline{S}_1[T_0,t]) + c_{fl}(S_2, t; \underline{S}_2[T_0,t]) + S_1 - S_2 - Ke^{-r(T-t)} \\ & + e^{-r(T-t)} \int_{\overline{S}_1[T_0,t]}^{\underline{S}_2[T_0,t]+K} P[\overline{S}_1[t,T] < \xi \leq \underline{S}_2[t,T] + K]\, d\xi. \end{aligned}$$

The last integral is related to the present value of replenishing premium.