1. Apply the exchange option price formula to price the floating strike Asian call option based on the knowledge of the price formula of the fixed strike Asian call option.

\[ \text{Hint: The covariance between } S_T \text{ and } G_T \text{ is equal to } \frac{\sigma^2(T-t)^2}{2}. \]

2. We define the geometric average of the price path of asset price \( S_i \), \( i = 1, 2 \), during the time interval \([t, t+T]\) by

\[ G_i(t+T) = \exp \left( \frac{1}{T} \int_0^T \ln S_i(t + u) \, du \right). \]

Consider an Asian option involving two assets whose terminal payoff is given by \( \max(G_1(t+T) - G_2(t+T), 0) \). Show that the price formula of this European Asian option at time \( t \) is given by

\[ V(S_1, S_2, t; T) = \tilde{S}_1 N(d_1) - \tilde{S}_2 N(d_2) \]

where

\[ \tilde{S}_i = S_i \exp \left( -\left( \frac{\sigma_i^2}{2} + \frac{\sigma^2}{12} \right) T \right), \quad i = 1, 2, \quad \sigma^2 = \frac{\sigma_1^2}{3} + \frac{\sigma_2^2}{3} - \frac{2}{3} \rho \sigma_1 \sigma_2, \]

\[ d_1 = \frac{\ln \tilde{S}_1 + \frac{\sigma^2}{2} T}{\sigma \sqrt{T}}, \quad d_2 = d_1 - \sigma \sqrt{T}. \]

3. Deduce the following put-call parity relation between the prices of European fixed strike Asian call and put options under continuously monitored geometric averaging

\[ c(S, G, t) - p(S, G, t) = e^{-r(T-t)} \left\{ G^{T/T} S^{(T-t)/T} \exp \left( (T-t) \left[ \frac{\sigma^2}{6} \left( \frac{T-t}{T} \right) \right]^2 \right. \right. \]

\[ \left. + \frac{r - q - \frac{\sigma^2}{2} T - t}{T} \right] - X \right\}. \]

4. Suppose continuous arithmetic averaging of the asset price is taken from \( t = 0 \) to \( T \), where \( T \) is the expiration time. The terminal payoff function of the floating strike Asian call and put options are, respectively,

\[ \max \left( S_T - \frac{1}{T} \int_0^T S_u \, du, 0 \right) \text{ and } \max \left( \frac{1}{T} \int_0^T S_u \, du - S_T, 0 \right). \]

Show that the put-call parity relation for the above pair of European floating strike Asian options is given by

\[ c - p = S e^{-q(T-t)} + \frac{S}{(r-q)T} \left[ e^{-r(T-t)} - e^{-q(T-t)} \right] - e^{-r(T-t)} A_t, \]
where

\[ A_t = \frac{1}{T} \int_0^t S_u \, du. \]

Suppose continuous geometric averaging of the asset price is taken, show that the corresponding put-call parity relation is given by

\[
c - p = S e^{-q(T-t)} - G^{T/T} S^{(T-t)/T} \left( \frac{\sigma^2 (T-t)^3}{6 T^2} + \frac{(r - q - q^2/2)}{2T} (T-t)^2 - r (T-t) \right),
\]

where

\[ G_t = \exp \left( \frac{1}{T} \int_0^t \ln S_u \, du \right). \]

5. Show that the put-call parity relations between the prices of floating strike and fixed strike Asian options at the start of the averaging period are given by

\[
p_{f \ell}(S_0, \lambda, r, q, T) - c_{f \ell}(S_0, \lambda, r, q, T) = S e^{-qT} - e^{-rT} \lambda S_0
\]

\[
c_{f ix}(X, S_0, r, q, T) - p_{f ix}(X, S_0, r, q, T) = S e^{-qT} - e^{-rT} \int X.
\]

By combining the above put-call parity relations with the fixed-floating symmetry relation between \(c_{f ix}\) and \(p_{f \ell}\), deduce the following symmetry relation between \(c_{f ix}\) and \(p_{f \ell}\):

\[
c_{f ix}(X, S_0, r, q, T) = p_{f \ell}\left( S_0, \frac{X}{S_0}, q, r, T \right).
\]

