Advanced Topics in Derivative Pricing Models

Topic 1 - Financial derivatives with embedded barrier features

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1.1 Product nature of barrier options

A *barrier option* is either nullified, activated or exercised when the underlying asset price breaches a *barrier* during the life of the option.

1. An *out-barrier option* (or knock-out option) is one where the option is nullified prior to expiration if the underlying asset price touches the barrier. The holder of the option may be compensated by a rebate payment for the cancellation of the option. An *in-barrier option* (or knock-in option) is one where the option only comes in existence if the asset price crosses the in-barrier. The holder has paid the option premium up-front since there can be potential positive payoff with zero chance of negative payoff.

2. When the barrier is upstream with respect to the asset price, the barrier option is called an *up-option*; otherwise, it is called a *down-option*.

One can identify eight types of European barrier options, such as down-and-out calls, up-and-out calls, down-and-in puts, down-and-out puts, etc.

$$\begin{cases} \text{up} \\ \text{down} \end{cases} \qquad \begin{cases} \text{in} \\ \text{out} \end{cases} \qquad \begin{cases} \text{call} \\ \text{put} \end{cases}$$

How do both buyer and writer benefit from the up-and-out call?

- With an appropriate rebate paid upon breaching the upside barrier, this type of barrier options provide the upside exposure for option buyer but at a lower cost.

- The option writer is not exposed to unlimited liabilities when the asset price rises significantly since the liability amount is capped at the payoff of the call at the upstream barrier.

Barrier options are attractive since they give investors more flexibility to express their view on the asset price movement in the option contract design.

In general, embedded barrier feature in a derivative refers to the trigger of certain event (say, knock-out with rebate, accumulation of coupons, doubling of purchase, etc.) upon breaching of a barrier level.

Discontinuity at the barrier (circuit breaker effect upon knock-out)

- Pitched battles often erupt around popular knock-out barriers in currency barrier options and these add much unwanted volatility to the markets.

- George Soros once said "knock-out options relate to ordinary options the way crack relates to cocaine."

Accumulators

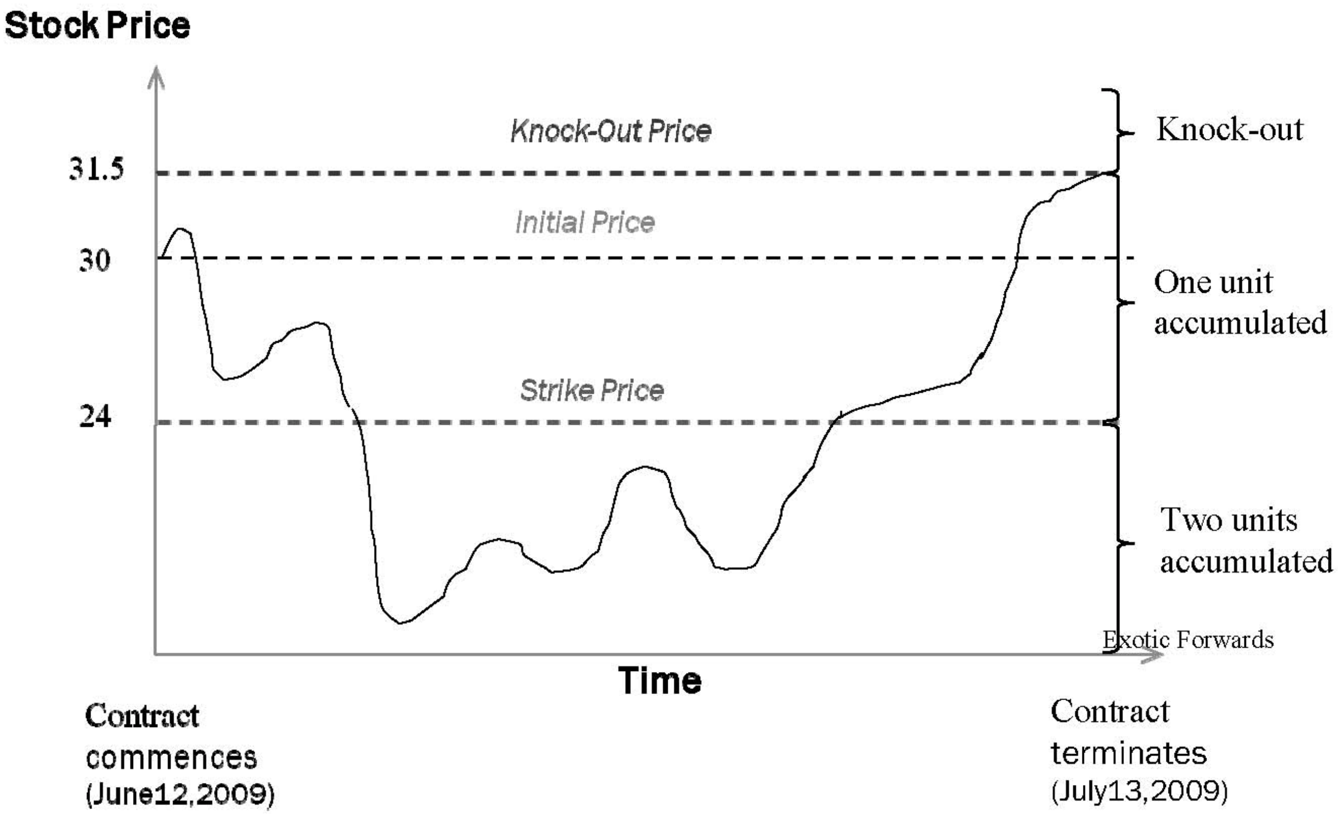
- Entails the investor entering into a commitment to purchase a fixed number of shares per day at a pre-agreed price (the "Accumulator Price"). This Price is set (typically 10-20%) below the market price of the shares at initiation. This is portrayed as the "discount" to the market price of the shares.

Example

Citic Pacific entered into an Australian dollar accumulator as hedges "with a view to minimizing the currency exposure of the company's iron ore mining project in Australia". The company benefits from strengthening in the A$ above the exchange rate of A$1 = US$0.87.

Example of an accumulator on China Life Insurance Company

- *Stock Price Movement of China Life Insurance Company Limited (June 12, 2009 - July 13, 2009)*



Decomposition of an accumulator under immediate settlement

Under the assumption of continuous monitoring of the upper knock-out barrier and immediate settlement of the accumulated stock, one can decompose an accumulator into a portfolio of up-and-out barrier call and put options. Let K = strike price and H = upper knock-out level, the payoff on the observation date t_i is given by

$$\begin{cases} 0 & \text{if } \max_{0 \le \tau \le t_i} S_\tau \ge H \\ S_{t_i} - K & \text{if } \max_{0 \le \tau \le t_i} S_\tau < H \text{ and } S_{t_i} \ge K \\ 2(S_{t_i} - K) & \text{if } \max_{0 \le \tau \le t_i} S_i < H \text{ and } S_{t_i} < K, \end{cases}$$

where $\max_{0 \le t \le t_i} S_\tau$ signifies continuous monitoring of barrier.

- When $S_{t_i} \ge K$, the t_i-maturity put option is out-of-the-money and the t_i-maturity call option has the payoff $S_{t_i} - K$.
- When $S_{t_i} < K$, the call option is out-of-the-money and the put option becomes in-the-money with payoff $K - S_{t_i}$. The two put options are in short position, the payoff is $-2(K - S_{t_i}) = 2(S_{t_i} - K)$.

Pricing formulas

$n =$ total number of observation dates
$c_{uo} =$ up-and-out barrier call option
$p_{uo} =$ up-and-out barrier put option

Fair value of an accumulator (continuous monitoring approximation) $= \sum_{i=1}^{n} c_{uo}(t_i; K, H) - 2p_{uo}(t_i; K, H)$.

- For the t_i-maturity call option, the payoff remains the same, independent of whether the knock-out event occurs on t_i or otherwise. This is an uncommon type of up-and-out call, where the call payoff is adopted as the rebate upon knock-out.

Delayed settlement

- To take care of the delayed delivery of the stocks, the present value of the purchase cost of each unit of stock on date t_i has to be adjusted by the time value of the strike price K paid on the delivery date (several business days after the ending date of the corresponding accumulation period). How to modify the formula?

- More precisely, the underlying asset of the t_i-maturity knock-out option should be the forward contract with delivery price K and maturity date T_i (T_i is a few business days after t_i), $i = 1, 2, \ldots, n$.

1.2 Partial differential equation approach and method of images

Pricing formulation of a European single-asset down-and-out call (continuous monitoring of barrier)

$$\frac{\partial c}{\partial \tau} = \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} + rS \frac{\partial c}{\partial S} - rc, \qquad S > B \text{ and } \tau \in (0, T],$$

subject to

$$\begin{array}{ll} \text{knock-out condition:} & c(B, \tau) = R(\tau) \\ \text{terminal payoff:} & c(S, 0) = \max(S - X, 0), \end{array}$$

Here, B is a down-barrier and $R(\tau)$ is the time-dependent rebate. Normally, B is set to be less than X; otherwise, the barrier is breached even when it is in-the-money. The rebate is set so as to avoid jump discontinuity in the payoff structure.

After applying the transformation of variable: $y = \ln S$, the barrier becomes the vertical line $y = \ln B$ in the (y, τ)-plane. The governing equation becomes

$$\frac{\partial c}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 c}{\partial y^2} + \left(r - \frac{\sigma^2}{2}\right)\frac{\partial c}{\partial y} - rc,$$

defined in the semi-infinite domain: $y > \ln B$ and $\tau \in (0, T]$.

The boundary condition and initial condition, respectively, become

$$c(\ln B, \tau) = R(\tau) \text{ and } c(y, 0) = \max(e^y - X, 0).$$

Since the down-and-out barrier call option becomes a forward contract at $S \to \infty$, the far field boundary condition is

$$\lim_{S\to\infty} c(S, \tau) = S - Xe^{-r\tau}.$$

Recall that the density function

$$u(x,t) = \frac{1}{\sigma\sqrt{2\pi t}} \exp\left(-\frac{(x-\mu t)^2}{2\sigma^2 t}\right)$$

satisfies

$$\frac{\partial u}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} - \mu\frac{\partial u}{\partial x} \quad \text{with} \quad u(x,0^+) = \delta(x).$$

Green function

Setting $\mu = r - \frac{\sigma^2}{2}$, the Green function in the infinite domain: $-\infty < y < \infty$ is given by

$$G_0(y,\tau;\xi) = \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \exp\left(-\frac{(y+\mu\tau-\xi)^2}{2\sigma^2\tau}\right),$$

where $G_0(y,\tau;\xi)$ satisfies the initial condition:

$$\lim_{\tau\to 0^+} G_0(y,\tau;\xi) = \delta(y-\xi).$$

Method of images

Assuming that the Green function in the semi-infinite domain takes the form

$$G(y,\tau;\xi) = G_0(y,\tau;\xi) - H(\xi)G_0(y,\tau;\eta), \quad y > \ln B,$$

we are required to determine $H(\xi)$ and η (in terms of ξ) such that the zero Dirichlet boundary condition $G(\ln B,\tau;\xi) = 0$ is satisfied.

Note that both $G_0(y,\tau;\xi)$ and $H(\xi)G_0(y,\tau;\eta)$ satisfy the differential equation. Also, provided that $\eta \notin (\ln B, \infty)$, then

$$\lim_{\tau\to 0^+} G_0(y,\tau;\eta) = 0 \text{ for all } y > \ln B.$$

By imposing the boundary condition at $y = \ln B$, one observes

$$H(\xi) = \frac{G_0(\ln B, \tau; \xi)}{G_0(\ln B, \tau; \eta)} = \exp\left(\frac{(\xi - \eta)[2(\ln B + \mu\tau) - (\xi + \eta)]}{2\sigma^2\tau}\right).$$

The assumed form of $G(y, \tau; \xi)$ is feasible only if the right hand side becomes a function of ξ only. This can be achieved by the judicious choice of

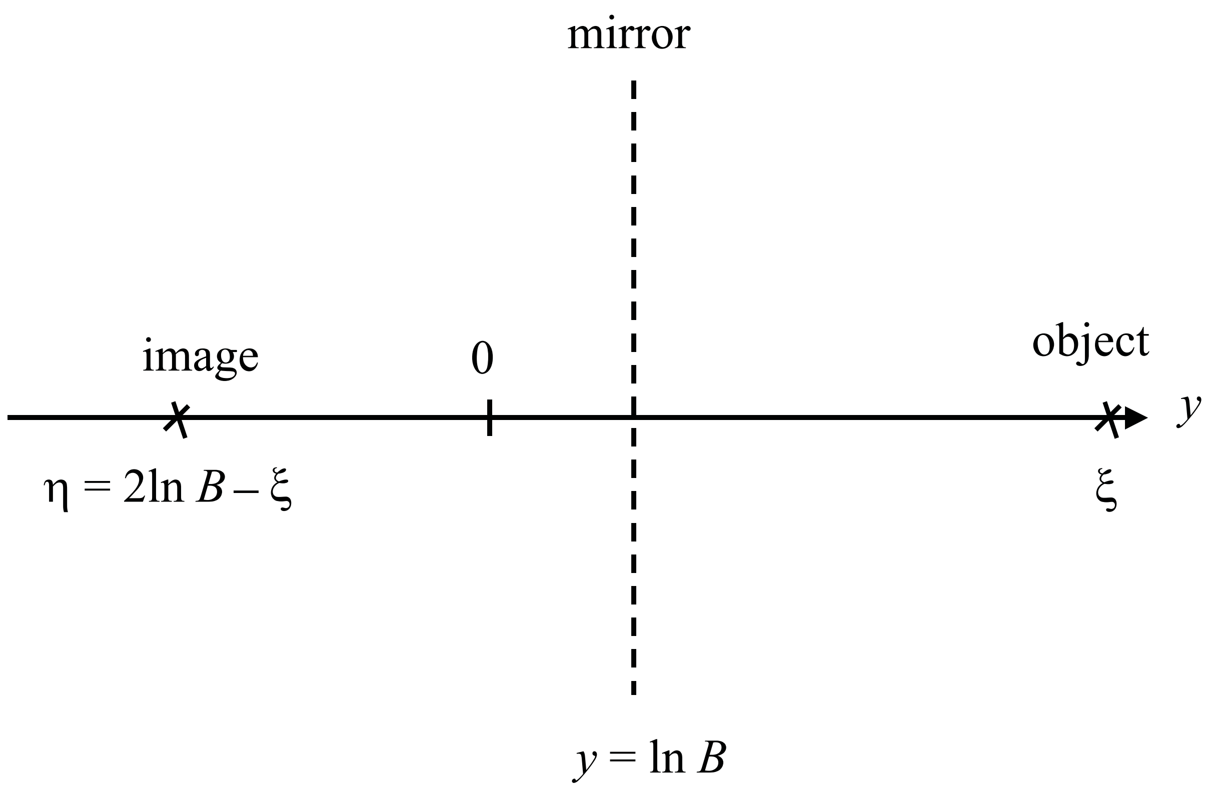
$$\eta = 2\ln B - \xi,$$

so that

$$H(\xi) = \exp\left(\frac{2\mu}{\sigma^2}(\xi - \ln B)\right).$$

- This method works only if μ/σ^2 is a constant, independent of τ. In other words, the method fails when the model parameters are time dependent.

- The parameter η lies outside $(\ln B, \infty)$. Actually, it can be visualized as the *mirror image* of ξ with respect to the barrier $y = \ln B$. In engineering perspective, an image sink of magnitude $H(\xi)$ is placed at the image point $\eta = 2\ln B - \xi$ so that the combination of the source of unit strength at ξ and image sink of strength $H(\xi)$ at η give zero value at the barrier $y = \ln B$.



Pictorial representation of the method of images. The mirror is placed along $y = \ln B$.

Once η and $H(\xi)$ are determined, we have

$$\begin{aligned}
& H(\xi)G_0(y,\tau;\eta) \\
= \;& \exp\left(\frac{2\mu}{\sigma^2}(\xi - \ln B)\right)\frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}}\exp\left(-\frac{[y+\mu\tau-(2\ln B-\xi)]^2}{2\sigma^2\tau}\right) \\
= \;& \left(\frac{B}{S}\right)^{2\mu/\sigma^2}\frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}}\exp\left(-\frac{[(y-\xi)+\mu\tau-2(y-\ln B)]^2}{2\sigma^2\tau}\right).
\end{aligned}$$

In the last expression, the scalar multiple of the Gaussian term is now independent of ξ so that integration with respect to ξ can be performed more effectively.

The Green function in the specified semi-infinite domain: $\ln B < y < \infty$ becomes

$$G(y,\tau;\xi) = \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}}\left\{\exp\left(-\frac{(u-\mu\tau)^2}{2\sigma^2\tau}\right) - \left(\frac{B}{S}\right)^{\lambda}\exp\left(-\frac{(u-2\beta-\mu\tau)^2}{2\sigma^2\tau}\right)\right\},$$

where $u = \xi - y$ and $\beta = \ln B - y = \ln\dfrac{B}{S}$. Also, $\lambda = \dfrac{2\mu}{\sigma^2} = \dfrac{2r}{\sigma^2} - 1 = \delta - 1$ with $\delta = \dfrac{2r}{\sigma^2}$.

Zero-rebate case

We consider the down-and-out barrier call option with zero rebate, where $R(\tau) = 0$, and let $K = \max(B, X)$, so $e^{\xi} - X > 0$ when $\xi \in (\ln K, \infty)$. The price of the zero-rebate European down-and-out call can be expressed as

$$\begin{aligned}
c_{do}(y,\tau) &= \int_{\ln B}^{\infty} \max(e^{\xi} - X, 0) G(y,\tau;\xi)\, d\xi \\
&= \int_{\ln K}^{\infty} (e^{\xi} - X) G(y,\tau;\xi)\, d\xi \\
&= \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_{\ln K/S}^{\infty} (Se^{u} - X) \left[\exp\left(-\frac{(u-\mu\tau)^2}{2\sigma^2\tau}\right)\right. \\
&\qquad \left. - \left(\frac{B}{S}\right)^{2\mu/\sigma^2} \exp\left(-\frac{(u-2\beta-\mu\tau)^2}{2\sigma^2\tau}\right)\right] du, \\
&\qquad\qquad \ln B < y < \infty, \quad \tau > 0.
\end{aligned}$$

The direct evaluation of the integral gives

$$
\begin{aligned}
c_{do}(S,\tau) &= S\left[N(d_1) - \left(\frac{B}{S}\right)^{\delta+1} N(d_3)\right] \\
&\quad -Xe^{-r\tau}\left[N(d_2) - \left(\frac{B}{S}\right)^{\delta-1} N(d_4)\right],
\end{aligned}
$$

where

$$
\begin{aligned}
d_1 &= \frac{\ln\frac{S}{K} + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}, \qquad d_2 = d_1 - \sigma\sqrt{\tau}, \\
d_3 &= d_1 + \frac{2}{\sigma\sqrt{\tau}}\ln\frac{B}{S}, \qquad d_4 = d_2 + \frac{2}{\sigma\sqrt{\tau}}\ln\frac{B}{S}, \qquad \delta = \frac{2r}{\sigma^2}.
\end{aligned}
$$

Suppose we define the modified European call price formula

$$\widetilde{c}_E(S,\tau;X,B) = SN(d_1) - Xe^{-r\tau}N(d_2),$$

then $c_{do}(S,\tau;X,B)$ can be expressed in the following succinct form

$$c_{do}(S,\tau;X,B) = \widetilde{c}_E(S,\tau;X,B) - \left(\frac{B}{S}\right)^{\delta-1}\widetilde{c}_E\left(\frac{B^2}{S},\tau;X,B\right).$$

One can show by direct calculation that the function $\left(\frac{B}{S}\right)^{\delta-1}\widetilde{c}_E\left(\frac{B^2}{S},\tau\right)$ satisfies the Black-Scholes equation identically. Also, we observe

$$\widetilde{c}_E\left(\frac{B^2}{S},0^+\right) = 0, \quad \ln B < S < \infty.$$

The above form allows us to observe readily the satisfaction of the boundary condition: $c_{do}(B,\tau) = 0$, and the terminal payoff condition.

Remarks

1. Closed form analytic price formulas for barrier options with exponential time dependent barrier, $B(\tau) = Be^{-\gamma\tau}$, can also be derived. However, when the barrier level has an arbitrary time dependence, the search for an analytic price formula for the barrier option fails.

2. Closed form price formulas for barrier options can also be obtained for other types of diffusion process followed by the underlying asset price. The types of processes include the square root constant elasticity of variance process (volatility is a power function of the stock price) and the double exponential jump diffusion process (to be discussed in Sec. 1.7).

3. The monitoring period for breaching of the barrier may be limited to only part of the life of the option. The pricing of this type of partial barrier option as a compound option is outlined in Problem 4 in Homework One.

4. Since the nullification of the out-option is compensated by the activation of the in-option counterpart, it is obvious that

$$c_{di}(S, \tau; X, B) + c_{do}(S, \tau; X, B) = c_E(S, \tau; X),$$

valid for either $B < X$ or $B \geq X$. Assuming $B < X$, so that $K = \max(B, X) = X$, the price of a down-and-in call option can be deduced to take the following simple form:

$$c_{di}(S, \tau; X, B) = \left(\frac{B}{S}\right)^{\delta - 1} c_E\left(\frac{B^2}{S}, \tau; X\right).$$

5. With a rebate $B(\tau)$ paid upon knock-out at $S = B$, the value of the rebate provision is given by

$$\int_0^\tau e^{-ru} \frac{\ln \frac{S}{B}}{\sqrt{2\pi}\sigma} \frac{\exp\left(-\frac{\left[\ln \frac{S}{B} + \left(r - \frac{\sigma^2}{2}\right)u\right]^2}{2\sigma^2 u}\right)}{u^{3/2}} R(\tau - u)\ du,$$

where u is the time lapsed from the current time (see Problem 3 in Homework One for the mathematical derivation using the pde approach).

Multistate models

Assume that there are m underlying risky assets, and let S_i denote the price process of asset i, $i = 1, 2, ..., m$. Let I denote an external barrier variable that determines whether the option is nullified or activated when I hits some prescribed threshold level B. Write $n = m + 1$. The terminal payoff function may not involve I.

Under a risk neutral measure Q, the dynamics of S_i, $i = 1, 2, ..., m$, and I are governed by

$$\frac{dS_i}{S_i} = (r - q_i)\ dt + \sigma_i\ dZ_i, \quad i = 1, 2, ..., m,$$
$$\frac{dI}{I} = (r - q_n)\ dt + \sigma_n\ dZ_n.$$

Let ρ_{ij} deonte the correlation coefficient between dZ_i and dZ_j, $i, j = 1, 2, ..., n$.

We apply the following transformation of variables:

$$x_i = \frac{1}{\sigma_i} \ln S_i,\ \ i = 1, 2, ..., m; \quad x_n = \frac{1}{\sigma_n} \ln \frac{I}{B}.$$

Let $V = V(x_1, x_2, ..., x_n)$ denote the price function of a down-and-out option on these m underlying assets and the external barrier variable. The governing partial differential equation for V with a downstream barrier is given by

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij}\frac{\partial^2 V}{\partial x_i \partial x_j} + \sum_{i=1}^{n}\mu_i\frac{\partial V}{\partial x_i} - rV,$$

$$-\infty < x_i < \infty,\ i = 1, 2, ..., m;\ 0 < x_n < \infty,\ \tau > 0,$$

where $\mu_i = \dfrac{r - q_i - \frac{\sigma_i^2}{2}}{\sigma_i}$, $i = 1, 2, ..., n$.

Two-step procedure to find the Green function of the governing equation:

1. Find the Green function for the infinite domain.
2. Use the method of images to find the Green function for the semi-infinite domain.

Green function for the infinite domain

We would like to derive the Green function of the following n-dimensional equation defined in the infinite domain:

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij}\frac{\partial^2 V}{\partial x_i \partial x_j} + \sum_{i=1}^{n}\mu_i\frac{\partial V}{\partial x_i} - rV, \ -\infty < x_i < \infty, \ i = 1, 2, ..., n.$$

Let $y_i = x_i + \mu_i \tau$, $i = 1, 2, ..., n$ and $\phi = e^{r\tau}V$, then ϕ is governed by

$$\begin{aligned}\frac{\partial \phi}{\partial \tau} &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij}\frac{\partial^2 \phi}{\partial y_i \partial y_j} \\ &= \frac{1}{2}\begin{pmatrix}\frac{\partial}{\partial y_1} & \frac{\partial}{\partial y_2} & \cdots \frac{\partial}{\partial y_n}\end{pmatrix} R \begin{pmatrix}\frac{\partial}{\partial y_1} \\ \frac{\partial}{\partial y_2} \\ \vdots \\ \frac{\partial}{\partial y_n}\end{pmatrix}\phi, \quad -\infty < y_i < \infty, \ i = 1, 2, ..., n.\end{aligned}$$

Here, R is the symmetric and semi-positive definite covariance matrix whose entries are $R_{ij} = \rho_{ij}$, $i, j = 1, 2, ..., n$. We rule out the unlikely case where R is singular. As a consequence, R becomes positive definite and all its eigenvalues are strictly positive.

Since R is symmetric and positive definite, there exists an orthonormal matrix Q such that $Q^TQ = QQ^T = I$ and

$$Q^TRQ = \Lambda,$$

where the columns of Q are the normalized eigenvectors of R and Λ is a diagonal matrix whose entries are the (positive) eigenvalues of R. Let $\Lambda^{-1/2}$ denote the inverse of the positive square root of the diagonal matrix Λ. Note that $R = Q\Lambda Q^T$ so that $R^{-1} = Q\Lambda^{-1}Q^T$.

We apply the following linear transformation of variables:

$$\begin{aligned}\boldsymbol{z} = (z_1 \quad z_2 \quad \cdots \quad z_n)^T &= \Lambda^{-1/2}Q^T(y_1 \quad y_2 \quad \cdots \quad y_n)^T \\ &= \Lambda^{-1/2}Q^T\boldsymbol{y},\end{aligned}$$

so that

$$\left(\frac{\partial}{\partial y_1} \quad \frac{\partial}{\partial y_2} \quad \cdots \frac{\partial}{\partial y_n}\right)^T = \Lambda^{-1/2}Q\left(\frac{\partial}{\partial z_1} \quad \frac{\partial}{\partial z_2} \quad \cdots \frac{\partial}{\partial z_n}\right)^T.$$

We then obtain

$$\frac{\partial \phi}{\partial \tau} = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2\phi}{\partial z_i^2}.$$

The fundamental solution to the above n-dimensional prototype equation in the infinite domain is given by

$$\phi(z_1, z_2, ..., z_n, \tau) = \frac{1}{(2\pi\tau)^{n/2}} \exp\left(-\frac{z_1^2 + z_2^2 + ... + z_n^2}{2\tau}\right),$$

$$-\infty < z_i < \infty, \quad i = 1, 2, ..., n.$$

Note that $\boldsymbol{z}^T\boldsymbol{z} = \boldsymbol{y}^T Q \Lambda^{-1} Q^T \boldsymbol{y} = \boldsymbol{y}^T R^{-1} \boldsymbol{y}$, and the Jacobian of the transformation is $\det(\Lambda^{-1/2} Q^T) = \frac{1}{\sqrt{\det R}}$.

The fundamental solution $\phi(y_1, y_2, ..., y_n, \tau)$ in the infinite domain can be expressed as

$$\phi(y_1, y_2, ..., y_n, \tau) = \frac{1}{(2\pi\tau)^{n/2}} \frac{1}{\sqrt{\det R}} \exp\left(-\frac{\boldsymbol{y}^T R^{-1} \boldsymbol{y}}{2\tau}\right),$$

$$-\infty < y_i < \infty, \quad i = 1, 2, ..., n.$$

Multistate Green function in the semi-infinite domain

Corresponding to the homogeneous boundary condition at $x_n = 0$, we seek the Green function in the semi-infinite domain represented by the form:

$$G(\boldsymbol{y}, \tau, \boldsymbol{\xi}) = \frac{e^{-r\tau}}{(2\pi\tau)^{n/2}} \frac{1}{\sqrt{\det R}} \Big[\text{exp} \left(-\frac{1}{2\tau} (\boldsymbol{y} - \boldsymbol{\xi})^T R^{-1} (\boldsymbol{y} - \boldsymbol{\xi}) \right) \\ - H(\boldsymbol{\xi}) \, \text{exp} \left(-\frac{1}{2\tau} (\boldsymbol{y} - \boldsymbol{\eta})^T R^{-1} (\boldsymbol{y} - \boldsymbol{\eta}) \right) \Big],$$

where $\boldsymbol{\eta}$ is to be determined so that the homogeneous condition at $x_n = 0$ is satisfied.

Extended method of images to multistate Green function

Observe that in general the following quantity

$$F = \left. \frac{\exp\left(-\frac{1}{2\tau}(\boldsymbol{y}-\boldsymbol{\xi})^T R^{-1}(\boldsymbol{y}-\boldsymbol{\xi})\right)}{\exp\left(-\frac{1}{2\tau}(\boldsymbol{y}-\boldsymbol{\eta})^T R^{-1}(\boldsymbol{y}-\boldsymbol{\eta})\right)} \right|_{x_n=0}$$

is a function of $\boldsymbol{\xi}$ and τ. We would like to examine whether an appropriate choice of $\boldsymbol{\eta}$ can be found such that F is a function of $\boldsymbol{\xi}$ only. If otherwise, the Green function does not admit the above analytic representation.

We consider

$$
\begin{aligned}
&-\frac{1}{2\tau}\left[(\boldsymbol{y}-\boldsymbol{\xi})^T R^{-1}(\boldsymbol{y}-\boldsymbol{\xi})-(\boldsymbol{y}-\boldsymbol{\eta})^T R^{-1}(\boldsymbol{y}-\boldsymbol{\eta})\right] \\
=&-\frac{1}{2\tau}\{(\boldsymbol{\xi}^T R^{-1}\boldsymbol{\xi}-\boldsymbol{\eta}^T R^{-1}\boldsymbol{\eta}) \\
&-2[y_1\boldsymbol{e}_1^T R^{-1}(\boldsymbol{\xi}-\boldsymbol{\eta})+\ldots+y_{n-1}\boldsymbol{e}_{n-1}^T R^{-1}(\boldsymbol{\xi}-\boldsymbol{\eta})+y_n\boldsymbol{e}_n^T R^{-1}(\boldsymbol{\xi}-\boldsymbol{\eta})]\},
\end{aligned}
$$

and observe that at $x_n = 0$, $y_n = \mu_n\tau$ so that the last term becomes $\mu_n\boldsymbol{e}_n^T R^{-1}(\boldsymbol{\xi}-\boldsymbol{\eta})$, which is independent of τ.

In order to make F to be independent of τ, we choose $\eta_1, \eta_2, \ldots, \eta_n$ such that

$$
\begin{aligned}
&\boldsymbol{\xi}^T R^{-1}\boldsymbol{\xi} = \boldsymbol{\eta}^T R^{-1}\boldsymbol{\eta} \\
&\boldsymbol{e}_i^T R^{-1}(\boldsymbol{\xi}-\boldsymbol{\eta}) = 0, \quad i = 1, 2, \ldots, n-1.
\end{aligned}
$$

The solution to the above algebraic system of equations is found to be

$$\boldsymbol{\eta} = \boldsymbol{\xi} - 2\xi_n R\boldsymbol{e}_n.$$

To verify the claim, we observe that

$$\boldsymbol{e}_i^T R^{-1}(\boldsymbol{\xi} - \boldsymbol{\eta}) = 2\xi_n \boldsymbol{e}_i^T R^{-1} R\boldsymbol{e}_n = 0, \quad i = 1, 2, ..., n-1,$$

and

$$\begin{aligned}\boldsymbol{\eta}^T R^{-1}\boldsymbol{\eta} &= (\boldsymbol{\xi} - 2\xi_n R\boldsymbol{e}_n)^T R^{-1}(\boldsymbol{\xi} - 2\xi_n R\boldsymbol{e}_n) \\ &= \boldsymbol{\xi}^T R^{-1}\boldsymbol{\xi} - 4\xi_n \boldsymbol{e}_n^T \boldsymbol{\xi} + 4\xi_n^2 \boldsymbol{e}_n^T R\boldsymbol{e}_n = \boldsymbol{\xi}^T R^{-1}\boldsymbol{\xi}.\end{aligned}$$

The corresponding value for $H(\boldsymbol{\xi})$ is found to be

$$H(\boldsymbol{\xi}) = \exp\left(\mu_n \boldsymbol{e}_n^T R^{-1}(\boldsymbol{\xi} - \boldsymbol{\eta})\right) = \exp(2\mu_n \xi_n).$$

As a result, the semi-infinite Green function can be expressed as

$$\begin{aligned} & G(\boldsymbol{y}, \tau; \boldsymbol{\xi}) \\ = \; & \frac{e^{-r\tau}}{(2\pi\tau)^{n/2}} \frac{1}{\sqrt{\det R}} \Big[\exp\left(-\frac{1}{2\tau}(\boldsymbol{y} - \boldsymbol{\xi})^T R^{-1}(\boldsymbol{y} - \boldsymbol{\xi})\right) \\ & - \exp(2\mu_n \xi_n) \exp\left(-\frac{1}{2\tau}(\boldsymbol{y} + 2\xi_n R\boldsymbol{e}_n - \boldsymbol{\xi})^T R^{-1}(\boldsymbol{y} + 2\xi_n R\boldsymbol{e}_n - \boldsymbol{\xi})\right) \Big]. \end{aligned}$$

The semi-infinite Green function takes the same form, independent of the external barrier variable being upstream or downstream.

Alternative representation

Observe that $\boldsymbol{y} - \boldsymbol{\xi} = \boldsymbol{x} + \boldsymbol{\mu}\tau - \boldsymbol{\xi} = \boldsymbol{x} - (\boldsymbol{\xi} - \boldsymbol{\mu}\tau)$ so that the drift $\boldsymbol{\mu}$ applied to $\boldsymbol{x}$ has to be swapped in sign when the drift is applied to $\boldsymbol{\xi}$. By virtue of the symmetry property of the Green function, an alternative representation of the semi-infinite Green function is given by

$$
\begin{aligned}
& G(\boldsymbol{y}, \tau; \boldsymbol{\xi}) \\
= \; & \frac{e^{-r\tau}}{(2\pi\tau)^{n/2}} \frac{1}{\sqrt{\det R}} \Big[\exp\left(-\frac{1}{2\tau} (\boldsymbol{y} - \boldsymbol{\xi})^T R^{-1} (\boldsymbol{y} - \boldsymbol{\xi}) \right) \\
& - \exp(-2\mu_n x_n) \exp\left(-\frac{1}{2\tau} (\boldsymbol{y} - 2x_n R\boldsymbol{e}_n - \boldsymbol{\xi})^T R^{-1} (\boldsymbol{y} - 2x_n R\boldsymbol{e}_n - \boldsymbol{\xi}) \right) \Big].
\end{aligned}
$$

This form is preferred since the option price formulas are derived based on the spatial integration with respect to the dummy variables: $\xi_1, \xi_2, ..., \xi_n$.

Two-state option model

We consider the valuation of a European out-option on one underlying risky asset and with single external barrier variable.

- Let θ denote a binary variable which equals 1 or -1, depending on whether the barrier is downstream or upstream, respectively. The interval of definition for ξ_2 is $(0, \infty)$ or $(-\infty, 0)$ corresponding to $\theta = 1$ or -1, respectively.

- Let η be a binary variable which equals 1 or -1, corresponding to the option being a call or a put, respectively. The terminal payoff is given by

$$\max(\eta(S_{1,T} - X), 0)),$$

where X is the strike price and $S_{1,T}$ is the asset price at maturity T.

The price of a European out-option with an external barrier is given by

$$
\begin{aligned}
& V(S_1, I, \tau) \\
= \; & \eta S_1 e^{-q_1\tau}\left[N_2(\eta\widehat{d}_1, -\theta\widehat{e}_1; \eta\theta\rho_{12}) - e^{-2(\mu_2+\rho_{12}\sigma_1)x_2} N_2(\eta\widehat{d}_1', -\theta\widehat{e}_1'; \eta\theta\rho_{12})\right] \\
& - \eta X e^{-r\tau}\left[N_2(\eta\widehat{d}_2, -\theta\widehat{e}_2; \eta\theta\rho_{12}) - e^{-2\mu_2 x_2} N_2(\eta\widehat{d}_2', -\theta\widehat{e}_2'; \eta\theta\rho_{12})\right],
\end{aligned}
$$

where

$$
\widehat{d}_2 = \frac{\ln\frac{S_1}{X} + \mu_1\sigma_1\tau}{\sigma_1\sqrt{\tau}}, \widehat{d}_1 = \widehat{d}_2 + \sigma_1\sqrt{\tau}, \widehat{d}_1' = \widehat{d}_1 - \frac{2\rho_{12}x_2}{\sqrt{\tau}}, \widehat{d}_2' = \widehat{d}_2 - \frac{2\rho_{12}x_2}{\sqrt{\tau}},
$$

$$
\widehat{e}_1 = -\frac{x_2 + (\mu_2 + \rho_{12}\sigma_1)\tau}{\sqrt{\tau}}, \widehat{e}_2 = -\frac{x_2 + \mu_2\tau}{\sqrt{\tau}}, \widehat{e}_1' = \widehat{e}_1 + \frac{2x_2}{\sqrt{\tau}}, \widehat{e}_2' = \widehat{e}_2 + \frac{2x_2}{\sqrt{\tau}}.
$$

1.3 Probabilistic approach: density functions of restricted Brownian motions and first passage time density functions

Realized extremum value of the asset price process

The realized maximum and minimum value of the asset price process from time zero to time t (under continuous monitoring) are defined by

$$\begin{aligned} m_0^t &= \min_{0 \leq u \leq t} S_u \\ M_0^t &= \max_{0 \leq u \leq t} S_u, \end{aligned}$$

respectively. The terminal payoffs of the various types of barrier options can be expressed in terms of m_0^T and M_0^T. For example, consider the down-and-out call and up-and-out put with barrier B (downstream or upstream), their respective terminal payoff can be expressed as

$$c_{do}(S_T, T; X, B) = \max(S_T - X, 0)\mathbf{1}_{\{m_0^T > B\}}$$
$$p_{uo}(S_T, T; X, B) = \max(X - S_T, 0)\mathbf{1}_{\{M_0^T < B\}}.$$

First passage time

Suppose B is the down-barrier, we define τ_B to be the stopping time at which the underlying asset price crosses the barrier and enters into the down-region (stopping event) for the first time:

$$\tau_B = \inf\{t | S_t \leq B\}, \quad S_0 = S.$$

Assume $S > B$ and asset price path continuity, we may express τ_B (commonly called the *first passage time*) as

$$\tau_B = \inf\{t | S_t = B\}.$$

In a similar manner, if B is the up-barrier and $S < B$, we have

$$\tau_B = \inf\{t | S_t \geq B\} = \inf\{t | S_t = B\}.$$

- A random variable $\tau : \Omega \to [0, \infty)$ is called a $\mathcal{F}_t$-stopping time if $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \in [0, \infty)$. That is, it is possible to decide whether $\{\tau \leq t\}$ has occurred on the basis of knowledge of $\mathcal{F}_t$.

Expectation representation of a European down-and-out call

Assuming $S > B$, it is easily seen that $\{\tau_B > T\}$ and $\{m_0^T > B\}$ are equivalent events if B is a down-barrier. By virtue of the risk neutral valuation principle, the price of a down-and-out call at time zero is given by

$$\begin{aligned} c_{do}(S,0;X,B) &= e^{-rT}E_Q[\max(S_T - X, 0)\mathbf{1}_{\{m_0^T > B\}}] \\ &= e^{-rT}E_Q[(S_T - X)\mathbf{1}_{\{S_T > X\}}\mathbf{1}_{\{\tau_B > T\}}]. \end{aligned}$$

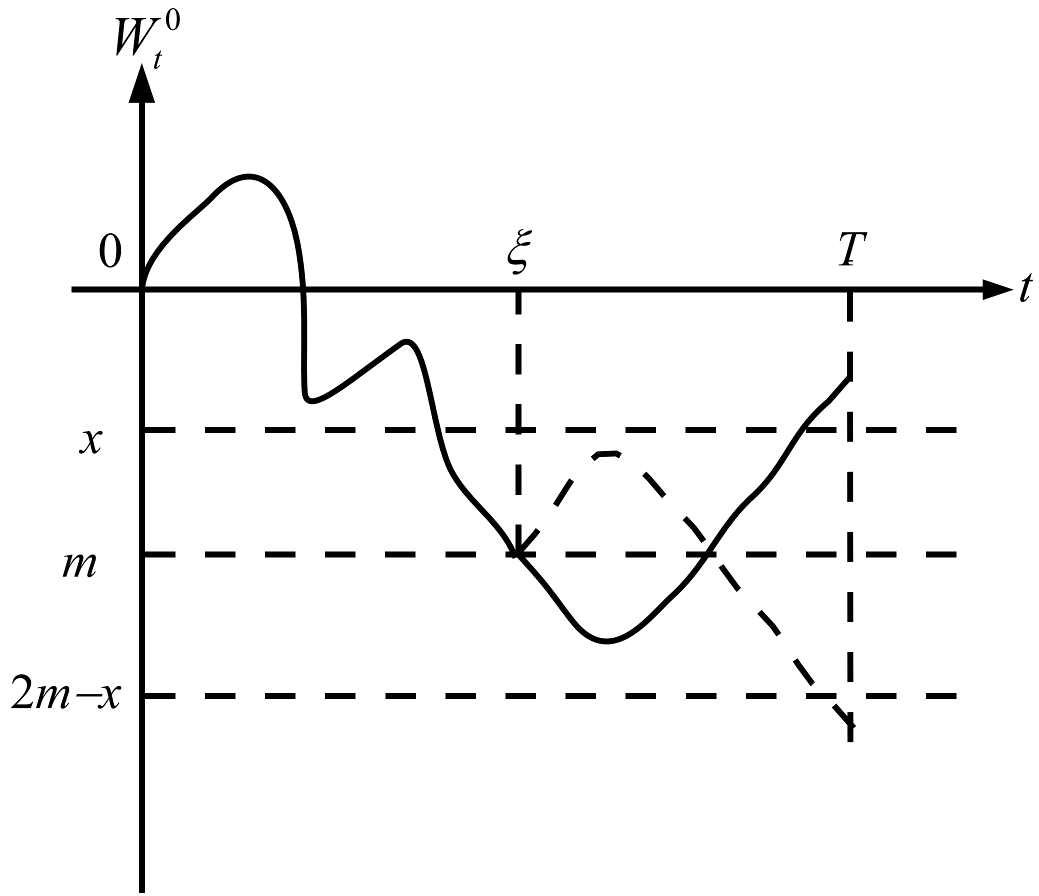
The determination of the price function $c_{do}(S,0;X,B)$ requires the determination of the joint density function of S_T and m_0^T.

Reflection principle

Let W_t^0 (W_t^μ) denote the Brownian motion that starts at zero, with constant volatility σ and zero drift rate (constant drift rate μ). We would like to find $P[m_0^T < m, W_T^\mu > x]$, where $x \geq m$ and $m \leq 0$.

Zero-drift Brownian motion W_t^0

Given that the minimum value m_0^T falls below m, then there exists some time instant $\xi, 0 < \xi < T$, such that ξ is the first time that W_ξ^0 equals m. Here, ξ is seen to be the first passage time to the down-barrier m. As Brownian paths are continuous, there exist some times during which $W_t^0 < m$. In other words, W_t^0 decreases at least below m and then increases at least up to level x (higher than or equal to m) at time T.



Pictorial representation of the reflection principle of the Brownian motion W_t^0. The dotted path after the stopping time ξ is the mirror reflection of the Brownian path at the level m. Suppose W_T^0 ends up at a value higher than x, then the reflected path $\widetilde{W}_T^0$ at time T has a value lower than $2m - x$.

Suppose we define the random process

$$\widetilde{W}_t^0 = \begin{cases} W_t^0 & \text{for } t < \xi \\ 2m - W_t^0 & \text{for } \xi \le t \le T, \end{cases}$$

that is, $\widetilde{W}_t^0$ is the mirror reflection of W_t^0 at the level m within the time interval between ξ and T.

- Note that W_t^0 is $\mathcal{F}_t$-Brownian and the first passage time ξ is a $\mathcal{F}_t$-stopping time. The strong Markov property of a Brownian motion states that for each stopping time ξ, the increment $W_{\xi+u}^0 - W_\xi^0$, $u \ge 0$, is a Brownian motion that is independent of the path history from time zero up to ξ.

- Though the stopping time ξ depends on the path history $\{W_t^0 : 0 \le t \le \xi\}$, it will not affect the Brownian motion at later times. The reflection of the Brownian path dictates that

$$\widetilde{W}_{\xi+u}^0 - \widetilde{W}_\xi^0 = -(W_{\xi+u}^0 - W_\xi^0), \quad u > 0.$$

- By the strong Markov property of Brownian motions, the two Brownian increments have the same distribution, and the distribution has zero mean and variance $\sigma^2 u$. In other words, for every Brownian path that starts at 0, travels at least m units (downward, $m \leq 0$) before T and later travels at least $x - m$ units (upward, $x \geq m$), there is an equally likely path that starts at 0, travels m units (downward, $m \leq 0$) some time before T and travels at least $m - x$ units (further downward, $m \leq x$).

- Hence, $\{W_T^0 > x\} \cap \{m_0^T < m\}$ is equivalent to $\{\widetilde{W}_T^0 < 2m - x\}$. Equivalently, we claim that the two events $\{W_T^0 > x\} \cap \{m_0^T < m\}$ and $\{2m - W_T^0 > x\}$ are equal in probability. We then have

$$
\begin{aligned}
P[W_T^0 > x, m_0^T < m] &= P[2m - W_T^0 > x] \\
&= P[W_T^0 < 2m - x] \quad \text{since } W_t^0 \text{ has zero drift} \\
&= N\left(\frac{2m - x}{\sigma\sqrt{T}}\right), \quad m \leq \min(x, 0).
\end{aligned}
$$

Non-zero drift Brownian motion W_t^μ

We apply the Girsanov Theorem to effect the change of measure for finding the above joint distribution when the Brownian motion has non-zero drift.

Suppose under the measure Q, W_t^μ is a Brownian motion with variance rate σ^2 and drift rate μ. We change the measure from Q to $\widetilde{Q}$ such that W_t^μ becomes a Brownian motion with variance rate σ^2 and zero drift under $\widetilde{Q}$. As an illustration, we consider

$$\begin{aligned}
P[W_T^\mu < y] &= E_Q[\mathbf{1}_{\{W_T^\mu < y\}}] = E_{\widetilde{Q}}\left[\mathbf{1}_{\{W_T^\mu < y\}} \exp\left(\frac{\mu W_T^\mu}{\sigma^2} - \frac{\mu^2 T}{2\sigma^2}\right)\right] \\
&= \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi\sigma^2 T}}\, e^{-\frac{z^2}{2\sigma^2 T}} e^{\frac{\mu z}{\sigma^2}} e^{-\frac{\mu^2 T}{2\sigma^2}}\, dz \\
&= \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi\sigma^2 T}}\, e^{-\frac{(z-\mu T)^2}{2\sigma^2 T}}\, dz = N\left(\frac{y - \mu T}{\sigma\sqrt{T}}\right).
\end{aligned}$$

- Note that the Radon-Nikodym derivative: $\exp\left(\frac{\mu W_T^\mu}{\sigma^2} - \frac{\mu^2 T}{2\sigma^2}\right)$ is appended in transforming from Q to $\widetilde{Q}$. Also, the density function of W_T^μ under $\widetilde{Q}$ is given by $\widetilde{Q}[W_T^\mu \in dz] = \frac{1}{\sqrt{2\pi\sigma^2 T}}\, e^{-z^2/2\sigma^2 T} dz$.

- When the μ-drift Brownian motion W_T^μ does not go beyond y, the zero-drift Brownian motion W_t^0 does not go beyond $y - \mu T$. This intuition gives $P[W_T^\mu < y] = N\left(\frac{y - \mu T}{\sigma\sqrt{T}}\right)$.

- In order that we can apply the reflection principle that is applicable under the zero-drift case, we perform all expectation calculations under $\widetilde{Q}$ whereby W_t^μ becomes a zero-drift Brownian motion.

Recall that the two events $\{W_T^0 > x\} \cap \{m_0^T < m\}$ and $\{2m - W_T^0 > x\}$ are equal in distribution. We transform from Q to $\widetilde{Q}$ by appending $\exp\left(\frac{\mu}{\sigma^2}(2m - W_T^\mu) - \frac{\mu^2 T}{2\sigma^2}\right)$ under which $2m - W_T^\mu$ becomes zero-drift Brownian motion. Also, W_T^μ is a Brownian motion with zero-drift under $\widetilde{Q}$. For $m \leq \min\{x, 0\}$, we then have

$$
\begin{aligned}
& P[W_T^\mu > x, m_0^T < m] \\
&= E_{\widetilde{Q}}\left[\mathbf{1}_{\{2m - W_T^\mu > x\}} \exp\left(\frac{\mu}{\sigma^2}(2m - W_T^\mu) - \frac{\mu^2 T}{2\sigma^2}\right)\right] \\
&= e^{\frac{2\mu m}{\sigma^2}} E_{\widetilde{Q}}\left[\mathbf{1}_{\{W_T^\mu < 2m - x\}} \exp\left(-\frac{\mu}{\sigma^2} W_T^\mu - \frac{\mu^2 T}{2\sigma^2}\right)\right] \\
&= e^{\frac{2\mu m}{\sigma^2}} \int_{-\infty}^{2m-x} \frac{1}{\sqrt{2\pi\sigma^2 T}} \exp\left(-\frac{z^2}{2\sigma^2 T} - \frac{\mu z}{\sigma^2} - \frac{\mu^2 T}{2\sigma^2}\right) dz \\
&= e^{\frac{2\mu m}{\sigma^2}} \int_{-\infty}^{2m-x} \frac{1}{\sqrt{2\pi\sigma^2 T}} \exp\left(-\frac{(z + \mu T)^2}{2\sigma^2 T}\right) dz \\
&= e^{\frac{2\mu m}{\sigma^2}} N\left(\frac{2m - x + \mu T}{\sigma\sqrt{T}}\right).
\end{aligned}
$$

Consider the restricted Brownian motion W_t^μ that has a downstream barrier m over the period $[0, T]$ so that $m_0^T > m$. Given that W_t^μ does not breach the barrier m, we would like to derive the joint distribution

$$P[W_T^\mu > x, m_0^T > m], \quad \text{and} \quad m \leq \min(x, 0).$$

By applying the law of total probabilities, we obtain

$$\begin{aligned}
& P[W_T^\mu > x, m_0^T > m] \\
= \; & P[W_T^\mu > x] - P[W_T^\mu > x, m_0^T < m] \\
= \; & N\left(\frac{-x + \mu T}{\sigma\sqrt{T}}\right) - e^{\frac{2\mu m}{\sigma^2}} N\left(\frac{2m - x + \mu T}{\sigma\sqrt{T}}\right), \quad m \leq \min(x, 0). \quad \text{(A)}
\end{aligned}$$

By setting $m = x$, and since $W_T^\mu > m$ is implicitly implied from $m_0^T > m$, we obtain the following distribution function for m_0^T:

$$P[m_0^T > m] = N\left(\frac{-m + \mu T}{\sigma\sqrt{T}}\right) - e^{\frac{2\mu m}{\sigma^2}} N\left(\frac{m + \mu T}{\sigma\sqrt{T}}\right).$$

Extension to upstream barrier

Consider the restricted Brownian motion W_t^μ that has an upstream barrier M over the period $[0, T]$ so that $M_0^T < M$, the joint distribution function of W_T^μ and M_0^T can be deduced using the following relation between M_0^T and m_0^T:

$$M_0^T = \max_{0 \le t \le T} (\sigma Z_t + \mu t) = - \min_{0 \le t \le T} (-\sigma Z_t - \mu t),$$

where Z_t is the standard Brownian motion. Since $-Z_t$ has the same distribution as Z_t, the distribution of the maximum value of W_t^μ is the same as that of the negative of the minimum value of $W_t^{-\mu}$.

By swapping $-\mu$ for $\mu, -M$ for m and $-y$ for x, we obtain

$$\begin{aligned}
&P[-W_T^\mu > -y, -M_0^T < -M] \\
=\,&P[W_T^\mu < y, M_0^T > M] \\
=\,&e^{\frac{2\mu M}{\sigma^2}} N\left(\frac{y - 2M - \mu T}{\sigma\sqrt{T}}\right), \quad M \geq \max(y, 0).
\end{aligned}$$

In a similar manner, we obtain

$$\begin{aligned}
&P[W_T^\mu < y, M_0^T < M] \\
= \quad & P[W_T^\mu < y] - P[W_T^\mu < y, M_0^T > M] \\
= \quad & N\left(\frac{y - \mu T}{\sigma\sqrt{T}}\right) - e^{\frac{2\mu M}{\sigma^2}} N\left(\frac{y - 2M - \mu T}{\sigma\sqrt{T}}\right), \quad M \geq \max(y, 0). \quad \text{(B)}
\end{aligned}$$

Lastly, by setting $y = M$, we obtain the following distribution function for M_0^T:

$$P[M_0^T < M] = N\left(\frac{M - \mu T}{\sigma\sqrt{T}}\right) - e^{\frac{2\mu M}{\sigma^2}} N\left(-\frac{M + \mu T}{\sigma\sqrt{T}}\right).$$

Density function of a restricted Brownian motion with one-sided downstream barrier

We define $f_{down}(x, m, T)$ to be the density function of W_T^μ with the downstream barrier m, where $m \leq \min(x, 0)$, that is,

$$f_{down}(x, m, T)\, dx = P[W_T^\mu \in dx, m_0^T > m].$$

By differentiating eq. (A) with respect to x and swapping the sign, we obtain

$$\begin{aligned} & f_{down}(x, m, T) \\ = \; & \frac{1}{\sigma\sqrt{T}}\left[n\left(\frac{x-\mu T}{\sigma\sqrt{T}}\right) - e^{\frac{2\mu m}{\sigma^2}} n\left(\frac{x-2m-\mu T}{\sigma\sqrt{T}}\right)\right] \mathbf{1}_{\{m \leq \min(x,0)\}}. \end{aligned}$$

Extension to upstream barrier

Similarly, we define $f_{up}(x, M, T)$ to be the density function of W_T^μ with the upstream barrier M, where $M > \max(y, 0)$. By differentiating eq. (B) with respect to y, we obtain

$$\begin{aligned}
& P[W_T^\mu \in dy, M_0^T < M] \\
= \; & f_{up}(y, M, T)\, dy \\
= \; & \frac{1}{\sigma\sqrt{T}}\left[n\left(\frac{y-\mu T}{\sigma\sqrt{T}}\right) - e^{\frac{2\mu M}{\sigma^2}} n\left(\frac{y-2M-\mu T}{\sigma\sqrt{T}}\right)\right] dy\,\mathbf{1}_{\{M \geq \max(y,0)\}}.
\end{aligned}$$

Transition density function of a restricted Geometric Brownian motion with downstream barrier

Suppose the asset price S_t follows the Geometric Brownian motion under the risk neutral measure such that $\ln \dfrac{S_t}{S} = W_t^\mu$, where S is the asset price at time zero and the drift rate $\mu = r - \dfrac{\sigma^2}{2}$. Let $\psi(S_T; S, B)$ denote the transition density of the asset price S_T at time T given the asset price S at time zero and conditional on $S_t > B$ for $0 \leq t \leq T$. Here, B is the downstream barrier. From the density function $f_{\text{down}}(x, m, T)$, we deduce that $\psi(S_T; S, B)$ is given by

$$
\begin{aligned}
\psi(S_T; S, B) \;=\; & \frac{1}{\sigma\sqrt{T} S_T}\left[n\left(\frac{\ln \frac{S_T}{S} - \left(r - \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}} \right) \right. \\
& \left. - \left(\frac{B}{S}\right)^{\frac{2r}{\sigma^2}-1} n\left(\frac{\ln \frac{S_T}{S} - 2\ln\frac{B}{S} - \left(r - \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}} \right) \right].
\end{aligned}
$$

First passage time density function of a Brownian motion

Let $Q(u;m)$ denote the density function of the first passage time at which the downstream barrier m is first hit by the Brownian path W_t^{μ}, that is, $Q(u;m)\,du = P[\tau_m \in du]$.

We determine the distribution function $P[\tau_m > u]$ by observing that $\{\tau_m > u\}$ and $\{m_0^u > m\}$ are equivalent events. This gives

$$
\begin{aligned}
P[\tau_m > u] &= P[m_0^u > m] \\
&= N\left(\frac{-m+\mu u}{\sigma\sqrt{u}}\right) - e^{\frac{2\mu m}{\sigma^2}} N\left(\frac{m+\mu u}{\sigma\sqrt{u}}\right).
\end{aligned}
$$

The first passage time density function $Q(u;m)$ associated with the downstream barrier is then given by

$$\begin{aligned} Q(u;m)\,du &= P[\tau_m \in du] \\ &= -\frac{\partial}{\partial u}\left[N\left(\frac{-m+\mu u}{\sigma\sqrt{u}}\right) - e^{\frac{2\mu m}{\sigma^2}} N\left(\frac{m+\mu u}{\sigma\sqrt{u}}\right)\right] du\,\mathbf{1}_{\{m<0\}} \\ &= \frac{-m}{\sqrt{2\pi\sigma^2 u^3}} \exp\left(-\frac{(m-\mu u)^2}{2\sigma^2 u}\right) du\,\mathbf{1}_{\{m<0\}}. \end{aligned}$$

Let $Q(u;M)$ denote the first passage time density associated with the upstream barrier M. In a similar manner, we obtain

$$\begin{aligned} Q(u;M) &= -\frac{\partial}{\partial u}\left[N\left(\frac{M-\mu u}{\sigma\sqrt{u}}\right) - e^{\frac{2\mu M}{\sigma^2}} N\left(-\frac{M+\mu u}{\sigma\sqrt{u}}\right)\right] \mathbf{1}_{\{M>0\}} \\ &= \frac{M}{\sqrt{2\pi\sigma^2 u^3}} \exp\left(-\frac{(M-\mu u)^2}{2\sigma^2 u}\right) \mathbf{1}_{\{M>0\}}. \end{aligned}$$

Now, we consider $\ln \dfrac{S_t}{S}$ to be a Brownian motion with drift $r - \dfrac{\sigma^2}{2}$. We write B as the option barrier, either upstream or downstream. The normalized barrier under the Brownian motion is $\ln \dfrac{B}{S}$. When the barrier is downstream (upstream), we have $\ln \dfrac{B}{S} < 0 \left(\ln \dfrac{B}{S} > 0 \right)$. The combined first passage time density function is given by

$$Q(u; B) = \frac{\left| \ln \frac{B}{S} \right|}{\sqrt{2\pi\sigma^2 u^3}} \exp \left(-\frac{\left[\ln \frac{B}{S} - \left(r - \frac{\sigma^2}{2} \right) u \right]^2}{2\sigma^2 u} \right).$$

Suppose a rebate $R(t)$ is paid to the option holder upon breaching the barrier at level B by the asset price path at time t. Since the expected rebate payment over the time interval $[u, u + du]$ is given by $R(u)Q(u; B)\, du$, so the expected present value of the rebate is given by

$$\text{rebate value } = \int_0^T e^{-ru} R(u) Q(u; B)\, du.$$

When $R(t) = R_0$, a constant value, direct integration of the above integral gives

$$\begin{aligned}\text{rebate value} \quad &= \quad R_0 \left[\left(\frac{B}{S} \right)^{\alpha_+} N \left(\delta \frac{\ln \frac{B}{S} + \beta T}{\sigma \sqrt{T}} \right) \right. \\ &\qquad \left. + \left(\frac{B}{S} \right)^{\alpha_-} N \left(\delta \frac{\ln \frac{B}{S} - \beta T}{\sigma \sqrt{T}} \right) \right],\end{aligned}$$

where

$$\beta = \sqrt{\left(r - \frac{\sigma^2}{2} \right)^2 + 2r\sigma^2}, \quad \alpha_\pm = \frac{r - \frac{\sigma^2}{2} \pm \beta}{\sigma^2},$$
$$\delta = \text{sign} \left(\ln \frac{S}{B} \right).$$

Here, δ is a binary variable indicating whether the barrier is down-stream ($\delta = 1$) or upstream ($\delta = -1$).

Two-sided barriers

We take the initial position $X_0 = 0$. Let $g(x, t; \ell, u)$ denote the density function of the restricted Brownian motion X_t with two-sided absorbing barriers at $x = \ell$ and $x = u$, where the barriers are positioned such that $\ell < 0 < u$.

Recall that $X_t = \ln \dfrac{S_t}{S}$, and if L and U are the absorbing barriers of the asset price process S_t, respectively, then $\ell = \ln \dfrac{L}{S}$ and $u = \ln \dfrac{U}{S}$.

The partial differential equation formulation for $g(x, t; \ell, u)$ is given by (see Problem 3.8 in Kwok's text)

$$\frac{\partial g}{\partial t} = -\mu \frac{\partial g}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 g}{\partial x^2}, \quad \ell < x < u, \quad t > 0,$$

with the homogeneous boundary conditions:

$$g(\ell, t) = g(u, t) = 0 \quad \text{and} \quad g(x, 0^+) = \delta(x).$$

Both $x = \ell$ and $x = u$ are the absorbing barriers (equivalent to say "particles are removed from the system once these barriers are hit"), so the probability of staying at each of these barriers is zero.

Defining the transformation

$$g(x,t) = e^{\frac{\mu x}{\sigma^2} - \frac{\mu^2 t}{2\sigma^2}} \widehat{g}(x,t),$$

we observe that $\widehat{g}(x,t)$ satisfies the forward Fokker-Planck equation with zero drift:

$$\frac{\partial \widehat{g}}{\partial t}(x,t) = \frac{\sigma^2}{2} \frac{\partial^2 \widehat{g}}{\partial x^2}(x,t).$$

Note that the factor $e^{\frac{\mu x}{\sigma^2} - \frac{\mu^2 t}{2\sigma^2}}$ resembles the Radon-Nikodym derivative: $\exp\left(\frac{\mu W_t^\mu}{\sigma^2} - \frac{\mu^2 t}{2\sigma^2}\right)$.

The auxiliary conditions for $\widehat{g}(x,t)$ are seen to remain the same as those for $g(x,t)$. Without the barriers, the infinite-domain fundamental solution to the above equation is known to be

$$\phi(x,t) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{x^2}{2\sigma^2 t}\right).$$

Like the one-sided barrier case, we try to add extra terms to the above solution such that the homogeneous boundary conditions at $x = \ell$ and $x = u$ are satisfied.

Method of images revisited

We attempt to add the pair of negative terms $-\phi(x-2\ell,t)$ and $-\phi(x-2u,t)$ whereby

$$[\phi(x,t)-\phi(x-2\ell,t)]\Big|_{x=\ell}=0 \quad \text{and} \quad [\phi(x,t)-\phi(x-2u,t)]\Big|_{x=u}=0.$$

Note that $\phi(x-2\ell,t)$ and $\phi(x-2u,t)$ correspond to the fundamental soluton with initial condition: $\delta(x-2\ell)$ and $\delta(x-2u)$, respectively. Writing the above partial sum with three terms as

$$\widehat{g}_3(x,t)=\phi(x,t)-\phi(x-2\ell,t)-\phi(x-2u,t),$$

we observe that the homogeneous boundary conditions are not yet satisfied since

$$\widehat{g}_3(\ell,t)=-\phi(x-2u,t)\Big|_{x=\ell}\neq 0$$
$$\widehat{g}_3(u,t)=-\phi(2-2\ell,t)\Big|_{x=u}\neq 0.$$

To nullify the non-zero value of $-\phi(x-2u,t)\Big|_{x=\ell}$ and $-\phi(x-2\ell,t)\Big|_{x=u}$, we add a new pair of positive terms $\phi(x-2(u-\ell),t)$ and $\phi(x+2(u-\ell),t)$. Similarly, we write the partial sum with five terms as

$$\widehat{g}_5(x,t) = \widehat{g}_3(x,t) + \phi(x-2(u-\ell),t) + \phi(x+2(u-\ell),t),$$

and observe that

$$\widehat{g}_5(\ell,t) = \phi(x-2(u-\ell),t)\Big|_{x=\ell} \neq 0$$
$$\widehat{g}_5(u,t) = \phi(x+2(u-\ell),t)\Big|_{x=u} \neq 0.$$

Whenever a new pair of positive terms or negative terms are added, the value of the partial sum at $x=\ell$ and $x=u$ becomes closer to zero. In a recursive manner, we add successive pairs of positive and negative terms so as to come closer to the satisfaction of the homogeneous boundary conditions at $x=\ell$ and $x=u$.

- The two absorbing barriers may be visualized as a pair of mirrors with the object placed at the origin (see Figure on the next page).

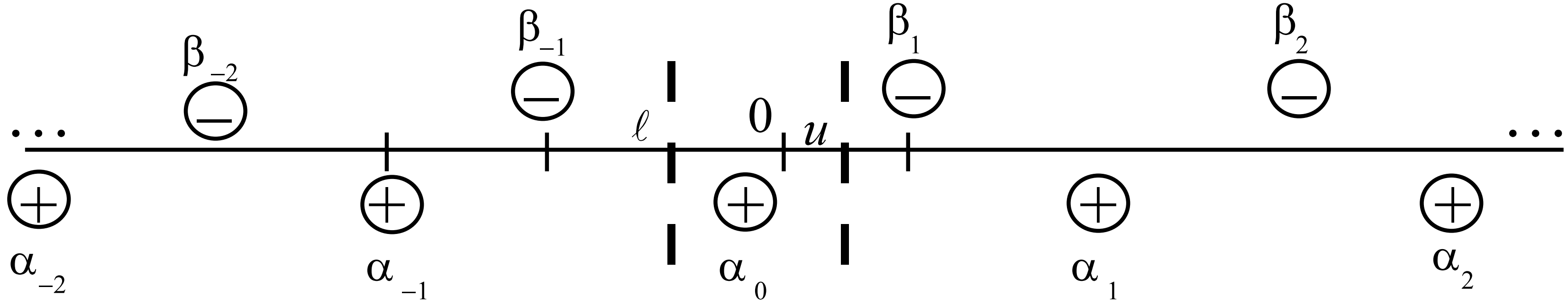
- The source at the origin generates a sink at $x = 2\ell$ due to the mirror at $x = \ell$ and another sink at $x = 2u$ due to the mirror at $x = u$.

- To continue, the sink at $x = 2\ell$ $(x = 2u)$ generates a source at $x = 2(u - \ell)$ $[x = 2(\ell - u)]$ due to the mirror at $x = u$ $(x = \ell)$.

- As the procedure continues, this leads to the sum of an infinite number of positive and negative terms.

Infinite number of images

The double-mirror analogy provides the intuitive argument showing why $g(x,t)$ involves an infinite number of terms.



A graphical representation of an infinite number of sources and sinks due to a pair of absorbing barriers (mirrors) with the object placed at the origin. The positions of the sources and sinks are $\alpha_j = 2(u-\ell)j$ and $\beta_j = 2\ell + 2(u-\ell)(j-1)$, respectively, $j = 0, \pm 1, \pm 2, \ldots$.

The solution to $g(x,t)$ is deduced to be

$$\begin{aligned}
g(x,t) &= e^{\frac{\mu x}{\sigma^2} - \frac{\mu^2 t}{2\sigma^2}} \widehat{g}(x,t) \\
&= e^{\frac{\mu x}{\sigma^2} - \frac{\mu^2 t}{2\sigma^2}} \sum_{n=-\infty}^{\infty} [\phi(x - 2n(u-\ell,t),t) - \phi(x - 2\ell - 2n(u-\ell),t)] \\
&= \frac{e^{\frac{\mu x}{\sigma^2} - \frac{\mu^2 t}{2\sigma^2}}}{\sqrt{2\pi\sigma^2 t}} \sum_{n=-\infty}^{\infty} \left[\exp\left(-\frac{[x - 2n(u-\ell)]^2}{2\sigma^2 t} \right) \right. \\
&\qquad \left. - \exp\left(-\frac{[(x-2\ell) - 2n(u-\ell)]^2}{2\sigma^2 t} \right) \right] \\
&= \sum_{n=-\infty}^{\infty} \left[\frac{e^{\frac{2\mu}{\sigma^2} n(u-\ell)}}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{[x - \mu t - 2n(u-\ell)]^2}{2\sigma^2 t} \right) \right. \\
&\qquad \left. - \frac{e^{\frac{2\mu}{\sigma^2}[\ell + n(u-\ell)]}}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{[(x - \mu t - 2\ell) - 2n(u-\ell)]^2}{2\sigma^2 t} \right) \right].
\end{aligned}$$

Alternative representation: eigenfunction expansion

Let $P(x,t;x_0,t_0)$ denote the transition density function of the restricted Brownian process $W_t^\mu = \mu t + \sigma Z_t$ with two absorbing barriers at $x = 0$ and $x = \ell$, where $\ell > 0$. We take the convenience of setting one of the absorbing barriers to be $x = 0$. Using the method of separation of variables, the solution to $P(x,t;x_0,t_0)$ admits the following eigenfunction expansion

$$P(x,t;x_0,t_0) = e^{\frac{\mu}{\sigma^2}(x-x_0)-\frac{\mu^2}{2\sigma^2}(t-t_0)}\frac{2}{\ell}\sum_{k=1}^{\infty} e^{-\lambda_k(t-t_0)} \sin\frac{k\pi x}{\ell} \sin\frac{k\pi x_0}{\ell}$$

where the eigenvalues are given by

$$\lambda_k = \frac{k^2\pi^2\sigma^2}{2\ell^2}.$$

$P(x,t;x_0,t_0)$ satisfies the forward Fokker-Planck equation with auxiliary conditions: $P(0,t) = P(\ell,t) = 0$ and $P(x,t_0^+;x_0,t_0) = \delta(x-x_0)$, $0 < x_0 < \ell$.

Proof (Separation of variables):

The eigenfunctions $\sin\frac{k\pi x}{\ell}$, $k = 1, 2, \ldots$, are seen to satisfy the homogeneous boundary conditions at $x = 0$ and $x = \ell$. The solution in the form of eigenfunction expansion assumes an infinite series of the form

$$P(x, t; x_0, t_0) = \sum_{k=1}^{\infty} A_k e^{-\lambda_k (t-t_0)} \sin\frac{k\pi x}{\ell},$$

where the eigenvalues λ_k, $k = 1, 2, \ldots$, are determined so that each term $e^{-\lambda_k(t-t_0)}\sin\frac{k\pi x}{\ell}$ satisfies the governing differential equation: $\frac{\partial P}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 P}{\partial x^2}$. This requires that the eigenvalues should be given by

$$-\lambda_k = -\frac{\sigma^2}{2}\frac{k^2\pi^2}{\ell^2} \quad \text{or} \quad \lambda_k = \frac{k^2\pi^2\sigma^2}{2\ell^2},\ k = 1, 2, \ldots.$$

Lastly, we determine the constants A_k, $k = 1, 2, \ldots$, using the initial condition:

$$\delta(x - x_0) = \sum_{k=1}^{\infty} A_k \sin \frac{k\pi x}{\ell}.$$

By virtue of the orthogonality of the eigenfunctions, we have

$$\int_0^{\ell} A_k \sin^2 \frac{k\pi x}{\ell} dx = \int_0^{\ell} \delta(x - x_0) \sin \frac{k\pi x}{\ell} dx, \;\; 0 < x_0 < \ell.$$

Lastly, we obtain $A_k = \dfrac{2}{\ell} \sin \dfrac{k\pi x_0}{\ell}$, $k = 1, 2, \ldots$.

The solution of the density function can be expressed either as an infinite series of Gaussian kernel functions using the method of images or the eigenfunction expansion approach. These two solutions are equivalent by virtue of the Poisson summation formula. It has been shown that the Gaussian kernel series has a faster rate of convergence to the exact value with respect to the number of terms n used.

The density function of the first passage time to either barrier is defined by

$$q(t;\ell,u)\,dt = P(\min(\tau_\ell,\tau_u) \in dt),$$

where $\tau_\ell = \inf\{t|X_t = \ell\}$ and $\tau_u = \inf\{t|X_t = u\}$. We consider the corresponding distribution function

$$P(\min(\tau_\ell,\tau_u) \le t) = 1 - P(\min(\tau_\ell,\tau_u) > t) = 1 - \int_\ell^u g(x,t)\,dx$$

where $\int_\ell^u g(x,t)\,dx$ is the total probability that W_t^μ stays within (ℓ,u). The density function of the first passage time is given by

$$q(t;\ell,u) = -\frac{\partial}{\partial t}\int_\ell^u g(x,t)\,dx = \frac{1}{\sqrt{2\pi\sigma^2 t^3}}$$
$$\left\{\sum_{n=-\infty}^{\infty}[2n(u-\ell)-\ell]\exp\left(\frac{\mu\ell}{\sigma^2}-\frac{\mu^2 t}{2\sigma^2}\right)\exp\left(-\frac{(2n(u-\ell)-\ell]^2}{2\sigma^2 t}\right)\right.$$
$$\left. + [2n(u-\ell)+u]\exp\left(\frac{\mu u}{\sigma^2}-\frac{\mu^2 t}{2\sigma^2}\right)\exp\left(-\frac{[2n(u-\ell)+u]^2}{2\sigma^2 t}\right)\right\}.$$

Exit time to a barrier

The density function of the exit time to the respective lower barrier and upper barrier are defined by

$$q^{-}(t;\ell,u)\,dt = P(\tau_\ell \in dt, \tau_\ell < \tau_u)$$
$$q^{+}(t;\ell,u)\,dt = P(\tau_u \in dt, \tau_u < \tau_\ell).$$

Since $\{\tau_\ell \in dt, \tau_\ell < \tau_u\} \cup \{\tau_u \in dt, \tau_u < \tau_\ell\} = \{\min(\tau_\ell, \tau_u) \in dt\}$, we deduce that

$$q(t;\ell,u) = q^{-}(t;\ell,u) + q^{+}(t;\ell,u).$$

A judicious decomposition of $q(t;\ell,u)$ into its two components would suggest

$$q^-(t;\ell,u) = \frac{1}{\sqrt{2\pi\sigma^2 t^3}} \sum_{n=-\infty}^{\infty} [2n(u-\ell)-\ell] \exp\left(\frac{\mu\ell}{\sigma^2} - \frac{\mu^2 t}{2\sigma^2}\right) \exp\left(-\frac{[2n(u-\ell)-\ell]^2}{2\sigma^2 t}\right)$$

$$q^+(t;\ell,u) = \frac{1}{\sqrt{2\pi\sigma^2 t^3}} \sum_{n=-\infty}^{\infty} [2n(u-\ell)+u] \exp\left(\frac{\mu u}{\sigma^2} - \frac{\mu^2 t}{2\sigma^2}\right) \exp\left(-\frac{[2n(u-\ell)+u]^2}{2\sigma^2 t}\right).$$

To show the claim, we define the probability flow by

$$J(x,t) = \mu g(x,t) - \frac{\sigma^2}{2}\frac{\partial g}{\partial x}(x,t),$$

where the negative sign is chosen for the diffusion term since the probability flow is in the negative direction when $\dfrac{\partial g}{\partial x} > 0$ (diffusion tends to make probability concentration to spread evenly). Also, recall that

$$q(t;\ell,u) = -\frac{\partial}{\partial t}\int_\ell^u g(x,t)\,dx = \int_\ell^u -\frac{\partial g}{\partial t}\,dx.$$

Since g satisfies the forward Fokker-Planck equation, we have

$$q(t;\ell,u) = \int_\ell^u \left(\mu\frac{\partial g}{\partial x} - \frac{\sigma^2}{2}\frac{\partial^2 g}{\partial x^2}\right)\,dx = J(u,t) - J(\ell,t).$$

One may visualize the probability flow across $x = \ell$ and $x = u$ as

$$\begin{aligned} -J(\ell,t) &= P(\tau_\ell \in dt, \tau_\ell < \tau_u) \\ J(u,t) &= P(\tau_u \in dt, \tau_u < \tau_\ell). \end{aligned}$$

Note that $J(\ell, t)$ is negative since the probability flow is outward from the interval (ℓ, u) through $x = \ell$ along the negative x-direction.

The exit time densities $q^-(t; \ell, u)$ and $q^+(t; \ell, u)$ are seen to satisfy

$$q^-(t; \ell, u) = -J(\ell, t) = -\left[\mu g(x,t) - \frac{\sigma^2}{2}\frac{\partial g}{\partial x}(x,t)\right]\Bigg|_{x=\ell}$$

$$q^+(t; \ell, u) = J(u, t) = \mu g(x,t) - \frac{\sigma^2}{2}\frac{\partial g}{\partial x}(x,t)\Bigg|_{x=u}.$$

Rebate payment

Suppose a rebate $R^-(t)\,[R^+(t)]$ is paid when the lower (upper) barrier is first breached during the life of the option, $0 < t < T$, the value of the rebate portion of the double-barrier option is then given by

$$\text{rebate value } = \int_0^T e^{-r\xi}[R^-(\xi)q^-(\xi; \ell, u) + R^+(\xi)q^+(\xi; \ell, u)]\, d\xi.$$

1.4 Approximation of probabilities of hitting time dependent barriers

Let $r(t)$ and $\sigma^2(t)$ be time dependent interest rate and exogenous volatility process. Under the risk neutral measure Q, the dynamics of S_t is governed by

$$\frac{dS_t}{S_t} = r(t)\ dt + \sigma(t)\ dW_t.$$

- The time dependence on $\sigma(t)$ can be resolved by applying the standard time-changed argument for Brownian motions. Instead of following the calendar time, we adopt the time frame where the physical time is shortened when the volatility level is high so that the volatility adjusted time is kept constant.
- At the end, the resulting barrier option price formulas can be deduced from those of "constant volatility σ" to "time dependent $\sigma(t)$, $0 \leq t \leq T$," by simply swapping $\sigma\sqrt{T}$ with $\sqrt{\int_0^T \sigma^2(t)\ dt}$, which is the total variance over $[0, T]$.

Assuming constant volatility σ, we obtain

$$\ln \frac{S_t}{S_0} = \sigma W_t - \frac{\sigma^2 t}{2} + R(t),$$

where W_t is the standard Brownian motion with $W_0 = 0$ and

$$R(t) = \int_0^t r(u)\ du.$$

Let $H(t)$ be the time dependent upstream barrier. We assume $S_0 < H(t)$ and define

$$f(t) = \frac{\ln \frac{H(t)}{S_0} + \frac{\sigma^2 t}{2} - R(t)}{\sigma},$$

then "S_t hitting $H(t)$" is equivalent to "W_t hitting $f(t)$".

Reference "Pricing barrier options with time dependent coefficients," G.O. Roberts and C.F. Shortland, *Mathematical Finance*, vol.7 (1997) p.83-93.

Let $\psi(S_T)$ denote the terminal payoff of the derivative security at time T. The value of the up-and-in barrier option is given by

$$V = e^{-R(T)} \int_{-\infty}^{\infty} \psi\left(S_0 e^{\sigma x + R(T) - \frac{\sigma^2 T}{2}}\right) Q\left[\tau_f^W < T \middle| W_T = x\right] n\left(\frac{x}{\sqrt{T}}\right) \frac{1}{\sqrt{T}}\, dx,$$

where $n(x)$ is the density of the standard normal variable and τ_f^W is the first passage time that the Brownian motion W hits the up-stream barrier f from below. Here, Q is a risk neutral measure.

Note that $Q[\tau_f^W < T|W_T = x] = 1$ when $x > H'$, where $H' = f(T)$ [equivalently, the terminal value of stock price is above $f(T)$].

We decompose V into two parts:

$$V_1 = e^{-R(T)} \int_{-\infty}^{H'} \psi\left(S_0 e^{\sigma x + R(T) - \frac{\sigma^2 T}{2}}\right) Q\left[\tau_f^W < T \middle| W_T = x\right] n\left(\frac{x}{\sqrt{T}}\right) \frac{1}{\sqrt{T}}\, dx;$$

$$V_2 = e^{-R(T)} \int_{H'}^{\infty} \psi\left(S_0 e^{\sigma x + R(T) - \frac{\sigma^2 T}{2}}\right) n\left(\frac{x}{\sqrt{T}}\right) \frac{1}{\sqrt{T}}\, dx.$$

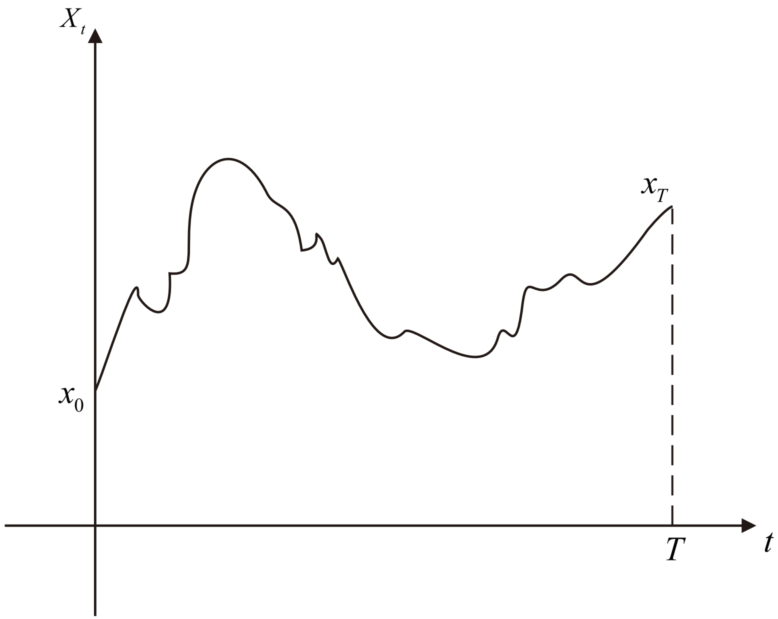
For $x < H'$, we use the Brownian bridge to compute $Q\left[\tau_f^W < T \middle| W_T = x\right]$.

Brownian bridge technique

The Brownian bridge X_t from $(0, x_0)$ to (T, x_T) can be visualized as a time-changed Brownian motion (see Appendix):

$$X_t = x_0 + \frac{x_T - x_0}{T} t + (T - t) W^*_{\frac{t}{T(T-t)}}, \quad \text{where } W^*_0 = 0.$$

Note that $X_0 = x_0$ and $X_T = x_T$, while $W^*_{\frac{t}{T(T-t)}}$ is the time-changed Brownian motion with variance rate equals $\dfrac{t}{T(T-t)}$.



In terms of the Brownian bridge X from $(0,0)$ to (T,x), we observe

$$Q[\tau_f^W < T \Big| W_T = x] = Q[\tau_f^X < T].$$

The adjusted time s is related to the calendar time t via

$$s = \frac{t}{T(T-t)} \Longleftrightarrow t = \frac{sT^2}{1+sT}.$$

Also, $T - t = \dfrac{T}{1+sT}$. Given $x_0 = 0$, when X_t hits $f(t)$, we relate W_s^* with f via $X_t = f(t) = \dfrac{x}{T}t + \dfrac{T}{1+sT}W_s^*$.

Note that $\dfrac{t}{T}\Big/\dfrac{T}{1+sT} = s$, so that X_t hits $f(t)$ when the unit-variance time changed Brownian motion hits the value

$$\frac{1+sT}{T}f\left(\frac{sT^2}{1+sT}\right) - xs.$$

Also, as t evolves from 0 to T, s evolves from 0 to ∞. Hence,

$$\{X_t \text{ hitting } f(t) \text{ in } (0,T)\} \Leftrightarrow \{W_s^* \text{ ever hitting } g(s)\},$$

where

$$g(s) = \frac{1+sT}{T} f\left(\frac{sT^2}{1+sT}\right) - xs.$$

We then have

$$Q[\tau_f^W < T | W_T = x] = Q[\tau_g^{W^*} < \infty].$$

Except for a few examples of $g(t)$ will the exact value of $Q[\tau_g^W < \infty]$ be available, so approximation techniques are required.

Hazard rate of the first exit time across an arbitrary boundary $\lambda(t)$

Define $h_\lambda(t)$ by

$$h_\lambda(t) = \lim_{\varepsilon \to 0^+} \frac{Q[\tau_\lambda^W \leq t + \varepsilon | \tau_\lambda^W > t]}{\varepsilon},$$

where

$$\tau_\lambda^W = \inf_{t>0} \{t : W_t \geq \lambda(t)\}.$$

In other words, $\varepsilon h_\lambda(t)$ gives the probability that W hits $\lambda(t)$ during the time interval $(t, t+\varepsilon]$, conditional on no hitting up to time t.

Survival function and hazard rate

Let $S(t)$ be the probability that there is no hitting of the barrier up to time t, that is, $S(t) = Q[\tau_g^W > t]$. we then have

$$\frac{S(t) - S(t + \Delta t)}{S(t)} = h_\lambda(t)\Delta t.$$

Taking the limit $\Delta t \to 0$, we obtain

$$\frac{dS(t)}{S(t)} = -h_\lambda(t)\ dt \text{ with } S(0) = 1.$$

Solving the differential equation, we have

$$S(t) = \exp\left(-\int_0^t h_\lambda(u)\ du\right).$$

Recall that the probability of ever hitting the boundary g from below is given by

$$Q[\tau_g^W < \infty] = 1 - \exp\left(-\int_0^\infty h_g(s)\ ds\right).$$

Let g be a C^2 boundary, $\{\ell_t(\cdot), t \geq 0\}$ and $\{u_t(\cdot), t \geq 0\}$ be collections of C^2 functions, where ℓ_t and u_t observe

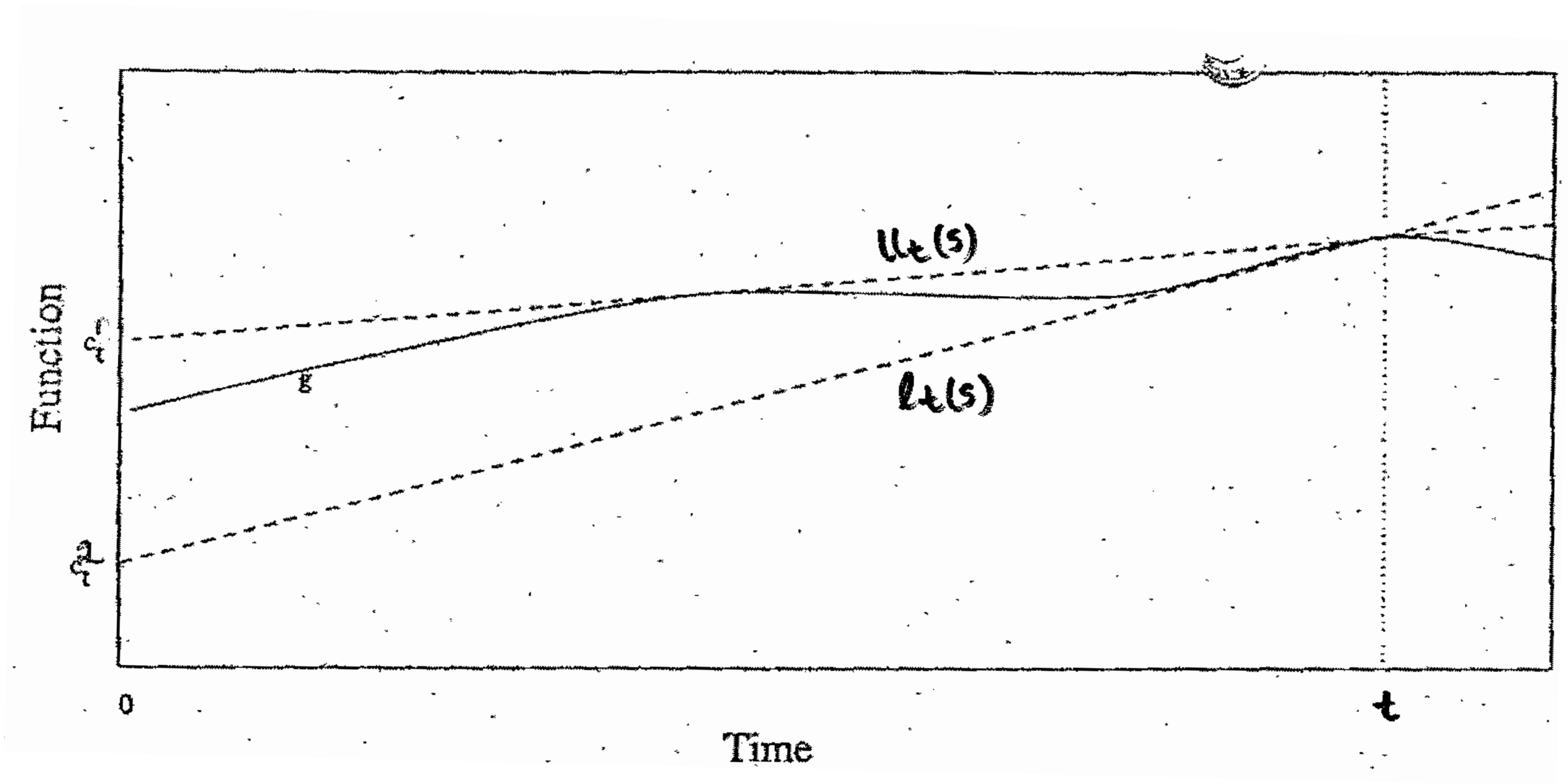
$$\ell_t(s) \leq g(s) \leq u_t(s) \quad \text{for all } s \leq t,$$

while $\ell_t(t) = g(t) = u_t(t)$. It is obvious that

$$Q[\tau_u^W < \infty] < Q[\tau_g^W < \infty] < Q[\tau_\ell^W < \infty].$$

Normally, we choose ℓ_t and u_t such that the exit distribution properties are known. A convenient choice is given by

$$\begin{aligned}
m_t^1 &= \inf_{s<t} \frac{g(t)-g(s)}{t-s}, & m_t^2 &= \sup_{s<t} \frac{g(t)-g(s)}{t-s}, \\
c_t^1 &= g(t) - m_t^1 t, & c_t^2 &= g(t) - m_t^2 t, \\
u_t(s) &= m_t^1 s + c_t^1, & l_t(s) &= m_t^2 s + c_t^2.
\end{aligned}$$



$g(s)$ lies within the envelope of straight lines bounded by $u_t(s)$ and $\ell_t(s)$. The line $\ell_t(s)$ [$u_t(s)$] has intercept c_t^2 (c_t^1) at $s = 0$.

With the above choices of u_t and ℓ_t, we obtain

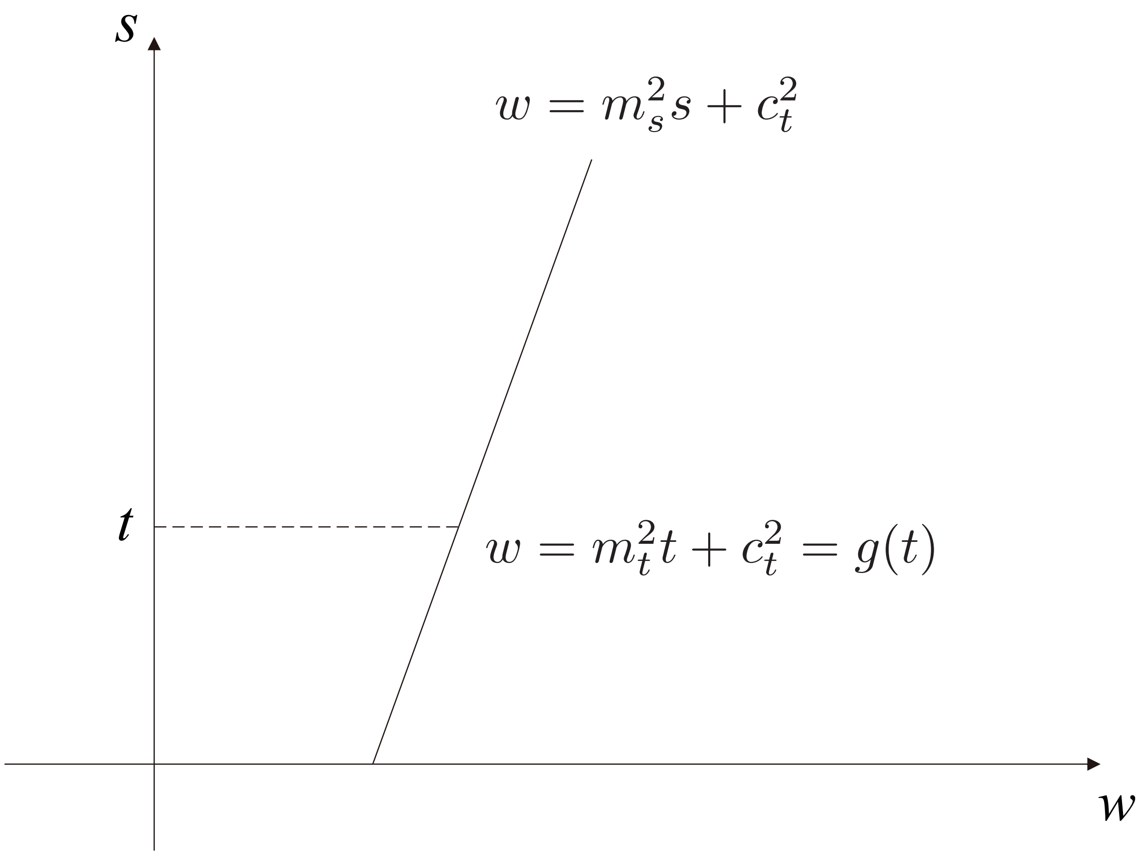
$$h_{l_t}(t) = \frac{\max(0, c_t^2) n\left(\frac{g(t)}{\sqrt{t}}\right)}{t\left[N\left(\frac{g(t)}{\sqrt{t}}\right) - e^{-2c_t^2 m_t^2} N\left(\frac{g(t)-2c_t^2}{\sqrt{t}}\right)\right]}$$

and

$$h_{u_t}(t) = \frac{\max(0, c_t^1) n\left(\frac{g(t)}{\sqrt{t}}\right)}{t\left[N\left(\frac{g(t)}{\sqrt{t}}\right) - e^{-2c_t^1 m_t^1} N\left(\frac{g(t)-2c_t^1}{\sqrt{t}}\right)\right]}.$$

Note that if c_t^1 or c_t^2 are negative, the straight lines used for comparison are negative at $t = 0$, so the first exit time is 0 by definition. Thus, the hazard rate at time t must be zero. The factors $\max(0, c_t^2)$ and $\max(0, c_t^1)$ incorporate these considerations. As a remark, the choice of a bounding line that has negative intercept would lead to meaningless approximation, where $Q[\tau_\ell^W < \infty] = 1$ for sure.

Proof of the formula for $h_{\ell_t}(t)$ *for* $c_t^2 > 0$



Recall the unconditional first passage time density function of the Brownian motion with up-barrier B and drift μ:

$$Q(t; B) = \frac{B}{\sqrt{2\pi}\sigma t^{3/2}} \exp\left(-\frac{(B-\mu t)^2}{2\sigma^2 t}\right).$$

The probability of reaching the time dependent barrier $w = m_t^2 s + c_t^2$ under zero-drift Brownian motion is equivalent to that of reaching the fixed barrier c_t^2 under the Brownian motion with drift $-m_t^2$.

In the present problem, we have

$$B = c_t^2 \text{ and } B - \mu t = c_t^2 + m_t^2 t = g(t).$$

The probability for the Brownian motion with drift μ not hitting B is given by

$$P\left[M_0^T < B\right] = N\left(\frac{B - \mu T}{\sigma\sqrt{T}}\right) - e^{\frac{2\mu B}{\sigma^2}} N\left(-\frac{B + \mu T}{\sigma\sqrt{T}}\right).$$

In the current problem, we set $B = c_t^2$, $\mu = -m_t^2$ and $\sigma = 1$.

The probability that the up-stream barrier is not hit from below up to time t is given by

$$\begin{aligned} Q\left[\tau_g^{W^*} < t\right] &= N\left(\frac{c_t^2 + m_t^2 t}{\sqrt{t}}\right) - e^{-2m_t^2 c_t^2} N\left(-\frac{c_t^2 - m_t^2 t}{\sqrt{t}}\right) \\ &= N\left(\frac{g(t)}{\sqrt{t}}\right) - e^{-2m_t^2 c_t^2} N\left(\frac{g(t) - 2c_t^2}{\sqrt{t}}\right). \end{aligned}$$

Recall that $Q[\tau_g^W < t]h_{\ell_t}(t)$ is the unconditional first passage time density to the barrier g. Lastly, we put all the results together to give

$$h_{\ell_t}(t) = \frac{c_t^2 n\left(\frac{g(t)}{\sqrt{t}}\right)}{tQ\left[\tau_g^{W^*} < t\right]} = \frac{c_t^2 n\left(\frac{g(t)}{\sqrt{t}}\right)}{t\left[N\left(\frac{g(t)}{\sqrt{t}}\right) - e^{-2c_t^2 m_t^2} N\left(\frac{g(t)-2c_t^2}{\sqrt{t}}\right)\right]}.$$

Bound on V_1

$$
\begin{aligned}
& e^{-R(T)} \int_{-\infty}^{H'} \psi \left(S_0 e^{\sigma x + R(T) - \frac{\sigma^2 T}{2}} \right) \\
& \qquad \left[1 - \exp\left(-\int_0^{\infty} h_{l_t}(t)\ dt \right) \right] n\left(\frac{x}{\sqrt{T}} \right) \frac{1}{\sqrt{T}}\ dx \\
\geq\ \ & V_1 \geq e^{-R(T)} \int_{-\infty}^{H'} \psi \left(S_0 e^{\sigma x + R(T) - \frac{\sigma^2 T}{2}} \right) \\
& \qquad \left[1 - \exp\left(-\int_0^{\infty} h_{u_t}(t)\ dt \right) \right] n\left(\frac{x}{\sqrt{T}} \right) \frac{1}{\sqrt{T}}\ dx.
\end{aligned}
$$

Note that both h_{l_t} and h_{u_t} depend on the value of x through $g(t)$ and cannot be factored out of the integrals.

Hazard rate tangent approximation

We adopt the approximation

$$h_g(t) \approx h_{T_t}(t),$$

where $T_t(s) = [g(t) - tg'(t)] + g'(t)s$ is the tangent to $g(s)$ at $s = t$.

By setting $c_t = g(t) - tg'(t)$ and $m_t = g'(t)$, the corresponding hazard rate is

$$h_{T_t}(t) = \frac{\max(0, g(t) - sg'(t))n\left(\frac{g(t)}{\sqrt{t}}\right)}{t\left[N\left(\frac{g(t)}{\sqrt{t}}\right) - e^{-2g'(t)[g(t)-tg'(t)]N\left(\frac{2tg'(t)-g(t)}{\sqrt{t}}\right)}\right]}.$$

The analytic approximation of V_1 according to the hazard rate tangent approximation is given by

$$e^{-R(T)} \int_{-\infty}^{H'} \psi\left(S_0 e^{\sigma x + R(T) - \frac{\sigma^2 T}{2}}\right)\left[1 - \exp\left(-\int_0^\infty h_{T_t}(t)\ dt\right)\right] n\left(\frac{x}{\sqrt{T}}\right)\frac{1}{\sqrt{T}}\ dx.$$

Illustrative example

We model the stock price by

$$dS_t = r(t)S_t\ dt + \sigma S_t\ dW_t,$$

with $S_0 = 10$, $\sigma = 0.1$, and $r(t) = r_0 + ae^{-t}$, with $r_0 = 0.1$ and $a = 0.05$. This represents the case where the risk-free interest rate has been perturbed and will return to its equilibrium rate in an exponential decay. We consider the analytic approximation to the value of an European up-and-in call option.

We set the strike price $c = 11$, maturity date $T = 1$ and knock-in upstream boundary at level $H = 12$.

We note that

$$R(t) = \int_0^t r(s)\ ds = rt + a(1 - e^{-t}),$$

so that

$$f(t; H) = \frac{\ln(H/S_0) + \sigma^2 t/2 - rt - a(1 - e^{-t})}{\sigma}.$$

For $T = 1$, we obtain

$$g(t) = \frac{(1+t)\ln(H/S_0) + \sigma^2 t/2 - rt - a(1+t)(1 - e^{-t/(1+t)})}{\sigma} - xt.$$

The lower and upper bounds l_t and u_t both have particularly simple forms due to mild upward concavity of $g(t)$:

$$l_t(s) = [g(t) - tg'(t)] + g'(t)s,$$
$$u_t(s) = g(0) + \frac{g(t) - g(0)}{t} s;$$

where $l_t(s)$ is the tangent line through $(t, g(t))$; and $u_t(s)$ is the line joining $(0, g(0))$ and $(t, g(t))$.

Finally, we define

$$c' = f(1;c) = \frac{\ln \frac{c}{S_0} - R(1) + \frac{\sigma^2}{2}}{\sigma} \quad \text{and} \quad H' = f(1;H) = \frac{\ln \frac{H}{S_0} - R(1) + \frac{\sigma^2}{2}}{\sigma}.$$

It is necessary to evaluate

$$\begin{aligned} V = \; & e^{-R(1)} \int_{c'}^{H'} (S_0 e^{\sigma x + R(1) - \sigma^2/2} - c') Q[\tau_g^{W^*} < \infty] n(x)\, dx \\ & + \; e^{-R(1)} \int_{H'}^{\infty} (S_0 e^{\sigma x + R(1) - \sigma^2/2} - c') n(x)\, dx \\ = \; & e^{-R(1)} \int_{c'}^{H'} (S_0 e^{\sigma x + R(1) - \sigma^2/2} - c') Q[\tau_g^{W^*} < \infty] n(x)\, dx \\ & + \; e^{-R(1)} \int_{H'}^{\infty} (S_0 e^{\sigma x + R(1) - \sigma^2/2} - c') n(x)\, dx. \end{aligned}$$

Using the bounding technique for the first integral and evaluating the second integral analytically, we obtain

$$0.516758 \leq V \leq 0.517968.$$

Mathematical Appendices – Brownian bridge

Gaussian processes

- A Gaussian process $X(t)$, $t \geq 0$, is a stochastic process that for arbitrary set of times: $0 < t_1 < t_2 < ..., t_n$, the random variables $X(t_1)$, $X(t_2)$,..., $X(t_n)$ are jointly normally distributed.

- Let $\Delta(t)$ be a non-random function of time, and define the integral

$$I(t) = \int_0^t \Delta(s)\ dW(s),$$

where $W(t)$ is a Brownian motion. It can be shown that $I(t)$ is a Gaussian process.

Brownian bridge as a Gaussian process

Let $W(t)$ be a Brownian motion. Fix $T > 0$, we define the Brownian bridge from 0 to 0 over $[0, T]$ to be the conditional Brownian motion

$$X(t) = W(t) - \frac{t}{T} W(T), \quad 0 \leq t \leq T.$$

- Note that $X(0) = X(T) = 0$, and $\frac{t}{T} W(T)$ as a function of t is the line from $(0, 0)$ to $(T, W(T))$.
- Since $W(T)$ enters into $X(t)$ for $0 \leq t \leq T$, so the Brownian bridge $X(t)$ is not adapted to the filtration $\mathcal{F}(t)$ generated by $W(t)$.
- For $0 < t_1 < t_2 < ... < t_n < T$, the random variables

$$X(t_1) = W(t_1) - \frac{t_1}{T} W(T), ..., X(t_n) = W(t_n) - \frac{t_n}{T} W(T),$$

  are jointly normal since $W(t_1), ..., W(t_n), W(T)$ are jointly normal. Therefore, the Brownian bridge from 0 to 0 is a Gaussian process.

- The Brownian bridge is a Gaussian process whose increments are not independent.

The mean of $X(t)$ is easily seen to be

$$m(t) = E[X(t)] = E\left[W(t) - \frac{t}{T}W(T)\right] = 0.$$

For $t_1, t_2 \in (0, T)$, the covariance function of $X(t)$ is given by

$$\begin{aligned}
c(t_1, t_2) &= E\left[\left(W(t_1) - \frac{t_1}{T}W(T)\right)\left(W(t_2) - \frac{t_2}{T}W(T)\right)\right] \\
&= E[W(t_1)W(t_2)] - \frac{t_1}{T}E[W(t_2)W(T)] \\
&\quad - \frac{t_2}{T}E[W(t_1)W(T)] + \frac{t_1 t_2}{T^2}E[W(T)^2] \\
&= \min(t_1, t_2) - \frac{2t_1 t_2}{T} + \frac{t_1 t_2}{T} \\
&= \min(t_1, t_2) - \frac{t_1 t_2}{T}.
\end{aligned}$$

It is not necessary to fix the starting point and ending point to be both at 0. More generally, we consider a Brownian bridge that starts at a at time 0 and ends at b at time T. The Brownian bridge from a to b on $(0, T)$ is the process

$$X^{a\to b}(t) = a + \frac{(b-a)t}{T} + X(t), \quad 0 \le t \le T,$$

where $X(t) = X^{0\to 0}(t)$.

Adding a non-random function to a Gaussian process gives another Gaussian process. The mean function becomes

$$m^{a\to b}(t) = a + \frac{(b-a)t}{T},$$

while the covariance function is not affected.

Brownian bridge as a scaled stochastic integral (time-changed Brownian motion)

Consider

$$Y(t) = \begin{cases} (T-t)\displaystyle\int_0^t \frac{1}{T-u}\, dW(u) & 0 \le t \le T \\ 0 & t = T \end{cases},$$

we would like to show that $Y(t)$ is a continuous Gaussian process on $[0,T]$ and has the same distribution as the Brownian bridge from 0 to 0 over $[0,T]$.

The process $Y(t)$ is adapted to the filtration generated by the Brownian motion $W(t)$. Also, the stochastic differential of $Y(t)$ is given by

$$dY(t) = -\frac{Y(t)}{T-t}\, dt + dW(t).$$

Effect of the drift term

- Suppose Y is positive as t approaches T, the drift term $-\frac{Y(t)}{T-t}dt$ becomes infinitely large in absolute value and is negative. This drives $Y(t)$ toward zero almost instantaneously.

- Similarly, suppose Y is negative, the drift term becomes infinitely large and positive, and this again drives $Y(t)$ toward zero.

- One can show rigorously that as $t \to T^-$, the process $Y(t)$ converges to zero almost surely.

Proof [$Y(t)$ and $X(t)$ are equal in distribution]

The integral

$$I(t) = \int_0^t \frac{1}{T-u}\, dW(u), \quad t < T,$$

is a Gaussian process. For $0 < t_1 < t_2 < ... < t_n < T$, the random variables:

$$Y(t_1) = (T-t_1)I(t_1),\ Y(t_2) = (T-t_2)I(t_2), ...,\ Y(t_n) = (T-t_n)I(t_n),$$

are jointly normal. Hence, $Y(t)$ is a Gaussian process.

The mean and covariance functions of I are

$$\begin{aligned} m^I(t) &= 0 \\ C^I(t_1,t_2) &= \int_0^{\min(t_1,t_2)} \frac{1}{(T-u)^2}\, du \\ &= \frac{1}{T-\min(t_1,t_2)} - \frac{1}{T}, \quad \text{for } t_1,t_2 \in [0,T]. \end{aligned}$$

Hence, $\text{var}(I(t)) = \dfrac{t}{T(T-t)}$, $0 \le t \le T$.

Similarly, the covariance function of Y is

$$
\begin{aligned}
C^Y(t_1, t_2) &= (T - t_1)(T - t_2)\left[\frac{1}{T - \min(t_1, t_2)} - \frac{1}{T}\right] \\
&= (T - t_1)(T - t_2)\frac{\min(t_1, t_2)}{T[T - \min(t_1, t_2)]} \\
&= \begin{cases} \dfrac{(T - t_2)t_1}{T} = t_1 - \dfrac{t_1 t_2}{T} & \text{if } t_1 \leq t_2 \\ \dfrac{(T - t_1)t_2}{T} = t_2 - \dfrac{t_1 t_2}{T} & \text{if } t_1 > t_2 \end{cases} \\
&= \min(t_1, t_2) - \frac{t_1 t_2}{T}, \quad t_1, t_2 \in [0, T).
\end{aligned}
$$

Remark

Now, variance of $X(t)$ is $t - \dfrac{t^2}{T} = \dfrac{t(T - t)}{T}$. In terms of time-changed Brownian motion, we may write

$$X(t) = (T - t)W^*_{\frac{t}{T(T-t)}}, \; W^*_0 = 0.$$

1.5 Barrier-type derivatives under stochastic interest rates

Up-and-in call option under stochastic interest rates

Let S_{max} be the realized maximum of the stock price over $[0, T]$ and H be the upstream barrier. Consider

$$c^{ui} = P(0,T)E_{Q_T}\left[(S_T - K)^+ \mathbf{1}_{\{S_{\text{max}}>H\}}\right],$$

where Q_T is the forward measure with riskfree unit par discount bond price $P(t,T)$ as the numeraire. By the martingale property, the time-t price of the contingent claim X_t is given by

$$\frac{X_t}{P(t,T)} = E^t_{Q_T}\left[\frac{X_T}{P(T,T)}\right] = E^t_{Q_T}[X_T].$$

We write

$$A = E_{Q_T}\left[S_T \mathbf{1}_{\{S_T>K\}} \mathbf{1}_{\{S_{\text{max}}>H\}}\right], B = Q_T[S_T > K, S_{\text{max}} > H].$$

Let γ denote the first passage time that the stock price hits the up-barrier H from below. Note that $\{S_{\text{max}} > H\}$ and $\{\gamma \leq T\}$ are equivalent events.

We let $\ell_t = \ln S_t$ and observe that the distribution of S_T would depend on the level of interest rate at the first passage time, so

$$
\begin{aligned}
A &= E_{Q_T}\left[S_T \mathbf{1}_{\{\ln S_T > \ln K\}} \mathbf{1}_{\{\gamma \le T\}}\right] = E_{Q_T}\left[e^{\ell_T} \mathbf{1}_{\{\ell_T > \ln K\}} \mathbf{1}_{\{\gamma \le T\}}\right] \\
&= \int_0^T \int_{-\infty}^{\infty} E_{Q_T}\left[e^{\ell_T} \mathbf{1}_{\{\ell_T > \ln K\}} \mathbf{1}_{\{\gamma \le T\}} | r_\gamma = r, \gamma = s\right] Q_T\left[r_\gamma \in dr, \gamma \in ds\right], \\
B &= \int_0^T \int_{-\infty}^{\infty} Q_T[\ell_T > \ln K | r_\gamma = r, \gamma = s] Q_T\left[r_\gamma \in dr, \gamma \in ds\right].
\end{aligned}
$$

We integrate the first passage time s over $[0, T]$ and r_γ over $(-\infty, \infty)$.

Joint distribution of γ and r_γ at time t under Q_T

The explicit expression of the joint distribution of (γ, r_γ) is not known. We approximate it by discretizing along the time and interest rate dimensions using the extended Fortet method (to be discussed later).

We assume the short rate process r_t to follow the Vasicek interest rate model so that $\ln S_t$ and r_t form a joint Gaussian process.

One-factor short rate models and bond prices

Assume that the short rate r_t under Q is governed by

$$dr_t = \mu(r_t, t)\ dt + \sigma_r(r_t, t)\ dZ_t.$$

The unit-par discount bond price function $P(t, T)$ is given by

$$P(t, T) = E_Q^t[e^{-\int_t^T r_u\ du}],$$

where E_Q^t is the expectation under Q conditional on the filtration $\mathcal{F}_t$.

Recall the Ito lemma, which gives the dynamics of $P(t, T)$ as follows:

$$dP(t, T) = \left(\frac{\partial P}{\partial t} + \mu \frac{\partial P}{\partial r} + \frac{\sigma_r^2}{2} \frac{\partial^2 P}{\partial r^2}\right) dt + \sigma_r \frac{\partial P}{\partial r}\ dZ_t.$$

Suppose we write formally

$$\frac{dP}{P} = \mu_P(r,t)\ dt + \sigma_P(r,t)\ dZ_t,$$

then

$$\begin{aligned}\mu_P(r,t) &= \frac{1}{P}\left(\frac{\partial P}{\partial t} + \mu\frac{\partial P}{\partial r} + \frac{\sigma_r^2}{2}\frac{\partial^2 P}{\partial r^2}\right)\\ \sigma_P(r,t) &= \frac{\sigma_r}{P}\frac{\partial P}{\partial r} = \sigma_r\frac{\partial}{\partial r}\ln P.\end{aligned}$$

For short rate models of the affine class, $P(t,T)$ admits solution of the affine form:

$$P(t,T) = e^{-B(t,T)r-\eta(t,T)},$$

then

$$\frac{\partial}{\partial r}\ln P = -B(t,T) \text{ so that } \sigma_P(t,T) = -\sigma_r B(t,T).$$

Note that $B(t,T)$ is a positive function [consistent with $P(t,T)$ being decreasing in r]. It is desirable to take the volatility of $P(t,T)$ to be $\sigma_r B(t,T)$, a positive quantity.

Accordingly, we adopt the convention that the dynamics of $P(t,T)$ is specified as

$$\frac{dP}{P} = \mu_P(t,T)\ dt - \sigma_P(t,T)\ dZ_t$$

with $\sigma_P(t,T) = \sigma_r B(t,T)$. The sign does not matter since Z_t is symmetric with respect to the value zero.

Furthermore, since the discounted price of the riskfree discount bond is Q-martingale, so $\mu_P(t,T) = r_t$. This gives the following governing equation for $P(t,T)$:

$$\frac{\partial P}{\partial t} + \mu\frac{\partial P}{\partial r} + \frac{\sigma_r^2}{2}\frac{\partial^2 P}{\partial r^2} - rP = 0,\ \ P(T,T) = 1.$$

Joint dynamics of the interest rate process and stock price process

Under the equivalent martingale pricing measure Q, the dynamics of $P(t,T)$ can be characterized by

$$\frac{dP(t,T)}{P(t,T)} = r_t\ dt - \sigma_P(t,T)\ dZ_1(t),$$

where $\sigma_P(t,T)$ is the volatility structure of $P(t,T)$ and r_t is the short rate process. Let S_t denote the price process of the underlying stock, and ρ be the correlation coefficient between S_t and r_t. The dynamics of S_t is given by

$$\frac{dS_t}{S_t} = r_t\ dt + \sigma\left[\rho\ dZ_1(t) + \sqrt{1-\rho^2}\ dZ_2(t)\right],$$

where Z_1 and Z_2 are a pair of uncorrelated Q-Brownian motions.

Ornstein-Uhlenbeck (OU) process

The dynamics of an OU process X_t is governed by

$$dX_t = a(\theta - X_t)\ dt + \sigma\ dW_t,$$

where $a > 0$, $\sigma > 0$ and θ are parameters, and W_t denotes the standard Brownian motion.

The parameter θ represents the mean value (or equilibrium) supported by fundamentals, σ is the degree of volatility around the mean value caused by shocks, and a is the rate by which these shocks dissipate and the variable X_t reverts towards the mean.

The OU process is an example of a Gaussian process that has a bounded variance and admits a stationary probability distribution.

Analytic formulas of the OU process

Consider $f(x_t, t) = e^{at} x_t$ so that

$$df(x_t, t) = a x_t e^{at}\ dt + e^{at}\ dx_t = a e^{at} \theta\ dt + \sigma e^{at}\ dW_t.$$

Integrating from 0 to t gives

$$x_t e^{at} = x_0 + \int_0^t a e^{as} \theta\ ds + \int_0^t \sigma e^{as}\ dW_s$$

so that

$$x_t = x_0 e^{-at} + \theta(1 - e^{-at}) + \int_0^t \sigma e^{a(s-t)}\ dW_s.$$

Mean

$$E[x_t] = x_0 e^{-at} + \theta(1 - e^{-at})$$

Covariance

$$\begin{aligned} \mathrm{cov}(x_s, x_t) &= E[\sigma^2 e^{-a(s+t)} \int_0^s e^{au} dW_u \int_0^t e^{av} dW_v] \\ &= \frac{\sigma^2}{2a}\, e^{-a(s+t)}[e^{2a\min(t,s)} - 1]. \end{aligned}$$

Variance

$$\mathrm{var}(x_t) = \frac{\sigma^2}{2a}\, e^{-2at}(e^{2at} - 1) = \frac{\sigma^2(1 - e^{-2at})}{2a}.$$

For a fixed value of t, x_t is a Gaussian distribution, where

$$X_t \sim N\left(x_0 e^{-at} + \theta(1 - e^{-at}), \frac{\sigma^2}{2a}(1 - e^{-2at})\right).$$

Density function of X_t is

$$P[X_t \in dx] = \sqrt{\frac{a}{\pi\sigma^2(1-e^{-2at})}} \exp\left(-\frac{x - [x_\theta e^{-at} + \theta(1-e^{-at})]}{\frac{\sigma^2}{a}(1-e^{-2at})}\right).$$

Remark

The CKLS (Chan-Karolyi-Longstaff-Sanders) process with the volatility term replaces by $\sigma x^\gamma \, dW_t$ can be solved in closed form for $\gamma = \frac{1}{2}$, 1, as well as $\gamma = 0$.

Vasicek short rate model

Under Q, the dynamics of r_t is given by

$$dr_t = a(\theta - r_t)\ dt + \sigma_r\ dZ_1(t).$$

The governing equation of $P(t,T)$ is given by

$$\frac{\partial P}{\partial t} + a(\theta - r)\frac{\partial P}{\partial r} + \frac{\sigma_r^2}{2}\frac{\partial^2 P}{\partial r^2} - rP = 0.$$

The bond price function admits $P(t,T) = e^{-B(T-t;a)r_t - \eta(T-t)}$. Substituting the assumed affine solution into the above differential equation, we obtain a coupled system of ordinary differential equations for $B(T-t)$ and $\eta(T-t)$. The auxiliary conditions are

$$B(0) = \eta(0) = 0 \quad \text{(since bond price equals one at maturity)}.$$

Closed form solution to $B(T-t)$ and $\eta(T-t)$ can be obtained since the drift term $\mu(r_t,t)$ in the Vasicek model is linear in r_t. [See p.395 in Kwok's text for details]. We obtain

$$B(u;a) = \frac{1-e^{-au}}{a},\ \eta(u) = \left(\theta - \frac{\sigma_r^2}{2a^2}\right)[u - B(u;a)] + \frac{\sigma_r^2}{4a}B(u;a)^2.$$

Under the Vasicek short rate model, the corresponding volatility structure $\sigma_P(t,T)$ is found to be

$$\sigma_P(t,T) = \sigma_r B(T-t;a) = \frac{\sigma_r}{a}[1 - e^{-a(T-t)}].$$

The risk neutral dynamics of S_t and $P(t,T)$ can be expressed as

$$S_t = S_0 \exp\left(\int_0^t r_u\ du - \frac{\sigma^2 t}{2} + \int_0^t \rho\sigma\ dZ_1(u) + \int_0^t \sigma\sqrt{1-\rho^2}\ dZ_2(u)\right)$$

and

$$P(t,T) = P(0,T)\exp\left(\int_0^t r_u\ du - \int_0^t \frac{\sigma_P^2(u,T)}{2}\ du - \int_0^t \sigma_P(u,T)\ dZ_1(u)\right).$$

Setting $T = t$, we can deduce

$$\frac{P(t,t)}{P(0,t)} = \frac{1}{P(0,t)} = \exp\left(\int_0^t r_u\ du - \int_0^t \frac{\sigma_P^2(u,t)}{2}\ du - \int_0^t \sigma_P(u,t)\ dZ_1(u)\right).$$

The last equation is useful in relating $\exp\left(\int_0^t r_u\ du\right)$ to the initial price and volatility term structure of the t-maturity discount bond.

T-forward measure

Let Q_T denote the T-forward measure, where $P(t,T)$ is used as the numeraire. Under Q^T, we observe the martingale property of the relative price $S_t/P(t,T)$:

$$\frac{S_t}{P(t,T)} = \frac{S_0}{P(0,T)} \exp\left(\int_0^t [\sigma_P(u,T)+\rho\sigma]\ dZ_1^T(u)+\int_0^t \sigma\sqrt{1-\rho^2}\ dZ_2^T(u)\right),$$

$$dZ_1^T(t) = dZ_1(t) + \sigma_P(t,T)\ dt \quad \text{and} \quad dZ_2^T(t) = dZ_2(t),$$

where $Z_1^T(t)$ and $Z_2^T(t)$ are a pair of uncorrelated Q_T-Brownian motions.

The following proof is adopted from Section 8.1 in Kwok's text.

Proof of the formula for affecting change of measure from Q to Q_T

We would like to illustrate how to effect the change of measure from the risk neutral measure Q to the T-forward measure Q_T. Let the dynamics of the T-maturity discount bond price $P(t, T)$ under Q be governed by

$$\frac{dP(t,T)}{P(t,T)} = r(t)\ dt - \sigma_P(t,T)\ dZ(t),$$

where $Z(t)$ is Q-Brownian.

By integrating the above equation and observing $\dfrac{M(t)}{M(0)} = \int_0^t r(u)\ du$, where $M(t)$ is the time-t value of the money market account, we obtain

$$\frac{P(t,T)}{M(t)} = \frac{P(0,T)}{M(0)} \exp\left(-\int_0^t \sigma_P(u,T)\ dZ(u) - \frac{1}{2}\int_0^t \sigma_P(u,T)^2\ du\right).$$

The Radon-Nikodym derivative $\dfrac{dQ_T}{dQ}$ conditional on $\mathcal{F}_T$ is found to be

$$\begin{aligned}\frac{dQ_T}{dQ} &= \frac{P(T,T)}{P(0,T)}\Bigg/\frac{M(T)}{M(0)}\\ &= \exp\left(-\int_0^T \sigma_P(u,T)\ dZ(u) - \frac{1}{2}\int_0^T \sigma_P(u,T)^2\ du\right).\end{aligned}$$

For a fixed T, we define the process

$$\xi_t^T = E_Q\left[\left.\frac{dQ_T}{dQ}\right|\mathcal{F}_t\right]$$

and since $M(0) = 1$ and $P(0,T)$ is known at time t, we obtain

$$\begin{aligned}\xi_t^T &= \frac{1}{P(0,T)}E_Q\left[\left.\frac{P(T,T)}{M(T)}\right|\mathcal{F}_t\right] = \frac{P(t,T)}{P(0,T)M(t)}\\ &= \exp\left(-\int_0^t \sigma_P(u,T)\ dZ(u) - \frac{1}{2}\int_0^t \sigma_P(u,T)^2\ du\right).\end{aligned}$$

By virtue of the Girsanov Theorem and observing the above result, we deduce that the process

$$Z^T(t) = Z(t) + \int_0^t \sigma_P(u, T)\ du$$

is Q_T-Brownian.

As an example, consider the Vasicek model where the short rate is modeled by

$$dr(t) = \alpha[\gamma - r(t)]dt + \sigma_r\ dZ(t),$$

where $Z(t)$ is Q-Brownian. The corresponding volatility function $\sigma_P(t, T)$ of the discount bond price process is known to be

$$\sigma_P(t, T) = \frac{\sigma_r}{\alpha}[1 - e^{-\alpha(T-t)}].$$

Under the T-forward measure Q_T, the dynamics of $r(t)$ is given by

$$dr(t) = \alpha \left\{ \gamma - \frac{\sigma_r^2}{\alpha^2}[1 - e^{-\alpha(T-t)}] - r(t) \right\} dt + \sigma_r \, dZ^T(t),$$

where $Z^T(t)$ is Q_T-Brownian. We integrate the above equation to obtain

$$\begin{aligned} r(t) &= r(s)e^{-\alpha(t-s)} + \left(\gamma - \frac{\sigma_r^2}{\alpha^2}\right)[1 - e^{-\alpha(t-s)}] \\ &\quad + \frac{\sigma_r^2}{2\alpha^2}\left[e^{-\alpha(T-t)} - e^{-\alpha(T+t-2s)}\right] + \sigma_r \int_s^t e^{-\alpha(t-u)} dZ^T(u). \end{aligned}$$

Under Q_T, the distribution of $r(t)$ conditional on $\mathcal{F}_s$ is normal with the following mean and variance

$$\begin{aligned} E_{Q_T}[r(t)|\,\mathcal{F}_s] &= r(s)e^{-\alpha(t-s)} + \left(\gamma - \frac{\sigma_r^2}{\alpha^2}\right)[1 - e^{-\alpha(t-s)}] \\ &\quad + \frac{\sigma_r^2}{2\alpha^2}\left[e^{-\alpha(T-t)} - e^{-\alpha(T+t-2s)}\right] \\ \text{var}_{Q_T}(r(t)|\,\mathcal{F}_s) &= \sigma_r^2 \int_s^t e^{-2\alpha(t-u)} du = \frac{\sigma_r^2}{2\alpha}[1 - e^{2\alpha(t-s)}], \qquad s \le t \le T. \end{aligned}$$

Bond price process

We would like to express $P(t,T)$ in terms of Z_1^T and bond prices $P(0,t)$ and $P(0,T)$ (initial bond prices with maturity dates t and T). Recall

$$
\begin{aligned}
P(t,T) &= P(0,T)\exp\left(\int_0^t r_u\, du - \int_0^t \frac{\sigma_P^2(u,T)}{2}\, du - \int_0^t \sigma_P(u,T)\, dZ_1(u)\right) \\
\frac{1}{P(0,t)} &= \exp\left(\int_0^t r_u\, du - \int_0^t \frac{\sigma_P^2(u,t)}{2}\, du - \int_0^t \sigma_P(u,t)\, dZ_1(u)\right)
\end{aligned}
$$

and $dZ_1(u) = dZ_1^T(u) - \sigma_P(u,T)\, du$.

Putting these results together, we obtain

$$
\begin{aligned}
P(t,T) = \frac{P(0,T)}{P(0,t)}\exp\biggl(&\int_0^t [\sigma_P(u,t) - \sigma_P(u,T)]\, dZ_1^T(u) \\
&+ \frac{1}{2}\int_0^t [(\sigma_P(u,T) - \sigma_P(u,t)]^2\, du\biggr).
\end{aligned}
$$

Stock price process

Recall the formulas:

$$
\begin{aligned}
S_t &= S_0 \exp\left(\int_0^t r_u \, du - \frac{\sigma^2 t}{2} + \int_0^t \rho\sigma \, dZ_1(u) + \int_0^t \sigma\sqrt{1-\rho^2} \, dZ_2(u)\right) \\
\frac{1}{P(0,t)} &= \exp\left(\int_0^t r_u \, du - \int_0^t \frac{\sigma_P^2(u,t)}{2} \, du - \int_0^t \sigma_P(u,t) \, dZ_1(u)\right) \\
dZ_1(u) &= dZ_1^T(u) - \sigma_P(u,T) \, du,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
S_t = \frac{S_0}{P(0,t)} \exp\Bigg(&\int_0^t \frac{\sigma_P^2(u,t) - \sigma^2}{2} \, du \\
&+ \int_0^t [\sigma_P(u,t) + \rho\sigma][dZ_1^T(u) - \sigma_P(u,T) \, du] \\
&+ \int_0^t \sigma\sqrt{1-\rho^2} \, dZ_2^T(u)\Bigg).
\end{aligned}
$$

The short rate r_t has been eliminated and it does not appear in the above expression.

The forward risk neutral solution of S_t is given by

$$S_t = \frac{S_0}{P(0,t)} \exp\left(\int_0^t \left(-\sigma_P(u,T)[\sigma_P(u,t) + \rho\sigma] + \frac{\sigma_P^2(u,t) - \sigma^2}{2} \right) \, du \right.$$
$$\left. + \int_0^t [\sigma_P(u,t) + \rho\sigma] \, dZ_1^T(u) + \int_0^t \sigma\sqrt{1-\rho^2} \, dZ_2^T(u) \right)$$

or equivalently

$$\begin{aligned} \ell_t &= \ln S_t \\ &= \ln \frac{S_0}{P(0,t)} + \int_0^t \left(-\sigma_P(u,T)[\sigma_P(u,t) + \rho\sigma] + \frac{\sigma_P^2(u,t) - \sigma^2}{2} \right) \, du \\ &\quad + \int_0^t [\sigma_P(u,t) + \rho\sigma] \, dZ_1^T(u) + \int_0^t \sigma\sqrt{1-\rho^2} \, dZ_2^T(u). \end{aligned}$$

Moments and conditional moments of ℓ_t under Q_T

Using the closed form expression of ℓ_t [in terms of $S_0/P(0,t)$] and volatility functions of the bond price, we obtain

$$\text{mean} = M(t) = \ln\left(\frac{S_0}{P(0,t)}\right) + \int_0^t \left(-\sigma_P(u,T)[\sigma_P(u,t) + \rho\sigma] + \frac{\sigma_P^2(u,t) - \sigma^2}{2}\right) du;$$

$$\begin{aligned}
\text{cov}(s,t) &= \text{cov}(\ell_s, \ell_t) \\
&= \int_0^s [[\sigma_P(u,t) + \rho\sigma][\sigma_P(u,s) + \rho\sigma] + \sigma^2(1-\rho^2)]\ du \\
&= \int_0^s \left\{\sigma^2 + \rho\sigma[\sigma_P(u,t) + \sigma_P(u,s)] + \sigma_P(u,s)\sigma_P(u,t)\right\} du, \ \ s < t;
\end{aligned}$$

$$\text{var}(\ell_t) = V(t) = \int_0^t [\sigma^2 + \sigma_P^2(u,t) + 2\sigma\rho\sigma_P(u,t)]\ du.$$

Explicit expressions of the moments and conditional moments of ℓ_t can be found, given an exponential structure of volatility which corresponds to the Vasicek model.

Conditional moments for the process ℓ_t [in terms of r_u instead of $P(u,t)$]

$$\begin{aligned}
E_{Q_T}[\ell_t|\mathcal{F}_u] = \; & \ell_u - \left(r_u + \frac{\sigma^2}{2} + \frac{\sigma\rho\sigma_r}{a} - \theta + \frac{\sigma_r^2}{a^2}\right)(t-u) - \frac{\sigma_r^2}{a^2}e^{-a(T-t)}B(t-u;2a) \\
& + \left(r_u - \theta + \frac{\sigma_r^2}{a^2} + \frac{\sigma_r^2}{a^2}e^{-a(T-t)} + \frac{\sigma\rho\sigma_r}{a}e^{-a(T-t)}\right)B(t-u;a) \\
\mathrm{var}_{Q_T}(\ell_t|\mathcal{F}_u) = & \left(\sigma^2 + 2\frac{\sigma\rho\sigma_r}{a} + \frac{\sigma_r^2}{a^2}\right)(t-u) - 2\left(\frac{\sigma_r^2}{a^2} + \frac{\sigma\rho\sigma_r}{a}\right)B(t-u;a) \\
& + \frac{\sigma_r^2}{a^2}B(t-u;2a), \\
\mathrm{cov}_{Q_T}(\ell_s,\ell_t|\mathcal{F}_u) = \; & \frac{\sigma_r^2}{a^2}e^{-a(t-s)}B(s-u;2a) + \left(\sigma^2 + 2\frac{\sigma\rho\sigma_r}{a} + \frac{\sigma_r^2}{a^2}\right)(s-u) \\
& - \left(\frac{\sigma_r^2}{a^2} + \frac{\sigma\rho\sigma_r}{a}\right)[e^{-a(t-s)} + 1]B(s-u;a), \quad s < t.
\end{aligned}$$

Recall: $P(u,t) = e^{-B(t-u)r_u - \eta(t-u)}$, $\sigma_P(u,t) = \frac{\sigma_r}{a}[1 - e^{-a(t-u)}]$, $t > u$.

Covariance between ℓ_t and $r_t = \sigma_r \int_u^t e^{as}[\sigma_P(s,t) + \rho s]\ ds$

$$\mathrm{cov}_{Q_T}(\ell_t, r_t|\mathcal{F}_u) = \left(\frac{\sigma_r^2}{a} + \rho\sigma\sigma_r\right) B(t-u;a) - \frac{\sigma_r^2}{a} B(t-u;2a).$$

Replacing u by 0 in the above expressions of the conditional moments of ℓ_t, we obtain the following formulas:

$$\begin{aligned}
M(t) = {} & \ln \frac{S_0}{P(0,t)} + \frac{\sigma_r^2}{4a^3} - \left(\frac{\sigma_r^2}{2a^2} + \frac{\rho\sigma\sigma_r}{a} + \frac{\sigma^2}{2}\right) t - \frac{\sigma_r^2}{4a^3} e^{-2at} \\
& + \left(\frac{\sigma_r^2}{2a^3} + \frac{\rho\sigma\sigma_r}{a^2}\right) e^{-a(T-t)} - \left(\frac{\sigma_r^2}{a^3} + \frac{\rho\sigma\sigma_r}{a^2}\right) e^{-aT} + \frac{\sigma_r^2}{2a^3} e^{-a(T+t)}, \\
V(t) = {} & \left(\sigma^2 + \frac{\sigma_r^2}{a^2} + \frac{2\rho\sigma\sigma_r}{a}\right) t - \frac{3\sigma_r^2}{2a^3} - \frac{2\rho\sigma\sigma_r}{a^2} + \frac{2\sigma_r(\sigma_r + a\rho\sigma)}{a^3} e^{-at} - \frac{\sigma_r^2}{2a^3} e^{-2at}, \\
\mathrm{cov}(u,t) = {} & -\left(\frac{\rho\sigma\sigma_r}{a^2} + \frac{\sigma_r^2}{a^3}\right) + \left(\sigma^2 + \frac{2\rho\sigma\sigma_r}{a} + \frac{\sigma_r^2}{a^2}\right) \sigma_r - \frac{\sigma_r^2}{2a^3} e^{-a(t+u)} \\
& + \left(\frac{\rho\sigma\sigma_r}{a^2} + \frac{\sigma_r^2}{a^3}\right) (e^{-au} + e^{-at}) - \left(\frac{\rho\sigma\sigma_r}{a^2} + \frac{\sigma_r^2}{2a^3}\right) e^{-a(t-u)}.
\end{aligned}$$

Projection Theorem

When X and Y is a bivariate normal distribution, their joint density is given by

$$\begin{aligned} f_{X,Y}(x,y) &= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right.\right. \\ &\quad \left.\left. -\frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]\right). \end{aligned}$$

The conditional density of Y, given $X=x$, is given by

$$\begin{aligned} f_Y(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\ &= \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2\sigma_y^2(1-\rho^2)}\left[y-\mu_y-\frac{\rho\sigma_y}{\sigma_x}(x-\mu_x)\right]^2\right). \end{aligned}$$

The Projection Theorem gives

$$E[Y|X=x] = \mu_Y + \frac{\rho\sigma_X\sigma_Y}{\sigma_X^2}(x-\mu_X)$$
$$\text{var}[Y|X=x] = \sigma_Y^2 - \frac{\rho^2\sigma_X^2\sigma_Y^2}{\sigma_X^2}.$$

The conditional law of $\ln S_t$ given $\ln S_s = \ln H$, where $\ln H$ is an arbitrary given level, is normal and possesses the following mean $\widehat{M}$ and variance $\widehat{V}$:

$$\widehat{M}(s,t) = M_t + \frac{\text{cov}(s,t)}{V_s}(\ln H - M_s),$$
$$\widehat{V}(s,t) = V_t - \frac{\text{cov}^2(s,t)}{V_s}.$$

One-dimensional Fortet method

For a one-factor continuous Markov process ℓ_t, we define $\pi(\ell_t, t|\ell_s, s)$ as the free transition density. Further, define $g(\ell_s = \underline{\ell}, s|\ell_0, 0)$ as the probability density that the first passage time through a constant boundary $\underline{\ell}$ occurs at date-s. An implicit formula for $g(\cdot)$ in terms of $\pi(\cdot)$ is given by

$$\pi(\ell_t, t|\ell_0, 0) = \int_0^t g(\ell_s = \underline{\ell}, s|\ell_0, 0)\pi(\ell_t, t|\ell_s = \underline{\ell}, s)\ ds, \text{ where } \ell_t > \underline{\ell} > \ell_0.$$

Note that ℓ_t and ℓ_0 are on the opposite sides of the boundary $\ell = \underline{\ell}$.

When the process ℓ_t is one-factor Markov, the above equation has a very intuitive interpretation: The only way that the process can start below the boundary ($\ell_0 < \underline{\ell}$) and end up above the boundary ($\ell_t > \underline{\ell}$) is that the process at some intermediate time s must pass through the boundary for the first time.

More rigorously, we can write for arbitrary $\{\ell_t, \underline{\ell}, \ell_0\}$, where $\widetilde{\tau}$ is the first passage time to $\underline{\ell}$:

$$\begin{aligned}
\pi(\ell_t, t|\ell_0, 0) &= \int_0^t \pi(\ell_t, t; \widetilde{\tau} = s|\ell_0, 0)\ ds + \pi(\ell_t, t; \widetilde{\tau} > t|\ell_0, 0) \\
&= \int_0^t \pi(\ell_t, t|\widetilde{\tau} = s; \ell_0, 0)\pi(\widetilde{\tau} = s|\ell_0, 0)\ ds \\
&\quad + \pi(\ell_t, t; \widetilde{\tau} > t|\ell_0, 0) \\
&= \int_0^t \pi(\ell_t, t|\ell_s = \underline{\ell}, s)g(\ell_s = \underline{\ell}, s|\ell_0, 0)\ ds \\
&\quad + \pi(\ell_t, t; \widetilde{\tau} > t|\ell_0, 0).
\end{aligned}$$

- We have used the strong Markov property in the last line, where the path history of ℓ_t prior to the stopping time $\widetilde{\tau}$ is irrelevant to the distribution of ℓ_t, $t > \widetilde{\tau}$.

- When $\ell_t > \underline{\ell} > \ell_0$, the last term vanishes.

Extended Fortet method

Reference: "Pricing derivatives with barriers in a stochastic interest rate environment," C. Bernard *et al.*, *Journal of Economic Dynamics and Control*, vol.32 (2008) P.2903-2938.

The interval $[0,T]$ is subdivided into n_T subintervals of length $\delta_t = T/n_T$, and the interest rate is subdivided between r_{min} and r_{max} into n_r intervals of length $\delta_r = (r_{\text{max}} - r_{\text{min}})/n_r$. We write

$$t_j = j\delta_t \quad \text{and} \quad r_i = r_{\text{min}} + i\delta_r$$

as the discretized values of time and interest rate. Write

$$q(i,j) \approx Q_T(r_\gamma \in [r_i, r_{i+1}], \gamma \in [t_j, t_{j+1}])$$

as the discretized approximation of the joint distribution of the first passage time γ and r_γ.

We would like to find a numerical procedure to compute $q(i,j)$.

Let the conditional mean and variance of ℓ_T be defined by $\widehat{\mu}_{s,T} = E_{Q_T}[\ell_T|\mathcal{F}_s]$ and $\widehat{\sum}_{s,T} = \text{var}_{Q_T}(\ell_T|\mathcal{F}_s)$.

Suppose $X \sim N(m, \sigma^2)$, then

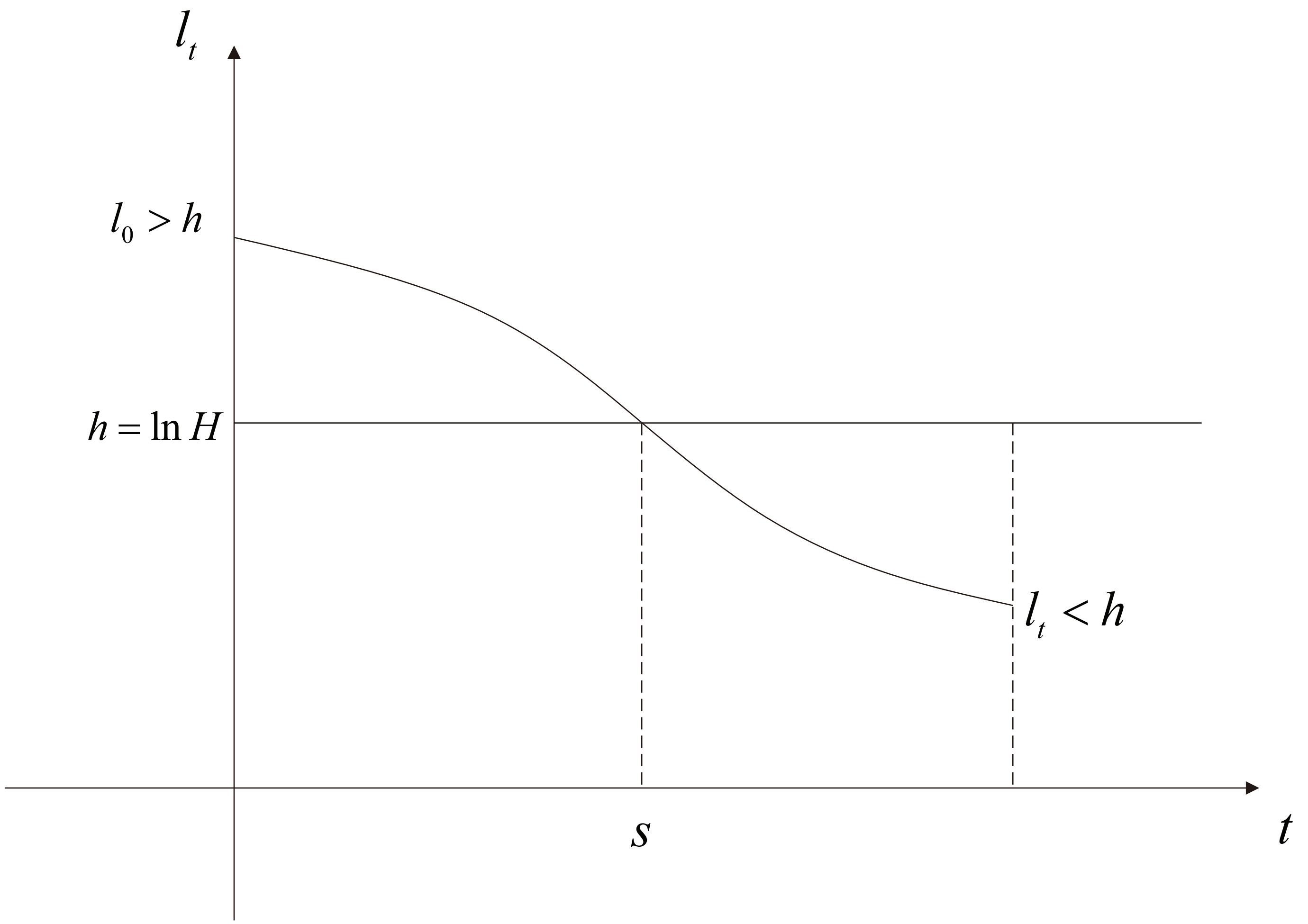
$$E[e^X \mathbf{1}_{\{X>\ln a\}}] = k(m, \sigma, a) = \exp\left(m + \frac{\sigma^2}{2}\right) N\left(\frac{m + \sigma^2 - \ln a}{\sigma}\right),$$

$$E[\mathbf{1}_{\{X>\ln a\}}] = N\left(\frac{m - \ln a}{\sigma}\right).$$

Recall

$$\begin{aligned} A &= \int_0^T \int_{-\infty}^{\infty} E_{Q_T}\left[e^{\ell_T}\mathbf{1}_{\{\ell_T>\ln K\}}\mathbf{1}_{\{\gamma\le T\}}\Big| r_\gamma = r, \gamma = s\right] Q_T[r_\gamma \in dr, \gamma \in ds] \\ &\approx \sum_{j=0}^{n_T}\sum_{i=0}^{n_r} k(\widehat{\mu}_{t_{j,T}}^{(i)}, \widehat{\sum}_{t_{j,T}}^{(i)}, K) q(i,j). \\ B &= \int_0^T \int_{-\infty}^{\infty} Q_T\left[\ell_T > \ln K | r_\gamma = r, \gamma = s\right] Q_T[r_\gamma \in dr, \gamma \in ds] \\ &\approx \sum_{j=0}^{n_T}\sum_{i=0}^{n_r} N\left(\frac{\widehat{\mu}_{t_{j,T}}^{(i)} - \ln K}{\sqrt{\widehat{\sum}_{t_{j,T}}^{(i)}}}\right) q(i,j). \end{aligned}$$

We assume a down-barrier, where one observes initially $\ell_0 > \ln H = h$. Suppose at time t, the process $\ell_t = \ell < h$, so the down-barrier must have been hit earlier. Also, we assume ℓ_t to be continuous.



Assuming $\ell_0 > h$ and $\ell_t < h$, the two-dimensional Fortet integral equation is given by

$$
\begin{aligned}
& Q_T[\ell_t \in [\ell, \ell + d\ell_t), r_t \in [r, r + dr)|\ell_0, r_0] \\
= & \int_0^t \int_{-\infty}^{\infty} Q_T[\ell_t \in [\ell, \ell + d\ell), r_t \in [r, r + dr)|\ell_s = h, r_s = r'] \\
& \qquad Q_T[r_\gamma \in [r', r' + dr'), \gamma \in [s, s + ds)].
\end{aligned}
$$

Next, we integrate with respect to ℓ from $-\infty$ to h and obtain

$$
\begin{aligned}
& Q_T[\ell_t \leq h, r_t \in [r, r + dr)|\ell_0, r_0] \\
= & \int_0^t \int_{-\infty}^{\infty} Q_T[\ell_t \leq h, r_t \in [r, r + dr)|\ell_s = h, r_s = r'] \\
& \qquad Q_T[r_\gamma \in [r', r' + dr'), \gamma \in [s, s + ds)].
\end{aligned}
$$

Write

$$
\begin{aligned}
& \Phi(r, t)\ dr = Q_T[\ell_t \leq h, r_t \in [r, r + dr]|\ell_0, r_0] \\
& \Psi(r, t, r', s)\ dr = Q_T[\ell_t \leq h, r_t \in [r, r + dr]|\ell_s = h, r_s = r'].
\end{aligned}
$$

When $t = s$, we have $\Psi(r, t, r', s)\ dr = \mathbf{1}_{\{r' \in [r, r+dr]\}}$ and $\Psi(r, s)\ dr = Q_T(r_s \in [r, r + dr],\ r = s)$.

Note that $X = (\ell, r)$ is a Gaussian process whose joint dynamics under Q_T is given by

$$dX_t = d\begin{pmatrix} \ell_t \\ r_t \end{pmatrix} = \begin{pmatrix} r_t \quad -\frac{\sigma^2}{2} - \sigma\rho\sigma_P(t,T) \\ a\left[\theta - \frac{\sigma_r}{a}\sigma_P(t,T) - r_t\right] \end{pmatrix} + \begin{pmatrix} \sigma\rho & \sigma\sqrt{1-\rho^2} \\ \sigma_r & 0 \end{pmatrix}\begin{pmatrix} dZ_1^T \\ dZ_2^T \end{pmatrix}.$$

We use f_{ℓ_t, r_t} to denote the density function of (ℓ_t, r_t) under Q_T. Thanks to the conditional results, one obtains

$$f_{\ell_t, r_t}(\ell, r) = f_{r_t}(r) f_{\ell_t | r_t}(\ell).$$

Let $\mathcal{F}_0$ and $\mathcal{F}_s$ represent the available information at time 0 and s, respectively.

Using the *strong Markov property of* (ℓ_t, r_t), conditioning on $\mathcal{F}_s$ is like conditioning on (ℓ_s, r_s), where s is the $\mathcal{F}_s$-stopping time. One then obtain Ψ and Φ:

$$\Phi(r,t) = f_{r_t}(r|\mathcal{F}_0) \int_{-\infty}^{h} f_{\ell_t|r_t}(\ell|\mathcal{F}_0)\ d\ell,$$
$$\Psi(r,t,r',s) = f_{r_t}(r|\mathcal{F}_s) \int_{-\infty}^{h} f_{\ell_t|r_t}(\ell|\mathcal{F}_s)\ d\ell.$$

Since the process (ℓ_t, r_t) is Gaussian, the conditional law of $\ell_t | r_t$ knowing the available information at time s is Gaussian. We denote $E_{Q_T}[\ell_t | r_t = r, \ell_s, r_s]$ and $\text{var}_{Q_T}[\ell_t | r_t = r, \ell_s, r_s]$ by $\mu(r, \ell_s, r_s)$ and $\sum^2(r, \ell_s, r_s)$, where $r_t = r$. By the projection Theorem:

$$\mu(r, \ell_s, r_s) = E_{Q_T}[\ell_t | \mathcal{F}_s] + \frac{\text{cov}(\ell_t, r_t | \mathcal{F}_s)}{\text{var}[r_t | \mathcal{F}_s]}(r - E_{Q_T}[r_t | \mathcal{F}_s]),$$
$$\sum{}^2(r, \ell_s, r_s) = \text{var}[\ell_t | \mathcal{F}_s] - \frac{\text{cov}(\ell_t, r_t | \mathcal{F}_s)^2}{\text{var}[r_t | \mathcal{F}_s]}.$$

The above moments have been computed. We then obtain

$$\Phi(r, t) = f_{r_t}(r | r_0) N\left(\frac{h - \mu(r, \ell_0, r_0)}{\sqrt{\sum^2(r, \ell_0, r_0)}}\right),$$
$$\Psi(r, t, r', s) = f_{r_t}(r | r_s = r') N\left(\frac{h - \mu(r, \ell_s = h, r')}{\sqrt{\sum^2(r, \ell_s = h, r')}}\right),$$

where f_{r_t} is the transition density of r_t.

Recall

$$\Phi(r,t)\ dr = Q_T[\ell_t \le h, r_t \in [r, r+dr)|\ell_0, r_0],$$
$$\Psi(r,t,r',s)\ dr = Q_T[\ell_t \le h, r_t \in [r, r+dr)|\ell_s = h, r_s = r'];$$

they observe the following integral equation for $Q_T[r_\gamma \in dr', \gamma \in ds]$

$$\Phi(r,t) = \int_{s\in[0,t]} \int_{r'\in\mathbb{R}} \Psi(r,t,r',s) Q_T[r_\gamma \in [r', r'+dr'), \gamma \in [s, s+ds)].$$

We start with $\ell_0 > h$ so that the first passage time cannot be zero. In discretized form, at $t = t_j$ and $r = r_i$, we have

$$\Phi(r_i, t_j) = \sum_{v=1}^{j} \sum_{u=0}^{n_r} \Psi(r_i, t_j, r_u, t_v) q(u, v).$$

In particular, when $j = 1$, the previous expression becomes

$$\Phi(r_i, t_1) = \sum_{u=1}^{n_r} \Psi(r_i, t_0, r_u, t_0) q(u, 0).$$

We then obtain the following expression:

$$q(i,1) = Q_T(r_\gamma \in [r_i, r_{i+1}], \gamma \in [t_0, t_1]).$$

Note that $\Psi(r_i, t_1, r_u, t_1) = \mathbf{1}_{\{r_i = r_u\}}$, one readily has $q(i,1) = \Phi(r_i, t_1)$.

Recursive scheme for the computation of $q(i,j)$

First, we compute $q(i,1)$. For $j > 1$, we use the relation:

$$\Phi(r_i, t_j) = \sum_{u=0}^{n_r} q(u,j)\Psi(r_i, t_j, r_u, t_j) + \sum_{v=1}^{j-1} \sum_{u=0}^{n_r} q(u,v)\Psi(r_i, t_j, r_u, t_v).$$

Thanks to $\Psi(r_i, t_j, r_u, t_j) = \mathbf{1}_{\{r_i = r_u\}}$, we deduce that

$$q(i,j) = \Phi(r_i, t_j) - \sum_{v=1}^{j-1} \sum_{n=0}^{n_r} q(u,v)\Psi(r_i, t_j; r_u, t_v).$$

Up-barrier case: summary of formulas

Starting with $q(i,1) = \Phi(r_i, t_1)$, we compute $q(i,j)$ recursively as follows:

$$q(i,j) = \Phi(r_i, t_j) - \sum_{k=1}^{j-1} \sum_{l=0}^{n_r} q(l,k)\Psi(r_i, t_j; r_l, t_k),$$

where

$$\Phi(r,t) = f_{r_t}(r|r_0) N \left(\frac{\mu(r, \ell_0, r_0) - h}{\sqrt{\sum^2(r, \ell_0, r_0)}} \right),$$

$$\Psi(r,t,r',s) = f_{r_t}(r|r_s = r') N \left(\frac{\mu(r, h, r') - h}{\sqrt{\sum^2(r, h, r')}} \right).$$

Review of the key results

- With regard to the knock-in condition, one has to find the joint distribution of ℓ_T and γ [note that $\{S_{\mathsf{max}} > H\}$ and $\{\gamma \leq T\}$ are equivalent events]. More specifically, we need to compute $Q_T[r_\gamma \in dr, \gamma \in ds]$. Goal: obtain an integral equation.

- We limit ourselves to the Vasicek interest rate process and Geometric Brownian motion for the stock price process. The joint process $\{\ell_t, r_t\}$ is two-dimensional Gaussian. The bond price process has exponential volatility structure, where

$$\sigma_P(t, T) = \frac{\sigma_r}{a}[1 - e^{-a(T-t)}].$$

The relation between the bond price $P(t,T)$ and the short rate r_t is given by

$$\ln P(t,T) = -B(T-t;a)r_t - \eta(T-t),$$

where

$$B(u;a) = \frac{1-e^{-au}}{a}, \;\; \eta(u) = \left(\theta - \frac{\sigma_r^2}{2a^2}\right)[u - B(u;a)] + \frac{\sigma_r^2}{4a}B(u;a)^2.$$

Recall the change of measure from Q to Q^T:

$$dZ_1^T(t) = \sigma_P(t,T)\ dt + dZ_1(t), \text{ where } Z_1^T(t) \text{ is } Q_T\text{-Brownian.}$$

Under Q_T, we have

$$\begin{aligned} dr_t &= a\left[\theta - \frac{\sigma_r^2}{a}B(T-t;a) - r_t\right]dt + \sigma_r\ dZ_1^T(t) \\ r_t &= e^{-at}\left[r_u e^{au} + a\int_u^t \widehat{\theta}_s e^{as}\ ds + \sigma_r \int_u^t e^{as}\ dZ_1^T(s)\right]. \end{aligned}$$

- *Bond price process*

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left(\int_0^t [\sigma_P(u,t) - \sigma_P(u,T)]\ dZ_1^T(u) + \frac{1}{2}\int_0^t [\sigma_P(u,T) - \sigma_P(u,t)]^2\ du\right)$$

- *Stock price process*

$$\begin{aligned}
\ell_t &= \ln S_t \\
&= \ln \frac{S_0}{P(0,t)} + \int_0^t \left\{-\sigma_P(u,T)[\sigma_P(u,t) + \rho\sigma] + \frac{\sigma_P^2(u,t) - \sigma^2}{2}\right\} du \\
&\quad + \int_0^t [\sigma_P(u,t) + \rho\sigma]\ dZ_1^T(u) + \int_0^t \sigma\sqrt{1-\rho^2}\ dZ_2^T(u) \\
d\ell_t &= \left[r_t - \frac{\sigma^2}{2} - \rho\sigma\sigma_P(t,T)\right] dt + \sigma\rho\ dZ_1^T(t) + \sigma\sqrt{1-\rho^2}\ dZ_2^T(t)
\end{aligned}$$

- To compute $E_{Q_T}[e^{\ell_T}\mathbf{1}_{\{\ell_T>\ln K\}}|r_\gamma = r, r = s]$, we use the formula:

$$k(m,\sigma,a) = E[e^X\mathbf{1}_{\{X>\ln a\}}] = \exp\left(m + \frac{\sigma^2}{2}\right) N\left(\frac{m+\sigma^2-\ln a}{\sigma}\right),$$

where $X \sim N(m,\sigma^2)$. At $\gamma = s$, we have $\ell_s = H$. By the strong Markov property of (ℓ_t, r_t), conditioning on (ℓ_s, r_s) is like conditioning on $\mathcal{F}_s$. Take the first passage time γ to be t_j, we obtain

$$E_{Q_T}[e^{\ell_T}\mathbf{1}_{\{\ell_T>\ln K\}}|\mathcal{F}_{t_j}] = k(\widehat{\mu}_{t_j,T}, \widehat{\sum}_{t_j,T}, K),$$

where

$$\widehat{\mu}_{s,T} = E_{Q_T}[\ell_T|\mathcal{F}_s] \text{ and } \widehat{\sum}_{t_j,T} = \text{var}_{Q_T}(\ell_T|\mathcal{F}_s).$$

Similarly,

$$E_{Q_T}[\mathbf{1}_{\{\ell_T>\ln K\}}|\mathcal{F}_{t_j}] = N\left(\frac{\widehat{\mu}_{t_j,T} - \ln K}{\widehat{\sum}_{t_j,T}}\right).$$

- $E_{Q_T}[\ell_t | r_t = r, \ell_s, r_s] = \mu(r, \ell_s, r_s)$

$$= E_{Q_T}[\ell_t | \mathcal{F}_s] + \frac{\mathrm{cov}(\ell_t, r_t | \mathcal{F}_s)}{\mathrm{var}(r_t | \mathcal{F}_s)}(r - E_{Q_T}[r_t | \mathcal{F}_s])$$

$$\mathrm{var}_{Q_T}[\ell_t | r_t = r, \ell_s, r_s] = \Sigma^2(r, \ell_s, r_s) = \mathrm{var}(\ell_t | \mathcal{F}_s) - \frac{\mathrm{cov}(\ell_t, r_t | \mathcal{F}_s)^2}{\mathrm{var}(r_t | \mathcal{F}_s)}.$$

- $\Phi(r,t)\ dr = Q_T[\ell_t \le h, r_t \in [r, r + dr) | \ell_0, r_0],\ \ell_0 > h$

  $\Psi(r, t, r', s)\ dr = Q_T[\ell_t \le h, r_t \in [r, r + dr) | \ell_s = h, r_s = r']$

  The discretized form of the integral equation is

$$\Phi(r_i, t_j) = \sum_{v=1}^{j} \sum_{u=0}^{n_r} \Psi(r_i, t_j, r_u, t_v) q(u, v),$$

  where

$$q(i,j) \approx Q_T[r_\gamma \in [r_i, r_{i+1}), \gamma \in [t_j, t_{j+1})].$$

1.6 Occupation time derivatives

Define the occupation time below the barrier B over the period $[0, T]$ by

$$\tau_B^- = \int_0^T \mathcal{H}(B - S_t)\, dt,$$

where $\mathcal{H}(x)$ is the Heavside step function. The following quantity

$$\exp(-\rho\tau_B^-) = \exp\left(-\rho \int_0^T \mathcal{H}(B - S_t)\, dt\right)$$

is the knock-out discount factor with knock-out rate ρ.

1. *Down-and-out proportional step call*

$$\text{terminal payoff} = \exp(-\rho\tau_B^-) \max(S_T - K, 0).$$

2. *Simple step call option with principal amortization*

$$\text{terminal payoff} = \max(1 - \rho\tau_B^-, 0) \max(S_T - K, 0).$$

3. *Delayed barrier call (also called cumulative Parasian call)*

$$\text{terminal payoff} = \mathbf{1}_{\{\tau_B^- < \alpha T\}} \max(S_T - K, 0).$$

Other contingent claims with dependence on the occupation time but no independence on the terminal stock price:

- *Switch option*
  Pays off a dollar amount proportional to the fraction of the contract life for which S_t lies above or below the barrier:

  $$A\tau_B^- \text{ or } A\tau_B^+, \text{ where } A \text{ is a notional constant.}$$

- *Day-in/day-out option* is the difference of 2 switch options

  $$A(\tau_B^- - \tau_B^+).$$

- *Occupation time option*

  $$\max(\tau_B^- - \alpha T, 0) \text{ or } \max(\alpha T - \tau_B^-, 0).$$

Quantile options

$$M(\alpha, T) = \inf \left\{ B : \frac{1}{T} \int_0^T \mathbf{1}_{\{S_t \leq B\}} dt \geq \alpha \right\},$$

which is the lowest barrier level B such that the occupation time τ_B^- is greater than or equal to a given fraction α of the option's life. Note that $M(\alpha, T)$ becomes the realized maximum of the asset price over $[0, T]$ when $\alpha = 1$; that is, $M(1, T) = S_{\max}^{[0,T]}$.

When B is taken to be below $M(\alpha, T)$, $\tau_B^- \geq \alpha T$ will not be satisfied, so $\{\tau_B^- < \alpha T\}$ and $\{M(\alpha, T) > B\}$ are equivalent events. We then have

$$P(\tau_B^- < \alpha T, W_T \in dz) = P(M(\alpha, T) > B, W_T \in dz).$$

One can obtain the joint law of the pair $(W_T, M(\alpha, T))$ by the known law of (W_T, τ_B^-).

Seasoned (in-progress) step option

Let t be the current time and recall

$$\tau_B^-(0,T) = \tau_B^-(0,t) + \tau_B^-(t,T),$$

where $\tau_B^-(0,t)$ is already known at time t. We then deduce that

$$c_\rho^-(S,\tau_B^-(0,t),t;T,K,B) = \exp(-\rho\tau_B^-(0,t))c_\rho^-(S;T-t,K,B),$$

where K is the strike price and B is the barrier.

The asset price path over $[0,t]$ determines how the terminal payoff is affected by the factor $\exp(-\rho\tau_B^-(0,t))$ while the terminal asset price at T does not depend on the path history over $[0,t]$. At time t, the time to expiry of the option is $T-t$.

Reference

"Step Options," V. Linetsky, *Mathematical Finance*, vol.9(1) (1999), P.55-96.

Partial differential equation formulation

We consider a contingent claim written at time $t = 0$ that pays $F(S_T, \tau_B^-)$ at time T. Let $f(S, I, t)$ denote its value at time t, $t \in [0, T]$, where I is the path dependent state variable. The occupation time $\tau_B^-(t, T)$ follows the process

$$d\tau_B^-(t,T) = \mathcal{H}(B - S_t)\, dt.$$

The function f solves the following terminal value problem:

$$\frac{\sigma^2}{2} S^2 \frac{\partial^2 f}{\partial S^2} + (r-q) S \frac{\partial f}{\partial S} + \mathcal{H}(B - S) \frac{\partial f}{\partial \tau_B^-} - rf = -\frac{\partial f}{\partial t}$$

subject to the terminal condition: $f(S, I, T) = F(S, I)$. Here, we assume S_t under Q follows the dynamics:

$$\frac{dS_t}{S_t} = (r-q)\, dt + \sigma\, dZ_t.$$

Special case: separable terminal payoff

Suppose the terminal payoff is separable, where

$$F(S, I) = e^{-\rho I}\Phi(S),$$

so that the solution f is also separable:

$$f(S, I, t) = e^{-\rho I} g(S, t).$$

For $I_t = \tau_B^-(t) = \int_0^t H(B - S_u)\,du$, we have

$$\frac{\partial f}{\partial I}\frac{dI}{dt} = -\rho\frac{dI}{dt}f = -\rho\mathcal{H}(B - S)f.$$

The governing equation for g is given by

$$\frac{\sigma^2}{2}S^2\frac{\partial^2 g}{\partial S^2} + (r - q)S\frac{\partial g}{\partial S} - [r + \rho\mathcal{H}(B - S)]g = -\frac{\partial g}{\partial t}$$

subject to the terminal condition:

$$g(S, T) = \Phi(S).$$

The discount rate becomes $r+\rho$ when $S \leq B$ and it is equal to r when $S > B$. The quantity $r+\rho\mathcal{H}(B-S)$ can be interpreted as the adjusted discount rate with killing rate ρ in the down-barrier region. Once $g(S,t)$ is obtained, $f(S,I,t) = e^{-\rho\tau_B^-(0,t)}g(S,t)$ (see the in-progress step option formula).

Remarks

- Consistency in the "separability" assumption is observed in the governing equation for $g(S,t)$.

- Numerical scheme can be constructed easily by adopting the adjusted discount rate $r+\rho$ in the "barrier" region.

- The discontinuity in the damping term leads to jump in $\dfrac{\partial^2 g}{\partial S^2}$.

Perpetual step options

Consider the price function $f(S)$ of a perpetual step option, whose governing equation reduces to the Euler equation

$$\frac{\sigma^2}{2}S^2\frac{d^2f}{dS^2}+(r-q)S\frac{df}{dS}-[r+\rho\mathcal{H}(B-S)]f=0$$

with the boundary conditions:

$$f(S)\to S^{\lambda_+ +1} \quad \text{as} \quad S\to\infty \quad \text{and} \quad f(S)=0 \quad \text{as} \quad S\to 0.$$

When $S>B$, the auxiliary equation is $\dfrac{\sigma^2}{2}x(x-1)+(r-q)x-r=0$ and whose roots are $\lambda_\pm+1$, where

$$\lambda_\pm=-\lambda\pm\sqrt{\lambda^2+\frac{2q}{\sigma^2}}, \quad \lambda=\frac{r-q}{\sigma^2}+\frac{1}{2}.$$

When $S \leq B$, the auxiliary equation is

$$\frac{\sigma^2}{2}x(x-1) + (r-q)x - (r+\rho) = 0,$$

and whose roots are $\pm\lambda_\rho + 1$, where

$$\lambda_\rho = -\lambda + \sqrt{\lambda^2 + \frac{2(q+\rho)}{\sigma^2}}.$$

The continuity boundary conditions at the barrier are

$$\lim_{\varepsilon\to 0^+} f(B+\varepsilon) = \lim_{\varepsilon\to 0^+} f(B-\varepsilon), \quad \lim_{\varepsilon\to 0^+} \frac{df}{dS}(B+\varepsilon) = \lim_{\varepsilon\to 0^+} \frac{df}{dS}(B-\varepsilon).$$

Solution

The general solution takes the form:

$$f = A_1 S^{\lambda_+ + 1} + A_2 S^{1-\lambda_-}, \; S > B;$$
$$f = B_1 S^{\lambda_\rho + 1} + B_2 S^{1-\lambda_\rho}, \; S < B.$$

The time-independent solution for the perpetual down-and-out step option is given by

$$f(S) = \begin{cases} S^{\lambda_+ + 1}\left[1 - \frac{\lambda_\rho - \lambda_+}{\lambda_\rho - \lambda_-}\left(\frac{B}{S}\right)^{\lambda_+ - \lambda_-}\right] & S > B \\ \frac{\lambda_+ - \lambda_-}{\lambda_\rho - \lambda_-} S^{\lambda_+ + 1}\left(\frac{B}{S}\right)^{\lambda_+ - \lambda_\rho} & S \leq B \end{cases}.$$

We consider the two asymptotic limits:

(i) $\rho \to 0$

$$\lim_{\rho \to 0} f(S) = S^{\lambda_+ + 1}$$

which is the stationary solution of the Black-Scholes formulation with continuous dividend yield q.

(ii) $\rho \to \infty$ (standard perpetual barrier option when the knock-out rate is infinite)

$$\lim_{\rho \to \infty} f(S) = S^{\lambda_+ + 1}\left[1 - \left(\frac{B}{S}\right)^{\lambda_+ - \lambda_-}\right], \quad S > B.$$

Proportional step options

$$C_\rho^-(S; T, K, B) = e^{-rT} E_S[e^{-\rho\tau_B^-} \max(S_T - K, 0)],$$

where E_S is the conditional expectation operator associated with a geometric Brownian motion S_t, $t \in [0, T]$, started at S at time $t = 0$. The governing dynamics is

$$dS_t = (r - q)S_t \; dt + \sigma S_t \; dZ_t.$$

Introduce the following notation

$$v = \frac{1}{\sigma}\left(r - q - \frac{\sigma^2}{2}\right), \quad \gamma = r + \frac{v^2}{2},$$
$$x = \frac{1}{\sigma}\ln\left(\frac{S}{B}\right), \qquad k = \frac{1}{\sigma}\ln\left(\frac{K}{B}\right).$$

The process S_t can be represented as

$$\begin{aligned} S_t &= Se^{\left(r-q-\frac{\sigma^2}{2}\right)t+\sigma Z_t} \\ &= Be^{\sigma(Z_t+x)}e^{\sigma vt} = Be^{\sigma(vt+W_t)}, \quad W_t = Z_t + x, \end{aligned}$$

where W_t is a Brownian motion started at x at time $t = 0$. By virtue of the Girsanov Theorem, we have

$$\begin{aligned} C_\rho^-(S;T,K,B) &= e^{-rT}E_x[e^{v(W_T-x)-\frac{v^2}{2}T-\rho\Gamma_T^-}(Be^{\sigma W_T} - K)\mathbf{1}_{\{W_T\geq k\}}] \\ &= e^{-\gamma T-vx}[B\Psi_\rho(v+\sigma;k,x,T) - K\Psi_\rho(v;k,x,T)]. \end{aligned}$$

The factor $e^{v(W_T-x)-\frac{v^2}{2}T}$ is the associated Radon-Nikodym derivative. The change of measure is effected by

$$e^{vZ_T-\frac{v^2}{2}T} = e^{v(W_T-x)-\frac{v^2}{2}T}.$$

When Z_t (W_t) is Brownian under the original measure, $\widetilde{Z}_t = Z_t + vt$ ($\widetilde{W}_t = W_t + vt$) is Brownian under the new measure. We then drop "tilde" for notational convenience.

The event $S_t = Be^{\sigma(vt+W_t)} \geq B \Leftrightarrow W_t + vt \geq 0$ in the original measure $\Leftrightarrow W_t \geq 0$ in the new measure. Similarly, $S_T = Be^{\sigma(W_T+vT)} \geq K$ in the original measure is equivalent to $Be^{\sigma W_T} - K \geq 0 \Leftrightarrow W_T \geq k$ in the new measure.

Here, Γ_T^- is the occupation time of $(-\infty, 0]$ until time T, and

$$\Gamma_T^- = \int_0^T \mathbf{1}_{\{W_t \leq 0\}} \, dt.$$

The occupation time of $S_t \leq B$ in the original measure is equivalent to the occupation time of W_t staying in $(-\infty, 0)$ in the new measure.

The function $\Psi_\rho(v; k, x, T)$ is defined by

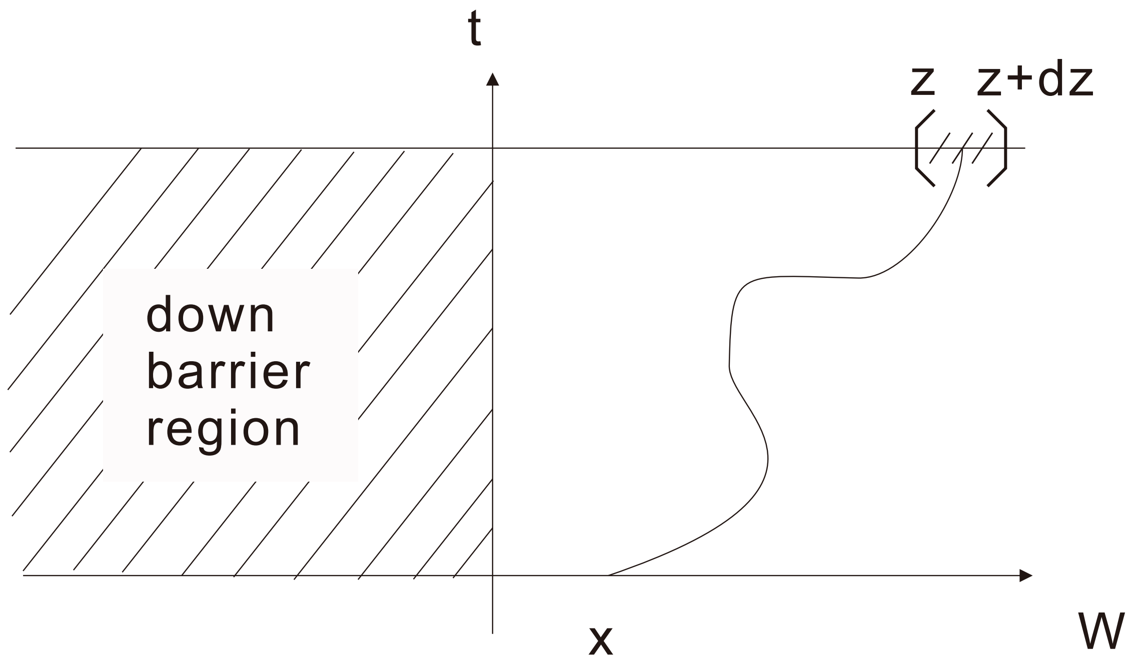
$$\Psi_\rho(v; k, x, T) = E_x[e^{vW_T - \rho\Gamma_T^-} \mathbf{1}_{\{W_T \geq k\}}] = \int_k^\infty e^{vz} E_x[e^{-\rho\Gamma_T^-}; W_T \in dz].$$

Here, E_x is associated with the Brownian motion W_t started at x at time 0.

Transition probability density of a Brownian motion with killing rate ρ

$$E_x[e^{-\rho\Gamma_T^-}; W_T \in dz] = \mathcal{K}_\rho(z, x; T)dz,$$

where $\mathcal{K}_\rho$ is the transition probability density of a Brownian motion started at x and killed at rate ρ below zero.



- Region I. $x \geq 0$, $z \geq 0$, $x + z > 0$: (initial stock price S is higher than or equal to barrier B)

  $$\mathcal{K}^I_\rho(z, x; T) = \mathcal{K}^-(z, x; T) + \int_0^T \frac{[1 - e^{-\rho(T-t)}](z + x)}{2\pi\rho(T - t)^{3/2} t^{3/2}} \; e^{-(z+x)^2/2t} dt,$$

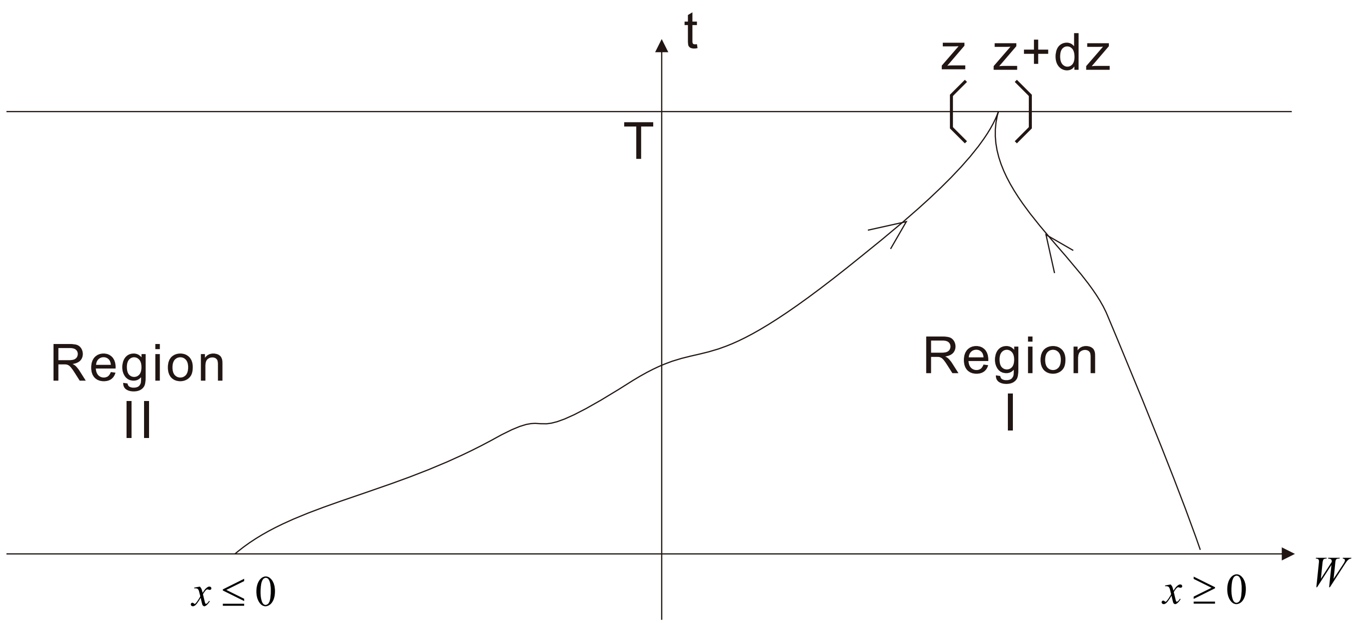
  where W_T stays outside the barrier region or at the barrier. The degenerate case $x = z = 0$ has a simpler form.

  Here, $\mathcal{K}^-$ is the transition probability density for a Brownian motion with absorbing barrier at zero and started at x

  $$\mathcal{K}^-(z, x; T) = \frac{1}{\sqrt{2\pi T}} \left[e^{-(z-x)^2/2T} - e^{-(z+x)^2/2T} \right].$$

- Region II. $x \le 0$, $z > 0$:

$$\mathcal{K}_\rho^{II}(z,x;T) = \int_0^T \frac{[1-e^{-\rho(T-t)}][z(1-x^2/(T-t))+x(1-z^2/t)]}{2\pi\rho(T-t)^{3/2}t^{3/2}} e^{-z^2/2t-x^2/[2(T-t)]}\,dt,$$



Introduce a Brownian motion $\widetilde{W}_t = -W_t$ so that

$$\begin{aligned}
\mathcal{K}_\rho(z,x;T)dz &= E_{-x}[e^{-\rho\widetilde{\Gamma}_T^+}; \widetilde{W}_T \in -dz] \\
&= e^{-\rho T} E_{-x}[e^{\rho\widetilde{\Gamma}_T^-}; \widetilde{W}_T \in -dz] \\
&= e^{-\rho T} \mathcal{K}_{-\rho}(-z,-x;T)dz.
\end{aligned}$$

- Region III. $x \geq 0$, $z < 0$:

$$\mathcal{K}_\rho^{III}(z,x;T) = e^{-\rho T}\mathcal{K}_{-\rho}^{II}(-z,-x;T);$$

- Region IV. $x \leq 0$, $z \leq 0$, $z + x < 0$:

$$\mathcal{K}_\rho^{IV}(z,x;T) = e^{-\rho T}\mathcal{K}_{-\rho}^{I}(-z,-x;T);$$

- $z = x = 0$:

$$\mathcal{K}_\rho(0,0;T) = \frac{1 - e^{-\rho T}}{\sqrt{2\pi}\rho T^{3/2}}.$$

Solution for G_ρ

Recall the forward Fokker-Planck equation for the dissipative density function $\mathcal{K}_\rho$:

$$\frac{\partial \mathcal{K}_\rho}{\partial T} = \frac{1}{2}\frac{\partial^2 \mathcal{K}_\rho}{\partial x^2} - \rho H(-x)\mathcal{K}_\rho$$

with terminal condition: $\mathcal{K}_\rho(z, x; T) = \delta(z - x)$. Here, T is the forward time variable and x is the diffusion state variable. The unit variance Brownian motion ends at the point z for sure at time T and dissipates at the rate ρ when $x \leq 0$.

The above governing differential equation resembles the option pricing equation with killing rate ρ in the downbarrier region. This is not surprising since $\mathcal{K}_\rho(z, x; T)dz$ gives the fair price of the contingent claim with terminal payoff $e^{-\rho\Gamma_T^-}dz\mathbf{1}_{\{W_T \in (z, z+dz)\}}$ subject to the amortization factor with killing rate ρ. We take the Laplace transform of $\mathcal{K}_\rho(z, x; T)$ with s as the dummy Laplace variable:

$$G_\rho(z, x; s) = \int_0^\infty e^{-sT}\mathcal{K}_\rho(z, x; T)\ dT.$$

When x tends to $-\infty$ or ∞, the transition density should tend to zero; so

$$\lim_{x\to\infty} \mathcal{K}_\rho(z,x;T) = \lim_{x\to-\infty} \mathcal{K}_\rho(z,x;T) = 0.$$

The pde is reduced to an ODE when we take the Laplace transform. Observe that the Laplace transform of $\dfrac{\partial \mathcal{K}_\rho}{\partial T}$ gives $sG_\rho - \delta(z-x)$. We then have

$$\frac{1}{2}\frac{\partial^2 G_\rho}{\partial x^2} - [s + \rho H(-x)]G_\rho = -\delta(z-x).$$

For the far field boundary conditions, by observing the corresponding far field boundary conditions for $\mathcal{K}_\rho(z,x;T)$, we observe

$$\lim_{x\to-\infty} G_\rho(z,x;s) = 0, \qquad \lim_{x\to\infty} G_\rho(z,x;s) = 0.$$

In the solution of the ODE in view of $s + \rho H(-x)$ in the coefficient of G_ρ, we observe that

(i) when $x \leq 0$, the fundamental solutions are $e^{\pm x\sqrt{2(s+\rho)}}$;

(ii) when $x > 0$, the fundamental solutions are $e^{\pm x\sqrt{2s}}$.

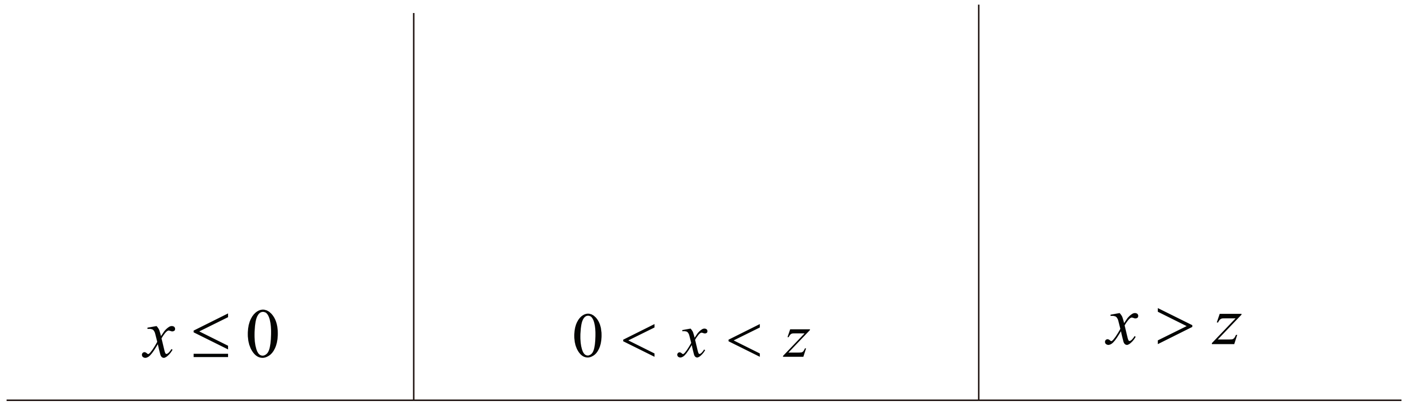
Jump conditions for G_ρ and $\dfrac{\partial G_\rho}{\partial x}$ at $x = 0$ and $x = z$

Due to the Heaviside term $H(-x)$ in the ODE, it remains to have continuity of G_ρ and $\dfrac{\partial G_\rho}{\partial x}$ at $x = 0$. However, there is a jump in $\dfrac{\partial G_\rho}{\partial x}$ at $x = z$ due to the Dirac term $\delta(z - x)$. We have

$$\lim_{\epsilon \to 0^+} [G_\rho(z, \epsilon; s) - G_\rho(z, -\epsilon; s)] = 0;$$
$$\lim_{\epsilon \to 0^+} \left[\frac{\partial G_\rho}{\partial x}(z, \epsilon; s) - \frac{\partial G_\rho}{\partial x}(z, -\epsilon; s)\right] = 0;$$
$$\lim_{\epsilon \to 0^+} [G_\rho(z, z+\epsilon; s) - G_\rho(z, z-\epsilon; s)] = 0;$$
$$\lim_{\epsilon \to 0^+} \frac{1}{2}\left[\frac{\partial G_\rho}{\partial x}(z, z+\epsilon; s) - \frac{\partial G_\rho}{\partial x}(z, z-\epsilon; s)\right] = -1.$$

It suffices to consider the case $z > 0$. The solution to $\mathcal{K}_\rho$ with $z < 0$ can be deduced from $\mathcal{K}_\rho$ obtained for $z > 0$ using the symmetry relation: $\mathcal{K}_\rho(z, x; T)dz = e^{-\rho T}\mathcal{K}_\rho(-z, -x; T)dz$. The special case $z = 0$ can be obtained in a separate (possibly simpler) procedure.

Solve the ODE for G_ρ in 3 separate segments:



For the determination of the arbitrary constants, we apply the two far field boundary conditions at $x \to \pm\infty$, and observe continuity of G and $\dfrac{\partial G_\rho}{\partial x}$ at $x = 0$, continuity of G_ρ at $x = z$ and jump of $\dfrac{\partial G_\rho}{\partial x}$ of amount 2 from $x = z - \epsilon$ to $x = z + \epsilon$.

Region I. $x > 0$, $z > 0$:

$$G_\rho^I(z,x;s) = \frac{1}{\sqrt{2s}}(e^{-|z-x|\sqrt{2s}} - \mathcal{R}_\rho(s)e^{-(z+x)\sqrt{2s}}),$$

where the coefficient $\mathcal{R}_\rho$ is given by

$$\mathcal{R}_\rho(s) = \frac{\sqrt{s+\rho}-\sqrt{s}}{\sqrt{s+\rho}+\sqrt{s}}.$$

The solution consists of the two exponential terms: $A_1 e^{-x\sqrt{2s}}$ and $A_2 e^{x\sqrt{2s}}$.

Note that $e^{-|z-x|\sqrt{2s}}$ becomes $e^{-(x-z)\sqrt{2s}}$ when $x - z > 0$, which is consistent with the requirement that the exponential term $A_2 e^{x\sqrt{2s}}$ should be excluded when $x - z > 0$.

The coefficients A_1 and A_2 are determined by the continuity conditions and jump conditions at $x = 0$ and $x = z$. This leads to the solution for $G_\rho^I(z,x;s)$ in the above form.

We rewrite

$$G^I_\rho(z,x;s) = G^-(z,x;s) + \frac{\sqrt{2}}{\sqrt{s+\rho}+\sqrt{s}}\ e^{-(z+x)\sqrt{2s}},$$

where G^- is the Laplace transform for the transition density of the unit variance Brownian motion with an absorbing barrier at zero and starting point at x (restricted Brownian motion). With both $x > 0$ and $z > 0$, we have

$$G^-(z,x;s) = \frac{1}{\sqrt{2s}}(e^{-|z-x|\sqrt{2s}} - e^{-(z+x)\sqrt{2s}}).$$

Performing the Laplace inversion and noting that

$$\mathcal{L}_t^{-1}\left\{\frac{\sqrt{2}}{\sqrt{s+\rho}+\sqrt{s}}\right\} = \frac{1-e^{-\rho t}}{\sqrt{2\pi}\rho t^{3/2}},$$

we obtain

$$\begin{aligned}\mathcal{K}^I_\rho(z,x;T) &= \frac{1}{\sqrt{2\pi T}}\left[e^{-\frac{(z-x)^2}{2T}} - e^{-\frac{(z+x)^2}{2T}}\right] \\ &\quad + \int_0^T \frac{1-e^{-\rho(T-t)}}{\sqrt{2\pi}\rho(T-t)^{3/2}} \frac{(z+x)e^{-\frac{(z+x)^2}{2t}}}{\sqrt{2\pi}t^{3/2}}\ dt.\end{aligned}$$

Some useful Laplace transform formulas

$$\mathcal{L}_T^{-1}\{e^{-a\sqrt{s}}\} = \frac{a}{2\sqrt{\pi}T^{3/2}}\ e^{-a^2/4T},\ a > 0;$$

$$\mathcal{L}_T^{-1}\left\{\frac{1}{\sqrt{s}}\ e^{-a\sqrt{s}}\right\} = \frac{e^{-a^2/4T}}{\sqrt{\pi T}}, \qquad a \geq 0;$$

$$\mathcal{L}_T^{-1}\left\{\frac{1}{\sqrt{s+a}+\sqrt{s}}\right\} = \mathcal{L}_T^{-1}\left\{\frac{\sqrt{s+a}-\sqrt{s}}{a}\right\} = \frac{1-e^{-aT}}{2a\sqrt{\pi}T^{3/2}},\ a \geq 0,$$

convolution formula:

$$\mathcal{L}\left\{\int_0^T g(t)h(T-t)\ dt\right\} = \mathcal{L}\{g(T)\}\mathcal{L}\{h(T)\}.$$

Note that $G^I_\rho(z,x;T)$ remains to be continuous at $x=0$. At $z=x=0$, $G^I_\rho(0,0;s)$ becomes $\dfrac{\sqrt{2}}{\sqrt{s+\rho}+\sqrt{s}}$, so that

$$\mathcal{K}^I_\rho(0,0;T) = \frac{1-e^{-\rho T}}{\sqrt{2\pi}\rho T^{3/2}}.$$

As a remark, the last integral term in $\mathcal{K}^I_\rho(z,x;T)$ can be expressed by

$$\int_0^T \mathcal{K}^I_\rho(0,0;T-t)\ \frac{(z+x)e^{\frac{(z+x)^2}{2t}}}{\sqrt{2\pi}t^{3/2}}\ dt,$$

which is the convolution between $\mathcal{K}^I_\rho(0,0;t)$ and the first passage time density function of a standard Brownian motion that starts at 0 and travels downstream to $-(z+x)$.

More precisely, the dummy variable t is the sum of t_1 and t_2, where t_1 is the first passage time of W_t to barrier $x=0$ with $W_0=x$ and t_2 is the last passage time to the barrier $x=0$ with $W_T=z$.

Under such scenario, W_t moves from position 0 and ends at position 0 over the remaining period $T-t$. The corresponding transition density with killing rate ρ is $\mathcal{K}^I_\rho(0,0;T-t)$.

1. When $\rho \to \infty$, we observe $\lim_{\rho\to\infty} \dfrac{1-e^{-\rho(T-t)}}{\rho(T-t)} = 0$ so that

$$\lim_{\rho\to\infty} \mathcal{K}^I_\rho(z,x;T) = \frac{1}{\sqrt{2\pi T}}\left[e^{-\frac{(z-x)^2}{2T}} - e^{-\frac{(z+x)^2}{2T}}\right].$$

   This is the same as the density function of the restricted Brownian motion with an absorbing barrier at $x=0$.

2. When $\rho \to 0$, we observe $\lim_{\rho\to 0} \dfrac{1-e^{-\rho(T-t)}}{\rho(T-t)} = 1$ so that

$$\begin{aligned}\lim_{\rho\to 0} \mathcal{K}^I_\rho(z,x;T) \;&=\; \frac{1}{\sqrt{2\pi T}}\left[e^{-\frac{(z-x)^2}{2T}} - e^{-\frac{(z+x)^2}{2T}}\right] \\ &\quad + \int_0^T \frac{1}{\sqrt{2\pi}(T-t)^{1/2}} \frac{(z+x)e^{-\frac{(z+x)^2}{2t}}}{\sqrt{2\pi}t^{3/2}}\,dt.\end{aligned}$$

By taking the Laplace transform of both functions and using the convolution formula, one can show easily that

free space density function of a standard Brownian motion starting at x and ending at $-z$

$$= \frac{1}{\sqrt{2\pi T}} \, e^{-\frac{(z+x)^2}{2T}}$$

$$= \int_0^T \frac{1}{\sqrt{2\pi}(T-t)^{1/2}} \frac{(z+x)e^{-\frac{(z+x)^2}{2t}}}{\sqrt{2\pi}t^{3/2}} \, dt.$$

Note that $\mathcal{L}_T^{-1}\left\{\frac{1}{\sqrt{2s}}\right\} = \frac{1}{\sqrt{2\pi T}}$, which is the density function of a standard Brownian motion that starts at 0 and ends at 0 again at T. Also,

$$\mathcal{L}_T^{-1}\left\{e^{-(z+x)\sqrt{2s}}\right\} = \frac{z+x}{\sqrt{2\pi}T^{3/2}} \exp\left(\frac{-(z+x)^2}{2T}\right),$$

which gives the first passage time density to the barrier $x = -z$ with $W_0 = x$.

The integral can be expressed as the product of two terms; namely,

$$P[\tau_{-z}^W \in dt; W_0 = x] = \frac{(z+x)e^{-\frac{(z+x)^2}{2t}}}{\sqrt{2\pi}t^{3/2}}\ dt$$

and

$$\begin{aligned} & P[W_T \in (-z, -z+dz) | \tau_{-z}^W = t; W_0 = x] \\ = \ & P[W_T \in (-z, -z+dz) | W_t = -z] \text{ (strong Markov property)} \\ = \ & \frac{1}{\sqrt{2\pi}\sqrt{T-t}}. \end{aligned}$$

We integrate over all first passage times over $[0, T]$ and obtain the integral as

$$\begin{aligned} & \int_0^T P[W_T \in (-z, -z+dz) | \tau_{-z}^W = t; W_0 = x] P[\tau_{-z}^W \in dt; W_0 = x] \\ = \ & \int_0^T P[W_T \in (-z, -z+dz) | W_t = -z] P[\tau_{-z}^W \in dt; W_0 = x]. \end{aligned}$$

This integral contributes to $\mathcal{K}_\rho^I(z, x; T)$ for $\rho = 0$ under the scenario where the downside barrier $x = 0$ has been breached at some time within $[0, T]$.

- Region II. $x \le 0$, $z \ge 0$:

$$G_\rho^{II}(z,x;s) = \frac{1}{\sqrt{2s}} \mathcal{T}_\rho(s) e^{x\sqrt{2(s+\rho)} - z\sqrt{2s}}.$$

where

$$\mathcal{T}_\rho(s) = 1 - \mathcal{R}_\rho(s) = \frac{2\sqrt{s}}{\sqrt{s+\rho} + \sqrt{s}}.$$

The solution consists of one exponential term: $A_3 e^{x\sqrt{2(s+\rho)}}$.

- Region III. $x \ge 0$, $z \le 0$:

$$G_\rho^{III}(z,x;s) = \frac{1}{\sqrt{2s}} \mathcal{T}_\rho(s) e^{z\sqrt{2(s+\rho)} - x\sqrt{2s}}.$$

- Region IV. $x \le 0$, $z \le 0$:

$$G_\rho^{IV}(z,x;s) = \frac{1}{\sqrt{2(s+\rho)}} \{ e^{-|z-x|\sqrt{2(s+\rho)}} + \mathcal{R}_\rho(s) e^{(z+x)\sqrt{2(s+\rho)}} \}.$$

This is obtained by swapping $z \to -z$, $x \to -x$, $s \to s+\rho$ and $s+\rho \to s$ in $G_\rho^I(z,x;s)$.

- Region I. $k \geq 0$ $(K \geq B)$ and $x \geq 0$ $(S \geq B)$:

$$
\begin{aligned}
\Psi_\rho^I(v;k,x,T) &= \int_k^\infty e^{vz}\mathcal{K}_\rho^I(z,x;T)\ dz \\
&= \frac{1}{\sqrt{2\pi T}}\int_k^\infty e^{[-(z-x)^2/2T]+vz}\ dz \\
&\quad - \frac{1}{\sqrt{2\pi T}}\int_k^\infty e^{[-(z+x)^2/2T]+vz}\ dz \\
&\quad + \int_0^T \frac{1-e^{-\rho(T-t)}}{\sqrt{2\pi}\rho(T-t)^{3/2}} \\
&\qquad \left(\frac{1}{\sqrt{2\pi}t^{3/2}}\int_k^\infty (z+x)e^{[-(z+x)^2/2t]+vz}dz\right) dt \\
&= e^{vx+v^2T/2}N(d_1) - e^{-vx+v^2T/2}N(d_3) \\
&\quad + e^{-vx}\int_0^T \frac{[1-e^{-\rho(T-t)}]e^{v^2t/2}}{\sqrt{2\pi}\rho(T-t)^{3/2}}[vN(d_5)+t^{-1/2}N'(d_5)]\ dt;
\end{aligned}
$$

where

$$
d_1 = \frac{-k+x+vT}{\sqrt{T}},\ d_3 = \frac{-k-x+vT}{\sqrt{T}},
$$
$$
d_5 = \frac{-k-x+vt}{\sqrt{t}},\ d_6 = d_5 + \sigma\sqrt{t}.
$$

The function $\Psi_\rho(v; k, x, T)$ is continuous for all $k \in \mathcal{R}$ and $x \in \mathcal{R}$.

Price of a down-and-out proportional step call at $t = 0$

- $K \geq B$ and $S \geq B$

$$
\begin{aligned}
C_\rho^-(S; T, K, B) &= e^{-\gamma T - vx}[B\Psi_\rho^I(v + \sigma; k, x, T) - K\Psi_\rho^I(v; k, x, T)] \\
&= \mathrm{DOC}(S; T, K, B) \\
&\quad + \left(\frac{B}{S}\right)^{2v/\sigma} \int_0^T \frac{[1 - e^{-\rho(T-t)}]e^{-\gamma(T-t)}}{\sqrt{2\pi}\rho(T-t)^{3/2}} \\
&\quad \left[(v + \sigma)e^{-qt}\left(\frac{B^2}{S}\right)N(d_6) - ve^{-rt}KN(d_5)\right] dt.
\end{aligned}
$$

$$\begin{aligned}\mathrm{DOC}(S;T,K,B) &= e^{-qT}SN(d_2) - e^{-rT}KN(d_1) \\ &\quad - \left(\frac{B}{S}\right)^{2v/\sigma}\left[e^{-qT}\left(\frac{B^2}{S}\right)N(d_4) - e^{-rT}KN(d_3)\right],\end{aligned}$$

$d_2 = d_1 + \sigma\sqrt{T}$, $d_4 = d_3 + \sigma\sqrt{T}$.

In the limit $\rho \to \infty$,

$$\lim_{\rho\to\infty} C_\rho^-(S;T,K,B) = \mathrm{DOC}(S;T,K,B);$$

and in the limit $\rho \to 0$,

$$\lim_{\rho\to 0} C_\rho^-(S;T,K,B) = C(S;T,K) \text{ (vanilla call option).}$$

- Region II. $k \geq 0$ ($K \geq B$) and $x \leq 0$ ($S \leq B$) :

$$\begin{aligned}
\Psi_\rho^{II}(v;k,x,T) &= \int_k^\infty e^{vz}\mathcal{K}_\rho^{II}(z,x;T)\ dz \\
&= \int_0^T \frac{1-e^{-\rho(T-t)}}{\sqrt{2\pi}\rho(T-t)^{3/2}} e^{-x^2/2(T-t)} \\
&\quad \left(\frac{1}{\sqrt{2\pi}t^{3/2}} \int_k^\infty [z(1-x^2/(T-t)) \right. \\
&\quad \left. + x(1-z^2/t)]e^{-z^2/2t+vz} dz\right) dt; \\
&= \int_0^T \frac{[1-e^{-\rho(T-t)}]e^{v^2t/2}}{\sqrt{2\pi}\rho(T-t)^{3/2}} \\
&\quad [vC_1N(d_7)+C_2N'(d_7)]e^{-x^2/[2(T-t)]}\ dt;
\end{aligned}$$

$$
\begin{aligned}
C_\rho^-(S;T,K,B) &= e^{-\gamma T - vx}[B\Psi_\rho^{II}(v+\sigma;k,x,T) - K\Psi_\rho^{II}(v;k,x,T)] \\
&= \left(\frac{B}{S}\right)^{v/\sigma} \int_0^T \frac{[1-e^{-\rho(T-t)}]e^{-\gamma(T-t)}}{\sqrt{2\pi}\rho(T-t)^{3/2}} \\
&\qquad [(v+\sigma)C_3 e^{-qt} BN(d_8) - vC_1 e^{-rt} KN(d_7) \\
&\qquad - \sigma x t^{-1/2} e^{-qt} BN'(d_8)] e^{-x^2/[2(T-t)]}\ dt;
\end{aligned}
$$

$$
d_7 = \frac{-k+vt}{\sqrt{t}},\ d_8 = d_7 + \sigma\sqrt{t};
$$

$$
C_1 = 1 - \frac{x^2}{T-t} - vx,\ C_2 = \frac{C_1}{\sqrt{t}} - \frac{xk}{t\sqrt{t}},\ C_3 = C_1 - 6x.
$$

Continuity of value function and delta; jump in gamma

$$
\lim_{\epsilon\to 0^+}[C_\rho^-(B+\epsilon;T,K,B) - C_\rho^-(B-\epsilon;T,K,B)] = 0
$$

$$
\lim_{\epsilon\to 0^+}[\Delta_\rho^-(B+\epsilon;T,K,B) - \Delta_\rho^-(B-\epsilon;T,K,B)] = 0
$$

$$
\lim_{\epsilon\to 0^+}[\Gamma_\rho^-(B+\epsilon;T,K,B) - \Gamma_\rho^-(B-\epsilon;T,K,B)] = \frac{2\rho}{\sigma^2 B^2} C_\rho^-(B;T,K,B).
$$

- Region III. $k \le 0$ $(K \le B)$ and $x \ge 0$ $(S \ge B)$:

$$
\begin{aligned}
\Psi_\rho^{III}(v;k,x,T) &= \int_0^\infty e^{vz}\mathcal{K}_\rho^{I}(z,x;T)\ dz + \int_k^0 e^{vz}\mathcal{K}_\rho^{III}(z,x;T)\ dz \\
&= \int_0^\infty e^{vz}\mathcal{K}_\rho^{I}(z,x;T)\ dz + \int_{-\infty}^0 e^{vz}\mathcal{K}_\rho^{III}(z,x;T)\ dz \\
&\quad -\int_{-\infty}^{k} e^{vz}\mathcal{K}_\rho^{III}(z,x;T)\ dz; \\
&= \Psi_\rho^{I}(v;0,x,T) \\
&\quad + e^{-\rho T}[\Psi_{-\rho}^{II}(-v;0,-x,T) - \Psi_{-\rho}^{II}(-v;-k,-x,T)];
\end{aligned}
$$

$$
C_\rho^-(S;T,K,B) = e^{-\gamma T - vx}[B\Psi_\rho^{III}(v+\sigma;k,x,T) - K\Psi_\rho^{III}(v;k,x,T)].
$$

We apply a useful symmetry property of the function $\mathcal{K}_\rho(z, x; T)$:

$$\mathcal{K}_\rho(z, x; T) = e^{-\rho T} \mathcal{K}_{-\rho}(-z, -x; T).$$

- Region IV. $k \leq 0$ $(K \leq B)$ and $x \leq 0$ $(S \leq B)$:

$$\begin{aligned} \Psi_\rho^{IV}(v; k, x, T) &= \int_0^\infty e^{vz} \mathcal{K}_\rho^{II}(z, x; T)\ dz + \int_k^0 e^{vz} \mathcal{K}_\rho^{IV}(z, x; T)\ dz \\ &= \int_0^\infty e^{vz} \mathcal{K}_\rho^{II}(z, x; T)\ dz + \int_{-\infty}^0 e^{vz} \mathcal{K}_\rho^{IV}(z, x; T)\ dz \\ &\quad - \int_{-\infty}^k e^{vz} \mathcal{K}_\rho^{IV}(z, x; T)\ dz. \\ &= \Psi_\rho^{II}(v; 0, x, T) \\ &\quad + e^{-\rho T}[\Psi_{-\rho}^{I}(-v; 0, -x, T) - \Psi_{-\rho}^{I}(-v; -k, -x, T)] \end{aligned}$$

$$C_\rho^-(S; T, K, B) = e^{-\gamma T - vx}[B\Psi_\rho^{IV}(v+\sigma; k, x, T) - K\Psi_\rho^{IV}(v; k, x, T)].$$

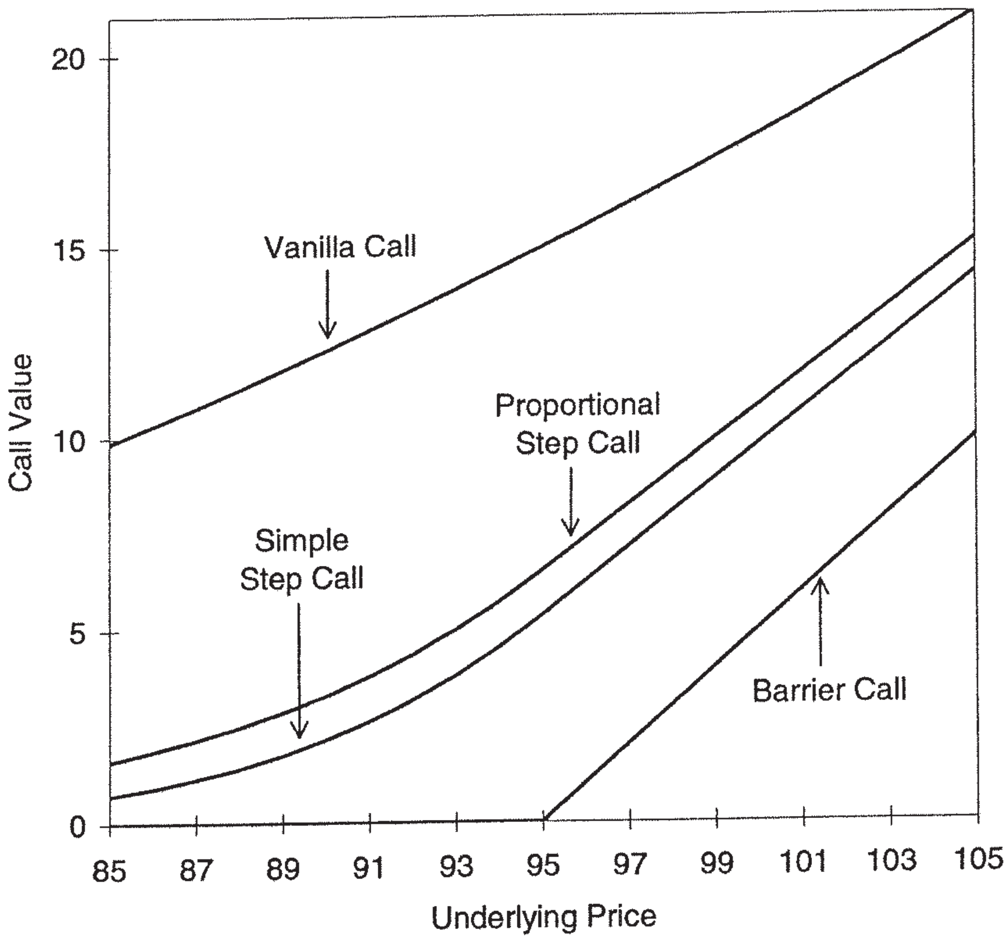
Vanilla, Down-and-Out Proportional Step, Simple Step, and Barrier Call Values and Deltas as Functions of the Underlying Asset Price S

	vanilla call		proportional step		simple step		barrier	
S	C	Δ	$C_\rho^{P,-}$	$\Delta_\rho^{P,-}$	$C_\rho^{s,-}$	$\Delta_\rho^{s,-}$	DOC	Δ^-
85	9.8517	0.4554	1.6062	0.2376	0.7200	0.1730	0	0
90	12.2641	0.5091	3.2951	0.4602	2.1528	0.4291	0	0
95	14.9373	0.5597	6.5008	0.8598	5.3548	0.8908	0	1.0058
96	15.5019	0.5694	7.3603	0.8591	6.2450	0.8895	1.0044	1.0029
97	16.0760	0.5790	8.2192	0.8587	7.1339	0.8884	2.0060	1.0003
102	19.0867	0.6247	12.5113	0.8589	11.5668	0.8855	6.9780	0.9892
105	20.9994	0.6503	15.0904	0.8607	14.2229	0.8855	9.9376	0.9841

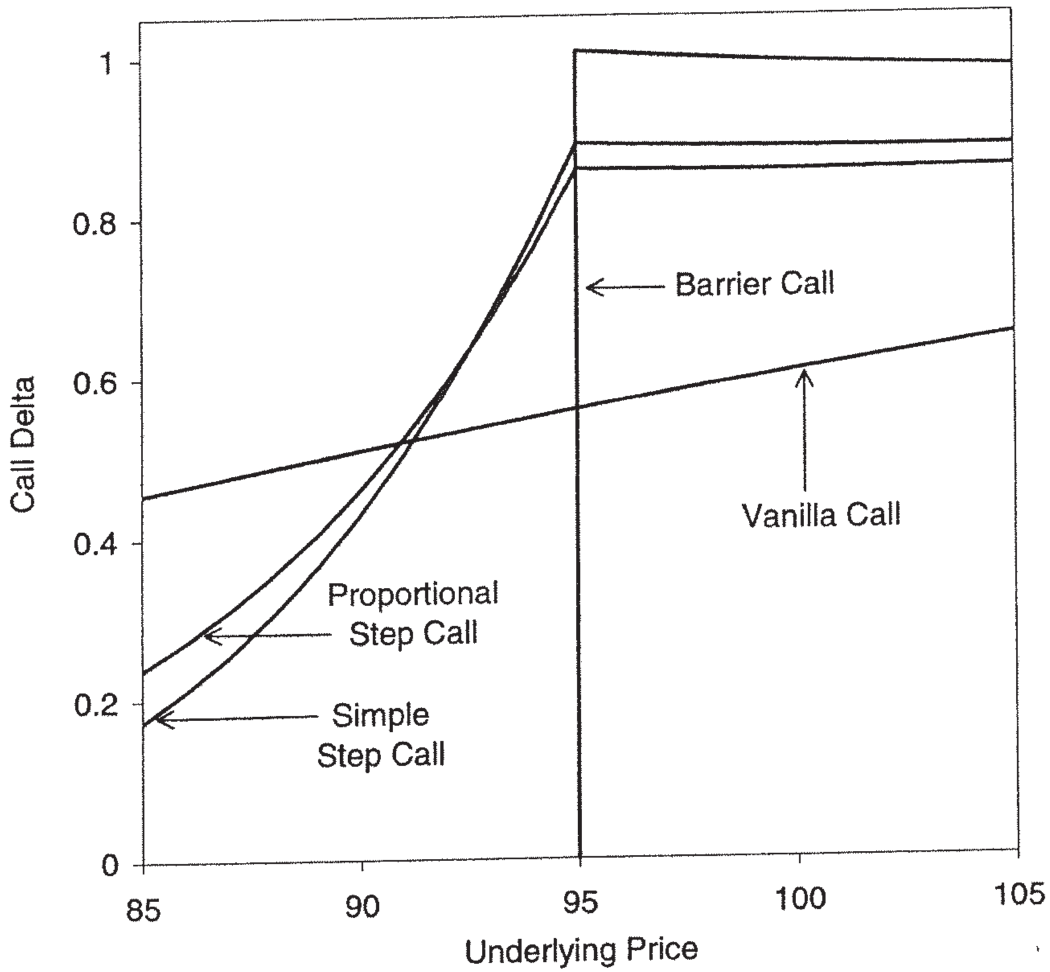
Option parameters: $K = 100$, $B = 95$, $\sigma = 0.6$, $r = 0.05$, $q = 0$, $T = 0.5$ (six months).

Proportional step call parameters: $\beta = 0.9$ ($\rho = 26.34$, $T_B^- = 21.85$ trading days).

Simple step call parameters: $\rho_d = 0.1$ ($\rho = 25$, $T_B^- = 10$ trading days, $\beta = 0.9$).



Vanilla, down-and-out proportional step, simple step, and barrier call values as functions of the asset price S. Option parameters: $K = 100$, $B = 95$, $\sigma = 0.6$, $r = 0.05$, $q = 0$, $T = 0.5$ (six months). Proportional step call parameters: $\beta = 0.9$ ($\rho = 26.34$, $T_B^- = 21.85$ trading days). Simple step call parameters: $\rho_d = 0.1$ ($\rho = 25$, $T_B^- = 10$ trading days, $\beta = 0.9$).



Vanilla, down-and-out proportional step, simple step, and barrier call values as functions of the current asset price S. Option parameters: $K = 100$, $B = 95$, $\sigma = 0.6$, $r = 0.05$, $q = 0$, $T = 0.5$ (six months). Proportional step call parameters: $\beta = 0.9$ ($\rho = 26.34$, $T_B^- = 21.85$ trading days). Simple step call parameters: $\rho_d = 0.1$ ($\rho = 25$, $T_B^- = 10$ trading days, $\beta = 0.9$).

Joint law of (W_T, Γ_T^-)

Define the joint density of terminal value of Brownian motion and occupation time by

$$p_x(W_T \in dz, \Gamma_T^- \in dt) = p_x(z,t;T)\ dzdt, \quad -\infty < z < \infty,\ t \leq T.$$

We would like to show: $p_x(z,t;T) = \mathcal{L}_t^{-1}\mathcal{K}_\rho(z,x;T)$. Consider

$$\begin{aligned}
& E_x[e^{-\rho\Gamma_T^-}; W_T \in dz] \\
= \ & \mathcal{K}_\rho(z,x;T)\ dz \\
= \ & \int_0^T e^{-\rho t} p_x(z,t;T)\ dtdz \\
= \ & \left(\int_0^\infty e^{-\rho t} p_x(z,t;T)\ dt\right) dz \quad \text{since } p_x(z,t;T) = 0 \text{ for } t > T.
\end{aligned}$$

Since the last integral can be visualized as the Laplace transform of $p_x(z,t;T)$ with the Laplace variable ρ, so

$$p_x(z,t;T) = \mathcal{L}_t^{-1}[\mathcal{K}_\rho(z,x;T)].$$

Region I. $x \geq 0$, $z \geq 0$, $z + x > 0$:

$$\begin{aligned} p_x^I(z,t;T) &= \mathcal{L}_t^{-1}\{\mathcal{K}_\rho^I(z,x;T)\} \\ &= \int_0^{T-t} \frac{(z+x)}{2\pi(T-u)^{3/2}u^{3/2}} \exp\left(-\frac{(z+x)^2}{2u}\right) du; \end{aligned}$$

or rewrite it as

$$p_x^I(z,t;T)\ dt = \int_0^{T-t} \frac{1}{\sqrt{2\pi}u^{1/2}} \frac{(z+x)}{\sqrt{2\pi}(T-u)^{3/2}} e^{-\frac{(z+x)^2}{2(T-u)}}\ \frac{dt}{u}\ du,$$

where u is the time variable lapsed backward from T. Note that u runs from 0 to $T-t$ (since $u > T-t$ means the calendar time is less than t and should be ruled out). Contribution to p_x^I arises only when the barrier $x = 0$ is breached.

Region II. $x \leq 0$, $z > 0$:

$$\begin{aligned} p_x^{II}(z,t;T) &= \mathcal{L}_t^{-1}\{\mathcal{K}_\rho^{II}(z,x;T)\} \\ &= \int_0^{T-t} \frac{\{z[1 - x^2/(T-u)] + x(1 - z^2/u)\}}{2\pi(T-u)^{3/2}u^{3/2}} \\ &\qquad \exp\left(-\frac{z^2}{2u} - \frac{x^2}{2(T-u)}\right) du. \end{aligned}$$

- Region III. $x \geq 0$, $z < 0$:

$$p_x^{III}(z,t;T) = \mathcal{L}_t^{-1}\{\mathcal{K}_\rho^{III}(z,x;T)\} = p_{-x}^{II}(-z,T-t;T);$$

where the last equality is deduced from $\mathcal{K}_\rho(z,x;T) = e^{-\rho T}\mathcal{K}_{-\rho}(-z,-x;T)$.

- Region IV. $x \leq 0$, $z \leq 0$, $z + x < 0$:

$$p_x^{IV}(z,t;T) = \mathcal{L}_t^{-1}\{\mathcal{K}_\rho^{IV}(z,x;T)\} = p_{-x}^{I}(-z,T-t;T);$$

- $x = z = 0$: $p_0(0,t;T) = \mathcal{L}_t^{-1}\{\mathcal{K}_\rho(0,0;T)\} = \dfrac{1}{\sqrt{2\pi}T^{3/2}}$;

- $t = 0$, $x \geq 0$, $z \geq 0$: $p_x(W_T \in dz, \Gamma_T^- = 0) = \mathcal{K}^-(z,x;T)\ dz$;

- $t = T$, $x \leq 0$, $z \leq 0$: $p_x(W_T \in dz, \Gamma_T^- = T) = \mathcal{K}^-(z,x;T)\ dz$.

Pricing of contingent claims with payoff $F(S_T, \tau_B^-)$

The price at $t = 0$ of a claim with the payoff $F(S_T, \tau_B^-)$ at time T and $S \geq B$ [corresponds to $x \geq 0$] is given by

$$\begin{aligned} C_F(S; T, B) \;&=\; e^{-\gamma T - vx} \left\{ \int_0^\infty F(Be^{\sigma z}, 0) e^{vz} \mathcal{K}^-(z, x; T)\; dz \right. \\ &\quad + \int_0^T \int_0^\infty F(Be^{\sigma z}, t) e^{vz} p_x^I(z, t; T)\; dz dt \\ &\quad \left. + \int_0^T \int_{-\infty}^0 F(Be^{\sigma z}, t) e^{vz} p_x^{III}(z, t; T)\; dz dt \right\}. \end{aligned}$$

Here, $\mathcal{K}^-(z, x; T) dz$ gives the probability that $W_T \in (z, z + dz)$ while the stock price never crosses the downstream barrier (corresponds to $\tau_B^- = 0$).

For $\tau_B^- > 0$, z can assume values from $-\infty$ to ∞. When $z \geq 0$, $p_x^I(z, t; T)$ is used; while when $z \leq 0$, we use $p_x^{III}(z, t; T)$.

Delayed barrier options and simple step options

With separable payoff: $f(\tau_B^-)\Phi(S_T)$, we have

$$\begin{aligned} C_f(S;T,K,B) &= e^{-rT}E_S[f(\tau_B^-)\Phi(S_T)] \\ &= e^{-rT}\int_0^T f(t)\int_{-\infty}^{\infty}\Phi(\delta)p_S(\delta,t;T)\ d\delta dt. \end{aligned}$$

Recall

$$\mathcal{L}_\rho\{p_S(\delta,t;T)\} = E_S[e^{-\rho\tau_B^-};S_T\in d\delta]$$

so that

$$\begin{aligned} C_f(S;T,K,B) &= \int_0^T f(u)\mathcal{L}_u^{-1}\{e^{-rT}E_S[e^{-\rho\tau_B^-}\Phi(S_T)]\}\ du \\ &= \int_0^T f(u)\mathcal{L}_u^{-1}\{C_\rho^-(S;T,K,B)\}\ du. \end{aligned}$$

Note that $\Psi_\rho(v;k,x,T)$ in the price function $C_\rho^-(S;T,K,B)$ invariably contains the factor $\dfrac{1-e^{-\rho(T-t)}}{\rho}$, which arises from the choice of $f(u)=e^{-\rho u}$.

A useful identity: Given $\mathcal{L}_u^{-1}\left\{\frac{e^{-\rho u_0}}{\rho}\right\} = H(u-u_0)$, we have

$$
\begin{aligned}
& \int_0^T f(u)\mathcal{L}_u^{-1}\left[\frac{1-e^{-\rho(T-t)}}{\rho}\right] du \\
= & \int_0^T f(u)[H(u) - H(u-(T-t))]\ du \\
= & \int_0^{T-t} f(u)\ du = F(T-t).
\end{aligned}
$$

By using the price function of the proportional step option with down-barrier $C_\rho^-(S;T,K,B)$, we obtain

$K \geq B$ and $S \geq B$:

$$
\begin{aligned}
C_f(S;T,K,B) = \ & f(0)\mathrm{DOC}(S;T,K,B) \\
& + \left(\frac{B}{S}\right)^{2v/\sigma} \int_0^T \frac{F(T-t)e^{-\gamma(T-t)}}{\sqrt{2\pi}(T-t)^{3/2}} \\
& \left[(v+\sigma)e^{-qt}\left(\frac{B^2}{S}\right)N(d_6) - ve^{-rt}KN(d_5)\right] dt,
\end{aligned}
$$

where the factor $F(T-t)$ reveals the functional dependence of the terminal payoff on τ_B^- of the occupation time derivative.

For proportional step options, delayed barrier options, and simple step options we have

$$
\begin{aligned}
F_p(T-t) &= \int_0^{T-t} e^{-\rho u}\, du = \frac{1-e^{-\rho(T-t)}}{\rho}, \\
F_d(T-t) &= \int_0^{T-t} \mathbf{1}_{\{u<\alpha T\}}\, du = \begin{cases} \alpha T, & 0 \le t \le (1-\alpha)T \\ T-t, & (1-\alpha)T < t \le T \end{cases}, \\
F_s(T-t) &= \int_0^{T-t} \max(1-\rho u, 0)\, du \\
&= \begin{cases} \frac{1}{2\rho}, & 0 \le t \le T-\frac{1}{\rho} \\ (T-t)[1-\frac{\rho}{2}(T-t)], & T-\frac{1}{\rho} < t \le T \end{cases}.
\end{aligned}
$$

1.7 Discretely monitored barrier options

Discrete and continuous monitoring of the asset price process

- The asset price process is monitored over the life of the option contract for breaching of a barrier level. In actual implementation, these monitoring procedures can only be performed at discrete time instants rather than continuously at all times.

- When the asset price path is monitored at discrete time instants, the analytic forms of the price formulas become quite daunting since they involve multi-dimensional cumulative normal distribution functions and the dimension is equal to the number of monitoring instants.

Correction formula for discretely monitored barrier options

Let $V(B;m)$ be the price of a discretely monitored knock-in or knock-out down call or up put option with constant barrier B and m monitoring instants. Let $V(B)$ be the price of the corresponding continuously monitored barrier option. We have

$$V(B;m) = V(Be^{\pm\beta\sigma\sqrt{\Delta t}}) + o\left(\frac{1}{\sqrt{m}}\right),$$

where $\beta = -\xi\left(\frac{1}{2}\right) \Big/ \sqrt{2\pi} \approx 0.5826$, ξ is the Riemann zeta function, σ is the volatility, Δt is the uniform time interval between two successive monitoring instants.

The "+" sign is chosen when $B > S$, while the "−" sign is chosen when $B < S$.

One observes that the correction shifts the barrier away from the current underlying asset price by a factor of $e^{\beta\sigma\sqrt{\Delta t}}$.

Numerical comparison

Up-and-out call price ($m = 50$, roughly daily monitoring)

barrier level	option price under continuous barrier	option price using correction formula	exact value
155	12.775	12.905	12.894
150	12.240	12.448	12.431
145	11.395	11.707	11.684
140	10.144	10.581	10.551
135	8.433	8.994	8.959
130	6.314	6.959	6.922
125	4.012	4.649	4.616
120	1.938	2.442	2.418
115	0.545	0.819	0.807

Option parameters: $S(0) = 110$, $K = 100$, $\sigma = 0.30$ per year, $r = 0.1$, $T = 0.2$ year (roughly 50 trading days).

- The errors in adopting the continuous barrier price formula as an approximation can be quite significant when the stock price is close to the barrier.

Formulation of discretely monitored barrier options

In the discretely monitoring case, at the n^{th} monitoring point $n\Delta t$ with $\Delta t = T/m$, the asset price under the risk neutral measure Q is given by

$$S_n = S(0)\exp\left(\mu n\Delta t + \sigma\sqrt{\Delta t}\sum_{i=1}^{n} Z_i\right) = S(0)\exp(W_n\sigma\sqrt{\Delta t}),\ \ n = 1,\ldots,m,$$

where the random walk W_n is defined by

$$W_n = \sum_{i=1}^{n}\left(Z_i + \frac{\mu}{\sigma}\sqrt{\Delta t}\right).$$

Here, the drift is given by $\mu = r - \sigma^2/2$ and Z_i's are independent standard normal random variables.

Intuition behind the continuity correction for random walk: Corrections to normal approximation are made to adjust for the "*overshoot*" effects when a discrete random walk crosses a barrier.

We *rescale* the breaching condition from the stock price process S to the Wiener process W. Let H be the barrier, we consider

$$
\begin{aligned}
S_n \geq H \;&\Leftrightarrow\; \exp(W_n \sigma \sqrt{\Delta t}) \geq \frac{H}{S(0)} \\
&\Leftrightarrow\; W_n \geq \frac{1}{\sigma\sqrt{\Delta t}} \ln \frac{H}{S(0)} = \frac{a\sqrt{m}}{\sigma\sqrt{T}}, \text{ where } a = \ln \frac{H}{S(0)}.
\end{aligned}
$$

Let τ' (integer valued) be the (discrete) first passage time to the barrier x. The barrier is not hit until maturity (m^{th} time step) if and only if

$$
\tau'\left(\underbrace{\frac{a}{\sigma\sqrt{T}}}_{x}, W\right) > m \Leftrightarrow W_n < \underbrace{\frac{a}{\sigma\sqrt{T}}}_{x}\sqrt{m} \quad \text{for } n = 1, 2, \ldots, m.
$$

In the present context, we consider a first passage problem for the random walk W_n with small drift $\left[\frac{\mu}{\sigma}\sqrt{\Delta t} \to 0 \text{ as } m \to \infty\right]$ to cross a high barrier $\left[\frac{a}{\sigma\sqrt{T}}\sqrt{m} \to \infty \text{ as } m \to \infty\right]$.

$$\tau'(H,S) \;=\; \text{(discrete) first passage time (in units of monitoring time intervals) that the stock price reaches } H \text{ or above; when } \tau'(H,S) \text{ assumes } k\text{, the calendar time is } k\Delta t.$$

$I(\tau'(H,S) > m)$ is the indicator function that the barrier call option survives up to the maturity date.

$m =$ number of monitoring instants

$S_m =$ stock price at the last monitoring instant (maturity date)

The price of the discrete up-and-out call option is given by

$$\begin{aligned} V_m(H) &= E^*[e^{-rT}(S_m - K)^+ I\{\tau'(H,S) > m\}] \\ &= E^*[e^{-rT}(S_m - K)^+ I\{\tau'(\frac{a}{\sigma\sqrt{T}}, W) > m\}], \end{aligned}$$

where $a = \ln \dfrac{H}{S(0)} > 0$, $\tau'(H,S) = \inf\{n \geq 1\colon\ S_n \geq H\}$, $\tau'(x,W) = \inf\{n \geq 1\colon\ W_n \geq x\sqrt{m}\}$, where $x = \dfrac{a}{\sigma\sqrt{T}}$.

We consider standardized quantities, where σ and T are set to be unity so that $\Delta t = \dfrac{1}{m}$, where m is the number of monitoring instants.

For unit variance $U(t)$ and $U_m(n)$, we have $U(t) = \mu t + B(t)$ and $U_m(n)$ is a random walk with a small drift (as $m \to \infty$),

$$U_m(n) = \sum_{i=1}^{n} \left(Z_i + \frac{\mu}{\sqrt{m}} \right),$$

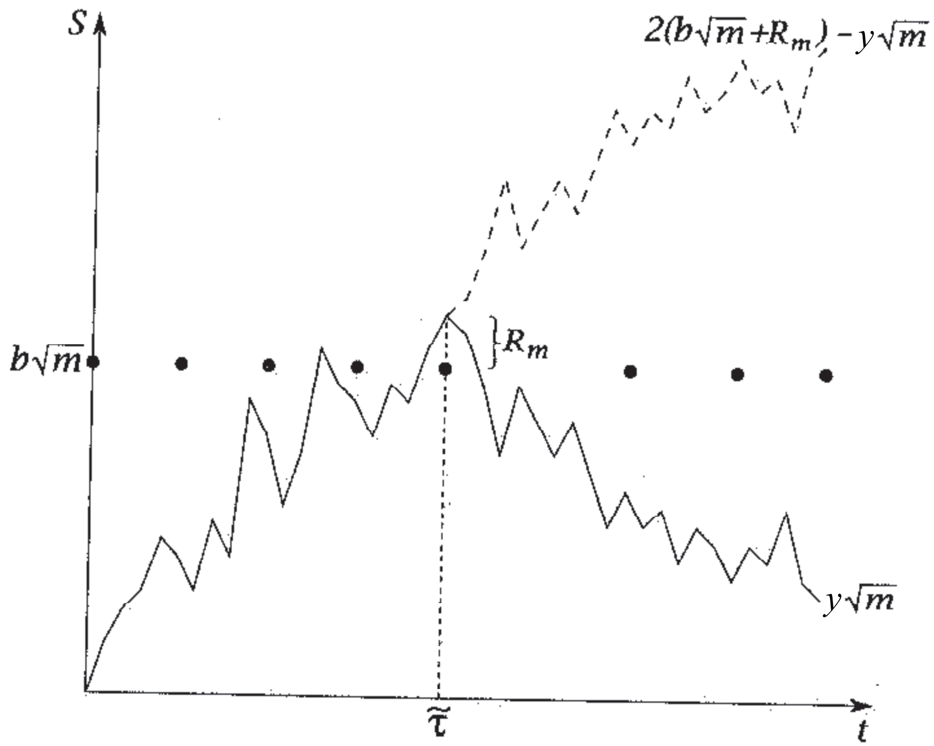
where Z_i's are independent standard normal random variables.

- n is the running index of the discrete random walk with m total increments
- t is the running time of the continuous Brownian motion up to time $T = 1$.

Reflection principle (discrete version)

The random overshoot of $U_m(\tau')$ over the barrier $b\sqrt{m}$ is defined by $R_m = U_m(\tau') - b\sqrt{m}$. The reflection principle for random walk should be

$$P[U_m < y\sqrt{m}, \tau'(b, U_m) \leq m] = P[U_m \geq 2(b\sqrt{m} + R_m) - y\sqrt{m}].$$



An illustration of the discrete reflection principle

Discrete Girsanov Theorem

For any probability measure P, let $\widehat{P}$ be defined by

$$\frac{d\widehat{P}}{dP} = \exp\left(\sum_{i=1}^{m} a_i Z_i - \frac{1}{2}\sum_{i=1}^{m} a_i^2\right),$$

where a_i, $i = 1, \ldots, n$, are arbitrary constants, and Z_i's are standard normal random variables under the probability measure P. Then under the probability measure $\widehat{P}$, for every $1 \le i \le m$, $\widehat{Z}_i := Z_i - a_i$ is a standard normal random variable.

Rescaling property

For Brownian motions with drifts $\alpha\mu$ and μ, and unit standard deviation, we have

$$P[W_{\alpha\mu}(1) \geq x, \tau(c, W_{\alpha\mu}) > 1] = P[W_\mu(\alpha^2) \geq \alpha x, \tau(\alpha c, W_\mu) > \alpha^2],$$

where $W_\mu(t)$ denotes the Brownian motion with drift μ and unit standard deviation.

Proof

Suppose μ is increased by a factor of α, then $W_{\alpha\mu}(t) = \alpha\mu t + B(t)$. Considering the increase of time by a factor of α^2, we observe

$$W_\mu(\alpha^2 t) = \mu(\alpha^2 t) + B(\alpha^2 t) = \alpha[\alpha\mu t + B(t)] = \alpha W_{\alpha\mu}(t);$$

and

$$W_{\alpha\mu}(1) \geq x \Leftrightarrow W_\mu(\alpha^2) \geq \alpha x.$$

Main Theorem

For $b \geq y$ and $b > 0$, the discrete joint distribution for U_m and $\tau'(b, U)$ and the continuous joint distribution for $U(1)$ and $\tau(b + \beta/\sqrt{m}, U)$ are related by

$$P[U_m < y\sqrt{m}, \tau'(b, U) \leq m] = P[U(1) \leq y, \tau(b{+}\beta/\sqrt{m}, U) \leq 1]{+}o(1/\sqrt{m}),$$

where $\beta = \dfrac{1}{\sqrt{2}}\left\{1 - \dfrac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\left[\dfrac{1}{\sqrt{n}} - \sqrt{\pi}\begin{pmatrix} -\frac{1}{2} \\ n \end{pmatrix}(-1)^n\right]\right\} = -\dfrac{\xi\left(\frac{1}{2}\right)}{\sqrt{2\pi}}$, and ξ is the Riemann-Zeta function.

Proof

Replacing the random overshoot R_m by its expectation $E[R_m]$, whose value can be shown to converge to $\beta = -\dfrac{\xi\left(\frac{1}{2}\right)}{\sqrt{2\pi}}$.

We then have

$$
\begin{aligned}
& P[U_m < y\sqrt{m}, \tau'(b,U) \le m] \\
\approx\ & P\left[U_m \ge 2\left(b + \frac{\beta}{\sqrt{m}}\right)\sqrt{m} - y\sqrt{m}\right] \text{ (renewal theory plus reflection principle)} \\
\approx\ & P\left[U(1) \ge 2\left(b + \frac{\beta}{\sqrt{m}}\right) - y\right] \\
=\ & P\left[U(1) \le y, \tau\left(b + \frac{\beta}{\sqrt{m}}, U\right) \le 1\right] \text{ (reflection principle)}
\end{aligned}
$$

where τ is the stopping time for the continuous counterpart.

Intuitive interpretation

The expectation of random overshoot is similar to the average of residual life, which is defined as the interval from time t until the next renewal event. For example, if we arrive at a bus stop at time t and buses arrive according to a renewal process, then the residual life is the time that we have to wait for a bus to arrive.

Limiting expectation of overshoot

The constant β is the limiting expectation of the overshoot, which can be viewed as an approximation to the average of the amount by which the random walk U_m exceeds the boundary $b\sqrt{m}$ the first time the random walk is above the boundary. By renewal theory, we have

$$\beta = \frac{E[A_N^2]}{2E[A_N]},$$

where the mean zero random walk A_n is defined as

$$A_n = \sum_{i=1}^{n} Z_i$$

and N is the first ladder height associated with A_n,

$$N = \min\{n \geq 1 : A_n > 0\}.$$

From Spitzer (1960), we have

$$E[A_N] = \frac{1}{\sqrt{2}}\ e^{w_0},\ E[A_N^2] = \left[w_2 + \frac{E[Z_1^3]}{3\sqrt{2}} - \sqrt{2}w_1\right] e^{w_0},$$

where

$$w_0 = \sum_{n=1}^{\infty} \frac{1}{n}\left[P[A_n \le 0] - \frac{1}{2}\right],\ w_1 = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}\left\{E\left[\left(\frac{A_n}{\sqrt{n}}\right)\right] - \frac{1}{\sqrt{2\pi}}\right\},$$

$$w_2 = 1 - \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \left[\frac{1}{\sqrt{n}} - \sqrt{\pi}\begin{pmatrix} -\frac{1}{2} \\ n \end{pmatrix}(-1)^n\right],$$

and $\begin{pmatrix} x \\ n \end{pmatrix} = \dfrac{x(x-1)\ldots(x-n+1)}{n!}$.

For normal random variables, we have $w_0 = 0$, $w_1 = 0$, $E[Z_1^3] = 0$, so

$$\begin{aligned}
\beta = \frac{E[A_N^2]}{2E[A_N]} &= \frac{\left[w_2 + \frac{E[Z_1^3]}{3\sqrt{2}} - \sqrt{2}w_1\right] e^{w_0}}{2\frac{1}{\sqrt{2}}e^{w_0}} = \frac{w_2}{\sqrt{2}} \\
&= \frac{1}{\sqrt{2}}\left[1 - \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \left[\frac{1}{\sqrt{n}} - \sqrt{\pi}\begin{pmatrix} -\frac{1}{2} \\ n \end{pmatrix}(-1)^n\right]\right].
\end{aligned}$$

From classical analysis, recall the property of the Riemann-Zeta function $\xi(s)$, where

$$\xi(s) = \lim_{x \uparrow 1} \sum_{n=1}^{\infty} \left\{ \frac{x^n}{n^s} - \Gamma(1-s) \left(\ln \frac{1}{x} \right)^{s-1} \right\},$$

where

$$\Gamma(1+s) = \int_0^{\infty} e^{-t} t^s \, dt.$$

Taking $s = 1/2$, and observing $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, we have

$$\lim_{x \uparrow 1} \sum_{n=1}^{\infty} \left\{ \frac{x^n}{\sqrt{n}} - \sqrt{\pi} \left[\ln \frac{1}{x} \right]^{-1/2} \right\} = \xi\left(\frac{1}{2}\right).$$

After same tedious manipulation, we obtain

$$\beta = -\frac{\xi\left(\frac{1}{2}\right)}{\sqrt{2\pi}}.$$

For details, see the proof in Appendix B in Kou's paper titled "Discrete barrier and lookback options" (2008).

Up-and-out call option

For valuation of up-and-out call, we need the following result:

For any constants $b \geq y$ and $b > 0$,

$$P(U_m \geq y\sqrt{m}, \tau'(b, U) > m) = P(U(1) \geq y, \tau(b + \beta/\sqrt{m}, U) > 1) + o(1/\sqrt{m}).$$

Simple algebra yields

$$\begin{aligned}
& P[U_m \geq y\sqrt{m}, \tau'(b, U) > m] \\
= \; & P[\tau'(b, U) > m] - P[U_m < y\sqrt{m}, \tau'(b, U) > m] \\
= \; & P[U_m < b\sqrt{m}, \tau'(b, U) > m] - P[U_m < y\sqrt{m}, \tau'(b, U) > m] \\
= \; & P[U_m < b\sqrt{m}] - P[U_m < b\sqrt{m}, \tau'(b, U) \leq m] - P[U_m < y\sqrt{m}] \\
& + P[U_m < y\sqrt{m}, \tau'(b, U) \leq m].
\end{aligned}$$

We use the Theorem to relate the distribution functions of the discrete random walks to those of the continuous Brownian motions and obtain

$$P[U_m < b\sqrt{m}, \tau'(b,U) \le m] = P[U(1) \le b, \tau(b+\beta/\sqrt{m}, U) \le 1] + o(1/\sqrt{m}),$$
$$P[U_m < y\sqrt{m}, \tau'(b,U) \le m] = P[U(1) \le y, \tau(b+\beta/\sqrt{m}, U) \le 1] + o(1/\sqrt{m}),$$

we have

$$\begin{aligned}
& P[U_m \ge y\sqrt{m}, \tau'(b,U) > m] \\
= \; & P[U(1) \le b] - P[U(1) \le b, \tau(b+\beta/\sqrt{m}, U) \le 1] - P[U(1) \le y] \\
& + \; P[U(1) \le y, \tau(b+\beta/\sqrt{m}, U) \le 1] + o(1/\sqrt{m}) \\
= \; & P[\tau(b+\beta/\sqrt{m}, U) > 1] - P[U(1) \le y, \tau(b+\beta/\sqrt{m}, U) > 1] + o(1/\sqrt{m}) \\
= \; & P[U(1) \ge y, \tau(b+\beta/\sqrt{m}, U) > 1] + o(1/\sqrt{m}).
\end{aligned}$$

Note that

$$\begin{aligned}
& E^*[e^{-rT}(S_m - K)^+ I(\tau'(H,S) > m)] \\
= \; & E^*[e^{-rT}(S_m - K) I(S_m \geq K, \tau'(H,S) > m)] \\
= \; & E^*[e^{-rT} S_m I(S_m \geq K, \tau'(H,S) > m)] - Ke^{-rT} P^*[S_m \geq K, \tau'(H,S) > m] \\
= \; & I - Ke^{-rT} \cdot II
\end{aligned}$$

Observing $e^{\mu m \Delta t - rT} = e^{\left(r - \frac{\sigma^2}{2}\right)T - rT} = e^{-\frac{\sigma^2}{2}T}$ and using the discrete Girsanov Theorem with $a_i = \sigma\sqrt{\Delta t}$, the first term is given by

$$\begin{aligned}
I \; = \; & E^*\left[e^{-rT} S(0) \exp\left(\mu m \Delta t + \sigma\sqrt{\Delta t}\sum_{i=1}^{m} Z_i\right) I(S_m \geq K, \tau'(H,S) > m)\right] \\
= \; & S(0) E^*\left[\exp\left(-\frac{1}{2}\sigma^2 T + \sigma\sqrt{\Delta t}\sum_{i=1}^{m} Z_i\right) I(S_m \geq K, \tau'(H,S) > m)\right] \\
= \; & S(0)\widehat{E}[I(S_m \geq K, \tau'(H,S) > m)] \\
= \; & S(0)\widehat{P}[S_m \geq K, \tau'(H,S) > m].
\end{aligned}$$

Under P, $\log S_m$ has mean $\mu m \Delta t$, where Z_i is a standard normal variable. Under $\widehat{P}$, $\sigma\sqrt{\Delta t} Z_i$ has mean $\sigma\sqrt{\Delta t} a_i = \sigma^2 \Delta t$; so $\sum_{i=1}^{m} \sigma\sqrt{\Delta t} Z_i$ has mean $\sigma^2 \sum_{i=1}^{m} \Delta t = \sigma^2 T$.

Under $\widehat{P}$, $\log S_m$ has a mean $\mu m \Delta t + \sigma\sqrt{\Delta t} \cdot m\sigma\sqrt{\Delta t} = (\mu + \sigma^2)T$ instead of μT under the measure P^*. Therefore, the price of the discrete up-and-out-call option is given by

$$
\begin{aligned}
& V_m(H) \\
= \; & S(0)\widehat{P}\left[W_m \geq \frac{\log(K/S(0))}{\sigma\sqrt{\Delta t}}, \tau'(a/(\sigma\sqrt{T}), W) > m\right] \\
& -Ke^{-rT}P^*\left[W_m \geq \frac{\log(K/S(0))}{\sigma\sqrt{\Delta t}}, \tau'(a/(\sigma\sqrt{T}), W) > m\right], \; a = \log\frac{H}{S(0)},
\end{aligned}
$$

where under $\widehat{P}$, $W_m = \sum_{i=1}^{m}(\widehat{Z}_i + \{\{(\mu + \sigma^2)/\sigma\}\sqrt{T}/m\})$. Under P^*, $W_m = \sum_{i=1}^{m}(Z_i + \{(\mu/\sigma)\sqrt{T}/m\})$, where $\widehat{Z}_i$ and Z_i being standard normal random variables under $\widehat{P}$ and P^*, respectively.

Recall

$$y = \frac{\log(K/S(0))}{\sigma\sqrt{T}},\ \ b = \frac{a}{\sigma\sqrt{T}} = \frac{\log(H/S(0))}{\sigma\sqrt{T}} \geq y,$$

as $m \to \infty$, we obtain

$$\begin{aligned} V_m(H) \ &= \ S(0)P\left[W_{\frac{(\mu+\sigma^2)\sqrt{T}}{\sigma}}(1) \geq \frac{\log(K/S(0))}{\sigma\sqrt{T}}, \tau(b+\beta/\sqrt{m}, W_{\frac{(\mu+\sigma^2)\sqrt{T}}{\sigma}}) > 1\right] \\ &\quad -Ke^{-rT}P\left[W_{\frac{\mu\sqrt{T}}{\sigma}}(1) \geq \frac{\log(K/S(0))}{\sigma\sqrt{T}}, \tau(b+\beta/\sqrt{m}, W_{\frac{\mu\sqrt{T}}{\sigma}}) > 1\right] \\ &\quad + o(1/\sqrt{m}), \end{aligned}$$

where $W_c(t)$ denotes a Brownian motion with drift c and unit standard deviation. By the rescaling property, we obtain

$$\begin{aligned} V_m(H) \ &= \ S(0)P\left[W_{\frac{\mu+\sigma^2}{\sigma}}(T) \geq \frac{\log(K/S(0))}{\sigma}, \tau(b\sqrt{T}+\beta\sqrt{T/m}, W_{\frac{\mu+\sigma^2}{\sigma}}) > T\right] \\ &\quad -Ke^{-rT}P\left[W_{\frac{\mu}{\sigma}}(T) \geq \frac{\log(K/S(0))}{\sigma}, \tau(b\sqrt{T}+\beta\sqrt{T/m}, W_{\frac{\mu}{\sigma}}) > T\right] \\ &\quad + o(1/\sqrt{m}). \end{aligned}$$

Lastly, we transform the barrier threshold and first passage time from W to S.

Since $\tau(b\sqrt{T}+\beta\sqrt{T/m}, W) = \tau(a/\sigma+\beta\sqrt{T/m}, W) = \tau(He^{\beta\sigma\sqrt{T/m}}, S)$, we have

$$\begin{aligned} V_m(H) &= S(0)P\left[S(0)e^{(\mu+\sigma^2)T+\sigma B(T)} \geq K, \tau(He^{\beta\sigma\sqrt{T/m}}, S) > T\right] \\ &\quad -Ke^{-rT}P\left[S(0)e^{\mu T+\sigma B(T)} \geq K, \tau(He^{\beta\sigma\sqrt{T/m}}, S) > T\right] + o(1/\sqrt{m}). \end{aligned}$$

Similarly, by using the continuous time Girsanov theorem, the continuous time price $V(H)$ can be written as

$$\begin{aligned} V(H) &= S(0)P\left[S(0)e^{(\mu+\sigma^2)T+\sigma B(T)} \geq K, \tau(H, S) > T\right] \\ &\quad -Ke^{-rT}P\left[S(0)e^{\mu T+\sigma B(T)} \geq K, \tau(H, S) > T\right]. \end{aligned}$$

Double-exponential fast Gauss transform algorithm

We set up the Black-Scholes framework for pricing a European barrier option with discrete monitoring dates. The risk neutral dynamics of the stock price process S_t follows

$$\frac{dS_t}{S_t} = (r-q)\,dt + \sigma\,dW_t.$$

We consider a time horizon $[0,T]$ and $n+1$ discrete time points $t_i = i\Delta t$, $i = 0,1,...,n$, where $\Delta t = \frac{T}{n}$, and denote S_{t_i} by S_i. The discretely monitored down-and-out call option with maturity T, monitoring dates $\{t_i\}_{i=1}^{n-1}$, barrier level H and strike price K has terminal payoff $(S_n - K)^+$ if $S_i > H$, $1 \le i \le n-1$, and zero otherwise.

Reference

M. Broadie and Y. Yamamoto, "A double-exponential fast Gauss transform algorithm for pricing discrete path-dependent options", *Operations Research*, vol.53(5) (2005) p.764-779.

We define the set of risk neutral probability density $\{P_i(S_i)\}_{i=1}^n$ such that $P_i(S)\,dS$ represents the probability that $S_j > H$, $1 \leq j \leq i$, and $S \leq S_i \leq S + dS$.

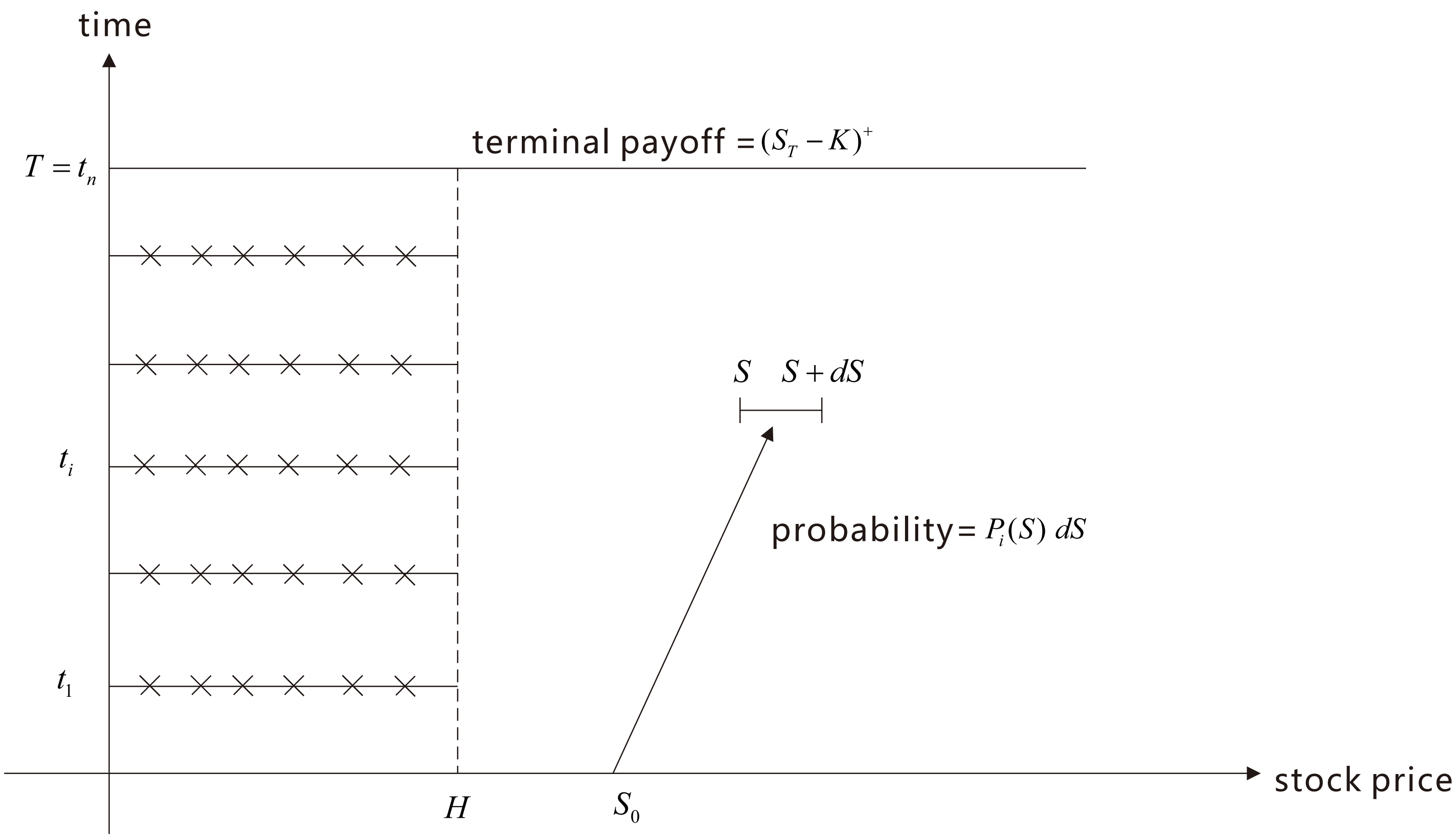
The recursive relation for finding $P_i(S_i)$ is seen to be

$$P_1(S_1) = \begin{cases} p(S_1|S_0) & \text{if } S_1 > H \\ 0 & \text{otherwise} \end{cases};$$

$$P_i(S_i) = \begin{cases} \int_H^\infty p(S_i|S_{i-1}) P_{i-1}(S_{i-1})\, dS_{i-1} & \text{if } S_i > H \\ 0 & \text{otherwise} \end{cases}.$$

The price Q_0^{DOC} of the discretely monitored down-and-out call at time 0 is given by

$$Q_0^{DOC}(S_0; K, H) = e^{-rT} \int_K^\infty P_n(S_n)(S_n - K)\, dS_n.$$



Discretely monitored down-and-out call option

Define

$$x_t = \ln S_t - \left(r - q - \frac{\sigma^2}{2}\right)t$$

so that x_t evolves according to

$$dx_t = \sigma\, dW_t.$$

In terms of x_i's, the transition probability density function is given by

$$p(x_i|x_{i-1}) = p^G(x_i - x_{i-1}) = \frac{1}{\sqrt{2\pi\Delta t}\sigma} \exp\left(-\frac{(x_i - x_{i-1})^2}{2\sigma^2\Delta t}\right),$$

which is a Gaussian density function. The option pricing formula becomes

$$Q_0^{DOC}(S_0; K, H) = e^{-rT}\int_k^\infty P_n(x_n)\left[\exp\left(x_n + \left(r - q - \frac{\sigma^2}{2}\right)T\right) - K\right] dx_n,$$

where $k = \ln K - \left(r - q - \frac{\sigma^2}{2}\right)T$.

The recursive scheme for the density functions becomes

$$P_1(S_1) = \begin{cases} p^G(x_1 - \ln S_0) & \text{if } x_1 > h_1 \\ 0 & \text{otherwise} \end{cases};$$

$$P_i(S_i) = \begin{cases} \int_{h_{i-1}}^{\infty} p^G(x_i - x_{i-1}) P_{i-1}(x_{i-1})\, dx_{i-1} & \text{if } x_i > h_i \\ 0 & \text{otherwise} \end{cases}.$$

Here, $h_i = \ln H - \left(r - q - \dfrac{\sigma^2}{2}\right) i\Delta t$.

The price of the down-and-out call can be computed by a series of convolution integrals of $P_i(x)$ and the Gaussian density function.

Unlike the time marching scheme in finite difference calculations, we can compute the density function at the next time step through one-step integration over $[t_{i-1}, t_i]$.

Double-exponential integration formula

Consider the integral

$$I = \int_c^\infty f(x)\,dx$$

with semi-infinite integration domain, it can be transformed into infinite domain by defining the following double exponential transformation:

$$x = c + \exp\left(\frac{\pi}{2}\sinh u\right).$$

The integral now becomes

$$I = \int_{-\infty}^\infty f\left(c + \exp\left(\frac{\pi}{2}\sinh u\right)\right)\exp\left(\frac{\pi}{2}\sinh u\right)\frac{\pi}{2}\cosh u\,du.$$

Applying the trapezoidal rule with step size h, we obtain

$$I_h = h\sum_{j=-\infty}^{\infty} f\left(c + \exp\left(\frac{\pi}{2}\sinh jh\right)\right)\exp\left(\frac{\pi}{2}\sinh ju\right)\frac{\pi}{2}\cosh jh.$$

The above trapezoidal sum can be truncated at a modest value of $|jh|$ without affecting too much on the accuracy.

Error estimation

The integrand $f(x)$ has to decrease faster than $\dfrac{1}{|x|}$ as $|x| \to \infty$ in order that $\displaystyle\int_c^\infty f(x)\ dx$ exists.

Suppose $f(x) \sim x^{-1-\alpha}$ as $x \to \infty$, where $\alpha > 0$. For $u > 0$, taking $u \to \infty$, the integrand becomes

$$\begin{aligned}
&\left(c + \exp\left(\frac{\pi}{2}\sinh u\right)\right)^{-1-\alpha} \exp\left(\frac{\pi}{2}\sinh u\right)\frac{\pi}{2}\cosh u \\
\sim\ & \exp\left(\frac{\pi}{2}\sinh u\right)^{-\alpha}\frac{\pi}{2}\cosh u \\
\sim\ & \exp\left(-\frac{\pi\alpha}{4}\exp u\right)\frac{\pi}{4}\exp u \\
=\ & \frac{\pi}{4}\exp\left(u - \frac{\pi\alpha}{4}\exp u\right).
\end{aligned}$$

which decays at the rate of double exponential.

Similar result can be deduced for $u < 0$, $u \to -\infty$. Take $\alpha \sim 1$, the above function becomes less than 10^{-16} at $u = 4$. Therefore, the infinite trapezoidal sum can be safety truncated at $|jh| \sim 4$ if double precision arithmetic is used.

- When the number of sample points N is increased in the double-exponential integration formula, its discretization error decreases faster than any negative power of N.

The fast Gauss transform (FGT)

We define the sample points a_j and weights w_j as follows:

$$I_h^N = \sum_{j=N^-}^{N^+} w_j f(a_j),$$

$$a_j = h_i + \exp\left(\frac{\pi}{2}\sinh jh\right), \quad w_j = h\exp\left(\frac{\pi}{2}\sinh jh\right)\frac{\pi}{2}\cosh jh,$$

where N^- and N^+ are determined so that

$$N^+ h \sim -N^- h \sim 4$$

and the total number of sample points is $N = N^+ - N^- + 1$. The convolution between $p^G(x_i - x_{i-1})$ and $P_{i-1}(x_{i-1})$ can be approximated by

$$P_i(a_j^i) = \sum_{j'=N^-}^{N^+} w_{j'} p^G(a_j^i - a_{j'}^{i-1}) P_{i-1}(a_{j'}^{i-1}), \quad j = N^-, ..., N^+.$$

- We do not include sample points in the region $x_i < h_i$ since $P_i(x_i)$ is always zero there.

Write $q_k = P_{i-1}(a_k^{i-1})w_k$ and $\delta = 2\sigma^2\Delta t$.

- The evaluation of $P_i(a_j^i)$ requires $O(N^2)$ computation for each time step.
- Fast Fourier transform cannot be used to reduce the computational work since the sample points $\{a_j^i\}$ and $\{a_{j'}^{i-1}\}$ are not equally spaced.
- The FGT can compute the discrete convolution of a given function with a Gaussian function in $O(N)$ work. We would like to calculate the sums

$$G(x_j) = \sum_{k=1}^{N} q_k \exp\left(-\frac{(x_j - y_k)^2}{\delta}\right), \quad j = 1, 2, ..., M.$$

As a result, the double-exponential fast Gauss transform algorithm has computational complexity of $O(Nn)$, where n is the number of monitoring dates and N is the number of sample points on each date.

Hermite functions

The Hermite polynomials $H_n(t)$ is defined by

$$H_n(t) = (-1)^n e^{t^2} \left(\frac{d}{dt}\right)^n e^{-t^2}.$$

The generating function for the Hermite polynomials is given by

$$e^{2ts-s^2} = \sum_{n=0}^{\infty} \frac{s^n}{n!} H_n(t).$$

Note that $H_n(t)$ are just the Taylor series coefficients of e^{2ts-s^2}. To verify the result, consider

$$\begin{aligned} \left.\frac{\partial^n}{\partial s^n} e^{2ts-s^2}\right|_{s=0} &= \left. e^{t^2} \frac{\partial^n}{\partial s^n} e^{-(t-s)^2}\right|_{s=0} \qquad \text{(next, set } u = t-s\text{)} \\ &= \left.(-1)^n e^{t^2} \frac{\partial^n}{\partial u^n} e^{-u^2}\right|_{u=t} = (-1)^n e^{t^2} \left(\frac{d}{dt}\right)^n e^{-t^2}. \end{aligned}$$

Multiplying each side by e^{-t^2}, we obtain

$$e^{-(t-s)^2} = \sum_{n=0}^{\infty} \frac{s^n}{n!}\ h_n(t),$$

where the Hermite functions $h_n(t)$ are defined by

$$h_n(t) = e^{-t^2} H_n(t).$$

Shifted and scaled version

$$\begin{aligned}
& e^{-(t-s)^2/\delta} \\
= \;& e^{-[(t-s_0)-(s-s_0)]^2/\delta} \\
= \;& \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{t_0 - s_0 - t + s}{\sqrt{\delta}}\right)^n h_n\left(\frac{t-s_0}{\sqrt{\delta}}\right) \\
= \;& \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} \left(\frac{t_0 - t}{\sqrt{\delta}}\right)^m \left(\frac{s-s_0}{\sqrt{\delta}}\right)^{n-m} h_n\left(\frac{t_0 - s_0}{\sqrt{\delta}}\right) \\
= \;& \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m!}\frac{1}{n!} \left(\frac{t_0 - t}{\sqrt{\delta}}\right)^m h_{m+n}\left(\frac{t_0 - s_0}{\sqrt{\delta}}\right) \left(\frac{t_0 - s_0}{\sqrt{\delta}}\right)^n .
\end{aligned}$$

Discrete Gauss transform

Consider

$$G(x_i) = \sum_{j=1}^{N} q_j \exp\left(-\frac{(x_i - y_j)^2}{\delta}\right), \quad i = 1, 2, \ldots, N';$$

which is called the discrete Gauss transform of $\{q_j\}_{j=1}^N$ with respect to the discrete exponential kernel with underlying point sets $\{x_i\}_{i=1}^{N'}$ and $\{y_j\}_{j=1}^N$. Apparently, it needs $O(NN')$ work to evaluate these sums based on the above definition.

Recall the following formula:

$$e^{-(x-y)^2} = \sum_{\alpha=0}^{\infty} \frac{y^\alpha}{\alpha!} h_\alpha(x), \quad \text{with } h_\alpha(x) = (-1)^\alpha \left(\frac{d}{dx}\right)^\alpha e^{-x^2}.$$

To compute the multiple sums efficiently, the FGT uses the following expansion of the Gaussian in terms of the Hermite functions $h_\alpha(x)$:

$$e^{-(x_j-y_k)^2/\delta} = \sum_{\beta=0}^{\infty}\sum_{\alpha=0}^{\infty}\frac{1}{\beta!}\frac{1}{\alpha!}\left(\frac{y_k-y_0}{\sqrt{\delta}}\right)^{\alpha} h_{\alpha+\beta}\left(\frac{x_0-y_0}{\sqrt{\delta}}\right)\left(\frac{x_j-y_0}{\sqrt{\delta}}\right)^{\beta}.$$

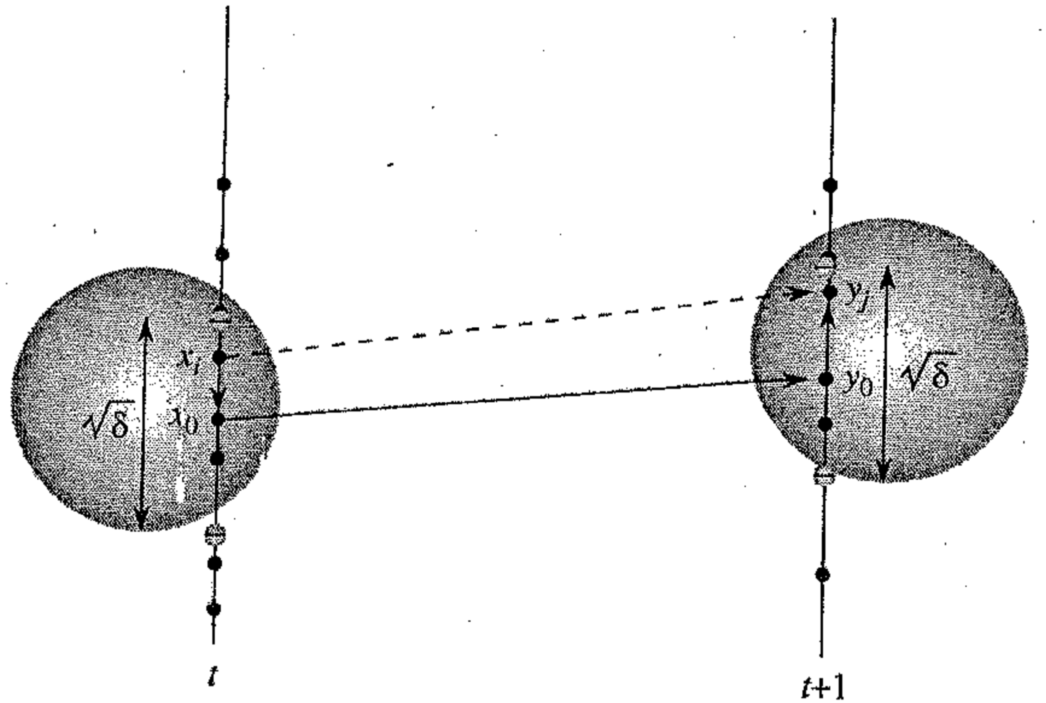
- This expansion converges very quickly and the double infinite sum over α and β can be truncated at a reasonably small integer, $\alpha = \beta = \alpha_{\text{max}}$. It is known that $\alpha_{\text{max}} = 8$ is sufficient to achieve a relative error of 10^{-8} when $|(y_k - y_0)/\sqrt{\delta}| < 1/2$ and $|(x_j - x_0)/\sqrt{\delta}| < 1/2$.

For each target point x_j and source point y_k, we choose an appropriate interval with length $\sqrt{\delta}$ and center $x_0^{(j)}$ and $y_0^{(k)}$, respectively, such that x_j and y_k fall within the intervals.

We can approximate $G(x_j)$ as

$$
\begin{aligned}
&G(x_j) \\
&\approx \sum_{k=1}^{N} q_k \sum_{\beta=0}^{\alpha_{\max}} \sum_{\alpha=0}^{\alpha_{\max}} \frac{1}{\beta!} \frac{1}{\alpha!} \left(\frac{y_k - y_0}{\sqrt{\delta}} \right)^{\alpha} h_{\alpha+\beta} \left(\frac{x_0 - y_0}{\sqrt{\delta}} \right) \left(\frac{x_j - x_0}{\sqrt{\delta}} \right)^{\beta} \\
&= \sum_{\beta=0}^{\alpha_{\max}} \frac{1}{\beta!} \left(\frac{x_j - x_0}{\sqrt{\delta}} \right)^{\beta} \left\{ \sum_{\alpha=0}^{\alpha_{\max}} h_{\alpha+\beta} \left(\frac{x_0 - y_0}{\sqrt{\delta}} \right) \underbrace{\left[\frac{1}{\alpha!} \sum_{k=1}^{N} q_k \left(\frac{y_k - y_0}{\sqrt{\delta}} \right)^{\alpha} \right]}_{\text{independent of } x_j} \right\}.
\end{aligned}
$$

Illustration of the FGT algorithm



The target points x_i and source points y_j lie in intervals of length $\sqrt{\delta}$ centered at x_0 and y_0, respectively.

The computation of $G(x_j)$ can be divided into three steps:

Step 1: Compute $A_\alpha = \dfrac{1}{\alpha!} \sum_{k=1}^{N} q_k \left(\dfrac{y_k - y_0}{\sqrt{\delta}} \right)^\alpha$ for $\alpha = 0, ..., \alpha_{\mathrm{max}}$.

Step 2: Compute $B_\beta = \sum_{\alpha=0}^{\alpha_{\mathrm{max}}} A_\alpha h_{\alpha+\beta} \left(\dfrac{x_0 - y_0}{\sqrt{\delta}} \right)$ for $\beta = 0, ..., \alpha_{\mathrm{max}}$.

Step 3: Compute $G(x_j) = \sum_{\beta=0}^{\alpha_{\mathrm{max}}} B_\beta \dfrac{1}{\beta!} \left(\dfrac{x_j - x_0}{\sqrt{\delta}} \right)^\beta$ for $j = 1, ..., N'$.

When α_{max} is fixed, Steps 1 and 3 require $O(N)$ and $O(N')$ computational effort, respectively, while Step 2 can be done in a constant time that does not depend either on N or N'.

In the general case, we divide the space into intervals of length $\sqrt{\delta}$ and apply the above method to each of the possible pairs of a source interval and a target interval. Let K and J denote the source interval and the target interval, respectively, and y_K and x_J denote their centers.

Step 1: Compute $A_{\alpha,K} = \dfrac{1}{\alpha!} \displaystyle\sum_{k=1}^{N} q_k \left(\frac{y_k - y_K}{\sqrt{\delta}} \right)^{\alpha}$ for $\alpha = 0, ..., \alpha_{\text{max}}$ and for each source interval K.

Step 2: Compute

$$B_{\beta,J} = \sum_{\alpha=0}^{\alpha_{\text{max}}} A_{\alpha,K} h_{\alpha+\beta} \left(\frac{x_J - y_K}{\sqrt{\delta}} \right) \quad \text{for } \beta = 0, ..., \alpha_{\text{max}}$$

and for each target interval J.

Step 3: Compute $G(x_j) = \sum_{\beta=0}^{\alpha_{\max}} B_{\beta,J} \frac{1}{\beta!} \left(\frac{x_j - x_J}{\sqrt{\delta}} \right)^{\beta}$ for $j = 1, ..., N'$.

Here, J is the target interval where x_j lies.

By applying the fast Gauss transform with source points $\{\alpha_{j'}^{i-1}\}$, target points $\{a_j^i\}$, and

$$
\begin{aligned}
&q_k = P_{i-1}(a_k^{i-1}) w_k, \\
&\delta = 2\sigma^2 \Delta t,
\end{aligned}
$$

we can compute the discrete convolution in $O(N)$ work.

There are other approaches to computing the discrete convolution with computational effort less than $O(N^2)$. One possibility is to use non-uniform FFTs or variants of the FFT for unequally spaced grids, which needs $O(N \log N)$ work when the number of grid points is N, as opposed to $O(N)$ work required by the FGT. In addition, non-uniform FFTs were seen to be about 10 times slower than FFTs for equally spaced grids.

- It is more efficient to use problem-specific convolution methods such as the FGT when they are available.

- Convolution based on non-uniform FFTs has a marked advantage that it can deal with a much wider class of transition probability density functions.

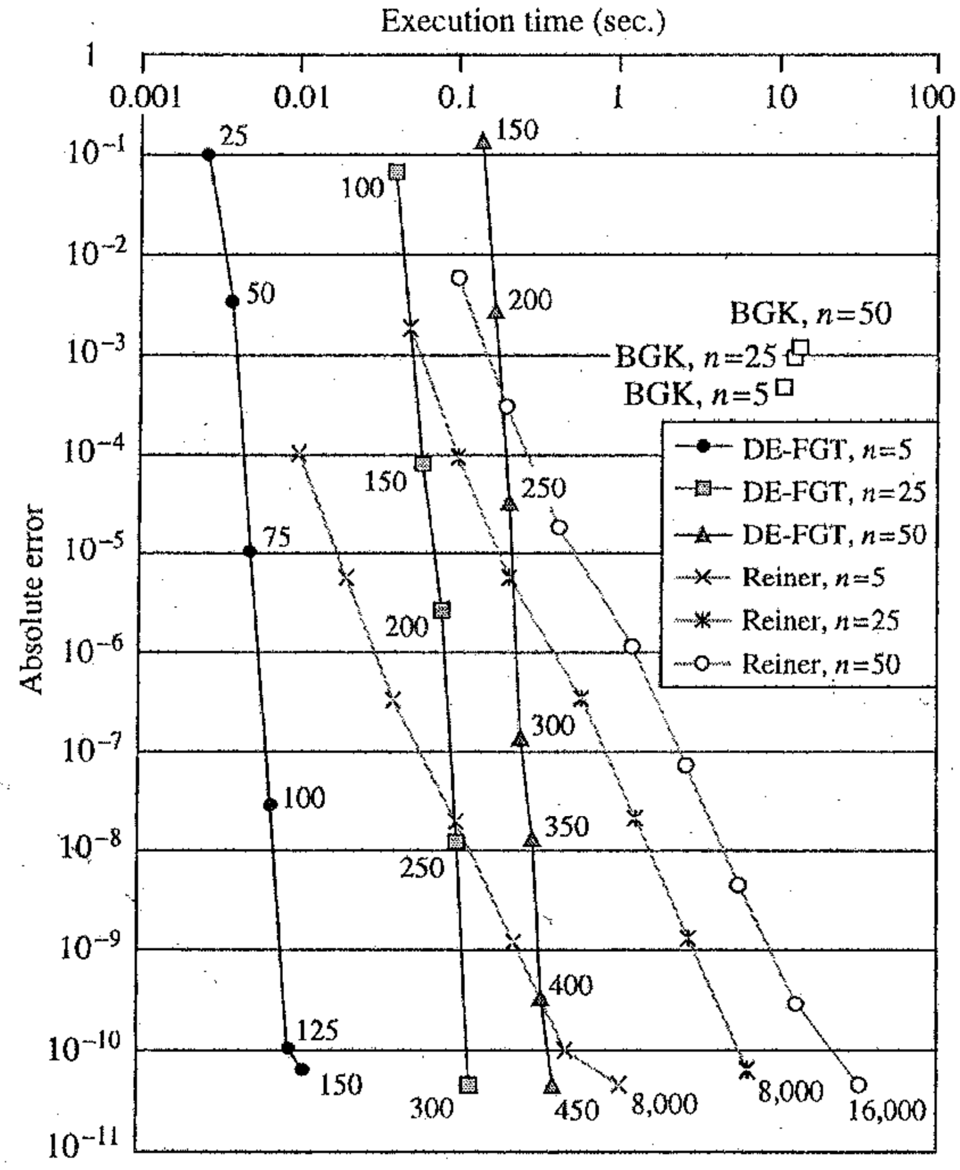
Down-and-out call option under the Black-Scholes model

We show results for European down-and-out call options under the Balck-Scholes model.

The parameters are $S_0 = K = 100$, $T = 0.2$, $r = 0.1$, $q = 0$, and $\sigma = 0.3$. We varied the barrier level from $H = 91$ to $H = 99$ in increments of 2 and set the number of monitoring dates to $n = 5$, 25, or 50.

	$n = 5$	$n = 25$	$n = 50$	$n = \infty$
$H = 91$	6.187290	6.032026	5.977069	5.807771
$H = 93$	5.999755	5.687532	5.584340	5.276814
$H = 95$	5.671105	5.081415	4.906789	4.397503
$H = 97$	5.167245	4.115815	3.833978	3.059563
$H = 99$	4.489172	2.812439	2.336387	1.170793

European down-and-out call option price with n monitoring instants under the Black-Scholes model.



BGK: trinomial tree method with $5,000$ time steps.

Reiner: FFT to compute the convolution integrals, use equally spaced grid points to discretize the log asset price.

- Errors are root mean square errors of 5 options with different barrier levels.

- Execution time is the time for computing one option price.

- The error of DE-FGT method decreases almost exponentially with the number of sample points N. As N is incremented by a constant, the position of the corresponding point in the graph moves downward by a constant distance.

Extension to Merton's jump diffusion model

The asset price follows the dynamics

$$S_i = S_{i-1} \exp\left\{\left(r - q - \frac{1}{2}\sigma^2 - \nu\lambda\right)\Delta t + \sigma\sqrt{\Delta t} z_0 + \sum_{l=1}^{N_i^P(\Delta t)} \left(\delta z_l + \gamma - \frac{1}{2}\delta^2\right)\right\},$$

where Δt is the time interval between t_{i-1} and t_i, $N_i^P(\Delta t)$ is the number of jumps during this interval, which follows a Poisson prcess with intensity λ, and z_l's are independent and follow the standard normal distribution $N(0,1)$.

The constants γ and δ determine the mean and the standard deviation of the jumps, respectively.

The compensator ν is given by $E[J-1] = e^\gamma - 1$, where J is the jump ratio in each independent jump.

- In this model, the market becomes incomplete due to the existence of jumps, and the standard argument for option pricing based on the replicating portfolio no longer holds.

- Merton derives an option-pricing formula under the assumption that jump risk is firm specific and uncorrelated with the market. In this case, the beta value of the derivative is zero. The expected rate of return of a zero-beta derivative is equal to the riskless interest rate.

- Others derive option-pricing formulas in representative agent general equilibrium models. The form of their pricing equations are identical to the Merton formula, but with altered parameter values that account for the market price of jump risk.

The pricing problems in these models are therefore equivalent from a computational viewpoint: One simply substitutes the appropriate "risk-adjusted" parameters into the risk-neutral pricing formula.

We can apply the change of variable

$$x_t = \ln S_t - \left(r - q - \frac{1}{2}\sigma^2 - \nu\lambda\right) t$$

and obtain an equation for x_i:

$$x_i = x_{i-1} + \sigma\sqrt{\Delta t} z_0 + \sum_{l=1}^{N_t^P(\Delta t)} \left(\delta z_l + \gamma - \frac{1}{2}\delta^2\right).$$

The Poisson probability can be written as

$$P[N_t^P(\Delta t) = n] = e^{-\lambda\Delta t}\frac{(\lambda\Delta t)^n}{n!}.$$

When the number of jumps is n, $x_i - x_{i-1}$ follows a Gaussian distribution with the variance and mean given by

$$\sigma_n^2 = \sigma^2 \Delta t + n\delta^2 \quad \text{and} \quad \mu_n = n\left(\gamma - \frac{1}{2}\delta^2\right),$$

respectively. We can then write

$$\begin{aligned} p(x_i|x_{i-1}) &= p^M(x_i - x_{i-1}) \\ &= \sum_{n=0}^{\infty} e^{-\lambda\Delta t} \frac{(\lambda\Delta t)^n}{n!} \frac{1}{\sqrt{2\pi}\sigma_n} \exp\left(-\frac{(x_i - x_{i-1} - \mu_n)^2}{2\sigma_n^2}\right). \end{aligned}$$

The probability density $p^M(x_i - x_{i-1})$ has the following expansion:

$$\begin{aligned} p^M(x_i - x_{i-1}) = &\sum_{\beta=0}^{\alpha_{\max}} \sum_{\alpha=0}^{\alpha_{\max}} \frac{1}{\beta!} \frac{1}{\alpha!} \left(\frac{x_i - x''}{\sqrt{2}\sigma}\right)^{\alpha} \\ &\cdot \left\{\sum_{n=0}^{\infty} e^{-\lambda\Delta t} \frac{(\lambda\Delta t)^n}{n!} \frac{1}{\sqrt{2\pi}\sigma_n} \left(\frac{\sigma}{\sigma_n}\right)^{\alpha+\beta} h_{\alpha+\beta}\left(\frac{x' - x'' + \mu_n}{\sqrt{2}\sigma_n}\right)\right\} \left(\frac{x_{i-1} - x'}{\sqrt{2}\sigma}\right)^{\beta}, \end{aligned}$$

where x' and x'' are the centers of intervals of length $\sqrt{2}\sigma$ containing x_{i-1} and x_i, respectively.

We can construct an algorithm similar to the FGT by replacing the Hermite function with a weighted sum of shifted and scaled Hermite functions. Specifically, we have only to replace the formula to compute B_β with the following:

$$B_\beta = \sum_{\beta=0}^{\alpha_{\max}} A_\alpha \left\{ \sum_{n=1}^{N_{\text{jump}}} e^{-\lambda\Delta t} \frac{(\lambda\Delta t)^n}{n!} \frac{1}{\sqrt{2\pi}\sigma_n} \left(\frac{\sigma}{\sigma_n}\right)^{\alpha+\beta} h_{\alpha+\beta}\left(\frac{x' - x'' + \mu_n}{\sqrt{2}\sigma_n}\right) \right\},$$

where we have truncated the sum over the number of jumps at N_{jump}. This algorithm enables us to compute the convolution of $p^M(x)$ and a given function almost as easily as in the Gaussian case.

Mathematical Appendices

Compensator of a Poisson process

Let $N(t)$ be a counting process with (possibly stochastic) intensity $\lambda(t)$. The probability of a jump in the next time interval Δt is proportional to Δt. For constant λ, we have

$$P[N(t + \Delta t) - N(t) = 1] = \lambda \Delta t.$$

We assume jumps in disjoint time intervals happen independent and jumps by more than once do not occur. Suppose we subdivide the interval $[t, T]$ into n subintervals of length $\Delta t = \dfrac{T - t}{n}$, the probability of zero number of jump within $[t, T]$ is given by

$$p[N(T) = N(t)] = (1 - \lambda \Delta t)^n \longrightarrow \exp(-\lambda(T - t)).$$

It is easy to see that

$$
\begin{aligned}
& E[[N(s) - \lambda s] - [N(t) - \lambda t] | N(t)] \\
= \; & E[[N(s) - \lambda s] - [N(t) - \lambda t]] \quad \text{(independent increment property)} \\
= \; & E[N(s) - N(t)] - \lambda(s - t) = 0
\end{aligned}
$$

so $N(t) - \lambda t$ is a martingale. The term λt is usually called the compensator of $N(t)$.

More generally, for inhomogeneous Poisson process, the compensator is $\int_0^t \lambda(u)\, du$, where

$$
M(t) = N(t) - \int_0^t \lambda(u)\, du
$$

is a martingale (with respect to its own filtration).

Counting processes may be characterized by

$$
N_t = \sum_{k=1}^{\infty} \mathbf{1}_{[T_k,\infty)}(t),\ t \in \mathcal{R}_+;\ \mathbf{1}_{[T_k,\infty)}(k) = \begin{cases} 1 & \text{if } t \geq T_k, \\ 0 & \text{if } 0 \leq t < T_k \end{cases} .
$$

Here, $(T_k)_{k \geq 1}$ is the increasing family of jump times of $(N_t)_{t \in \mathcal{R}_+}$.

Renewal processes are counting processes in which the holding times $\tau_k = T_{k+1} - T_k$, $k \in \mathcal{N}$, is a sequence of independent and identically distributed (iid) random variables.

- Poisson processes are renewal processes with exponential distributed holding times τ_k for all k.

Compound Poisson process

Let $(Z_k)_{k\geq 1}$ denote an iid sequence of random variables with probability distribution $\nu(dy)$ on $\mathcal{R}$, independent of the Poisson process $(N_t)_{t\in\mathcal{R}_+}$. We have

$$P[Z_k \in [a,b]] = \nu([a,b]) = \int_a^b v(dy), \ \ -\infty < a \leq b < \infty.$$

The process

$$Y_t = \sum_{k=1}^{N_t} Z_k, \ t \in \mathcal{R}_+,$$

is called a compound Poisson process.

The mean of Y_t is found to be

$$
\begin{aligned}
E[Y_t] &= E\left[E\left[\sum_{k=1}^{N_t} Z_k \middle| N_t\right]\right] \\
&= e^{-\lambda t}\sum_{n=0}^{\infty}\frac{\lambda^n t^n}{n!}E\left[\sum_{k=1}^{n} Z_k \middle| N_t = n\right] \\
&= e^{-\lambda t}\sum_{n=0}^{\infty}\frac{\lambda^n t^n}{n!}nE[Z_1] \\
&= \lambda t e^{-\lambda t}E[Z_1]\sum_{n=1}^{\infty}\frac{(\lambda t)^{n-1}}{(n-1)!} = \lambda t E[Z_1].
\end{aligned}
$$

Hence, the compensated compound Poisson process $M_t = Y_t - \lambda t E[Z_1]$ is a martingale.

The compound Poisson processes only have a finite number of jumps on any interval. They belong to the general gamily of L$\acute{e}$vy process which may have an infinite number of jumps on any finite time interval.

Characteristic function of the increment $Y_T - Y_t$

The characteristic function of $Y_T - Y_t$ is defined to be the Fourier transform of the density function of $Y_T - Y_t$. For any $t \in [0, T]$, we have

$$
\begin{aligned}
& E[\exp(i\alpha(Y_T - Y_t))] \\
= \; & \exp(\lambda(T-t)\int_{-\infty}^{\infty}(e^{i\alpha y} - 1)\nu(dy)), \text{ where } \alpha \in \mathcal{R}.
\end{aligned}
$$

Proof

Since N_t is a Poisson distribution that is independent of Z_k, $k \geq 1$; by conditioning, for all values of $\alpha \in \mathcal{R}$, we have

$$
\begin{aligned}
& E[\mathrm{exp}(i\alpha(Y_T - Y_t))] \\
= \; & \sum_{n=0}^{\infty} E\left[\mathrm{exp}\left(i\alpha \sum_{k=1}^{n} Z_k\right)\right] P[N_T - N_t = 0] \\
= \; & e^{-\lambda(T-t)} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}(T-t)^n E\left[\mathrm{exp}\left(i\alpha \sum_{k=1}^{n} Z_k\right)\right] \\
= \; & e^{-\lambda(T-t)} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}(T-t)^n \left(E\left[\mathrm{exp}\left(i\alpha Z_1\right)\right]\right)^n \\
= \; & \mathrm{exp}(\lambda(T-t)E[\mathrm{exp}(i\alpha Z_1)] - 1) \\
= \; & \mathrm{exp}(\lambda(T-t)\int_{-\infty}^{\infty}(e^{i\alpha y} - 1)\nu(dy)) \quad \text{[note that } \int_{-\infty}^{\infty} \nu(dy) = 1\text{].}
\end{aligned}
$$

Like the Poisson process N_t, $t \in \mathcal{R}_+$, the compound Poisson process Y_t, $t \in \mathcal{R}_+$, has independent increments. To show the claim, let $0 \le t_0 \le t_1 \le \cdots \le t_n$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathcal{R}$ and consider

$$\begin{aligned}
& E\left[\prod_{k=1}^{n} e^{i\alpha_k (Y_{t_k} - Y_{t_{k-1}})}\right] \\
= \; & \exp\left(\lambda \sum_{k=1}^{n} (t_k - t_{k-1}) \int_{-\infty}^{\infty} (e^{i\alpha_k y} - 1)\nu(dy)\right) \\
= \; & \prod_{k=1}^{n} \exp\left(\lambda (t_k - t_{k-1}) \int_{-\infty}^{\infty} (e^{i\alpha_k y} - 1)\nu(dy)\right) \\
= \; & \prod_{k=1}^{n} E\left[e^{i\alpha (Y_{t_k} - Y_{t_{k-1}})}\right].
\end{aligned}$$

Stochastic differential equation with jumps

Let $\eta \in \mathcal{R}$ be a constant coefficient and consider

$$dS_t = \eta S_{t^-} \ dN_t.$$

When the Poisson process has a jump at time t, we have $\Delta N_t = N_t - N_t = 1$, so

$$dS_t = S_t - S_{t^-} = \eta S_{t^-}, \ t > 0.$$

By performing integration, we obtain

$$S_t = (1 + \eta) S_{t^-}, \ t > 0;$$

and deductively,

$$S_t = S_0 (1 + \eta)^{N_t}, \ t \in \mathcal{R}_+.$$

Extending to time-dependent η_t, we consider

$$dS_{T_k} = \eta_t S_{t^-} \ dN_t.$$

At each jump time T_k, we obtain $S_{T_k} = (1 + \eta_{T_k}) S_{T_k^-}$.

Deductively, taking $k = 1, 2, \ldots, N_t$, we obtain

$$S_t = S_0 \prod_{k=1}^{N_t} (1 + \eta_{T_k}) = S_0 \prod_{\substack{\Delta N_s = 1 \\ 0 \le s \le t}} (1 + \eta_s), \ t \in \mathcal{R}_+.$$

For the more general case, suppose

$$dS_t = \mu_t S_t \ dt + \eta_t S_{t^-} (dN_t - \lambda \ dt),$$

then the solution can be expressed as

$$S_t = S_0 \exp\left(\int_0^t \mu_s \ ds - \lambda \int_0^t \eta_s \ ds\right) \prod_{k=1}^{N_t} (1 + \eta_{T_k}), \ t \in \mathcal{R}_+.$$

We randomize η_{T_k} and let $1 + Z_k$ denote the random jump ratio at T_k, so

$$S_t = S_0 \exp\left(\int_0^t \mu_s \ ds + \int_0^t dY_s - \lambda E