6. Consider a self financing trading strategy of a portfolio with a dividend paying asset and money market account over the time horizon \([0, T]\). Under the risk neutral measure \(Q\), let the dynamics of the asset price \(S_t\) be governed by

\[
\frac{dS_t}{S_t} = (r - q) dt + \sigma dZ_t,
\]

where \(q\) is the dividend yield, \(q \neq r\). We adopt the trading strategy of holding \(n_t\) units of the asset at time \(t\), where

\[ n_t = \frac{1}{(r - q) T} \left[ e^{-q(T-t)} - e^{-r(T-t)} \right]. \]

Let \(X_t\) denote the portfolio value at time \(t\), whose dynamics is then given by

\[
dX_t = n_t dS_t + r(X_t - n_t S_t) dt + q n_t S_t dt.
\]

The initial portfolio value \(X_0\) is chosen to be

\[ X_0 = n_0 S_0 - e^{-rT} X. \]

Show that

\[ X_T = \frac{1}{T} \int_0^T S_t \, dt - X. \]

Defining \(Y_t = \frac{X_t}{e^{qt} S_t}\), show that

\[ dY_t = -(Y_t - e^{-qt} n_t) \sigma dZ_t^*, \]
where \( Z_t^* = Z_t - \sigma t \) is a Brownian process under \( Q^* \)-measure with \( e^{q t} S_t \) as the numeraire. Note that the price function of the fixed strike Asian call option with strike \( X \) is given by

\[
c_{\text{fix}}(S_0, 0; X) = e^{-r T} E_Q[\max(X_T, 0)] = S_0 e^{-r T} E_{Q^*}[\max(Y_T, 0)],
\]

with

\[
Y_0 = \frac{X_0}{S_0} = \frac{e^{-q T} - e^{-r T}}{(r - q) T} - e^{-r T} \frac{X}{S_0}.
\]

Show that

\[
c_{\text{fix}}(S_0, 0; X) = S_0 u(Y_0, 0),
\]

where \( u(y, t) \) satisfies the following one-dimensional partial differential equation:

\[
\frac{\partial u}{\partial t} + \frac{1}{2} (y - e^{-q t} n_t)^2 \sigma^2 \frac{\partial^2 u}{\partial y^2} = 0
\]

with \( u(y, T) = \max(y, 0) \).

7. Consider the European continuously monitored arithmetic average Asian option with terminal payoff: \( \max(A_T - X_1 S_T - X_2, 0) \), where

\[
A_T = \frac{1}{T} \int_0^T S_u \, du.
\]

At the current time \( t > 0 \), the average value \( A_t \) over the time period \([0, t]\) has been realized. Let \( V(S, \tau; X_1, X_2) \) denote the price function of the Asian option at the start of the averaging period. Show that the value of the in-progress Asian option is given by

\[
\frac{T - t}{T} V\left(S_t, T - t; \frac{X_1 T}{T - t}, \frac{X_2 T}{T - t} - \frac{A_{t} - t}{T - t}\right).
\]

8. Let \( Z_t \) denote the standard Brownian process. Show that the covariance matrix of the bivariate Gaussian random variable \( \left(Z_t, \int_0^1 Z_u \, du\right) \) is given by

\[
E \left[ \left(Z_t, \int_0^1 Z_u \, du\right)^T \left(Z_t, \int_0^1 Z_u \, du\right) \right] = \begin{pmatrix} t & t \left(1 - \frac{t}{2}\right) \\ t \left(1 - \frac{t}{2}\right) & t \left(1 - \frac{t}{2}\right) \end{pmatrix}.
\]

Also, show that the conditional distribution of \( Z_t \) given \( \int_0^1 Z_u \, du = z \) is normal with mean \( 3t \left(1 - \frac{t}{2}\right) z \) and variance \( t - 3t^2 \left(1 - \frac{t}{2}\right)^2 \). With \( Y \) chosen to be \( \int_0^1 Z_u \, du \) and \( T = 1 \), show that

\[
c_{\text{fix}}(S, I, 0) \geq e^{-r} \int_{-\infty}^{\infty} \sqrt{3} n(\sqrt{3} z) \int_0^1 \max\left(S e^{3 t (1 - t/2) z + (r - q) t + \frac{z^2}{2} \left[t - 3t^2 (1 - \frac{t}{2})^2\right]} - X, 0\right) \, dt \, dz,
\]

where \( n(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \).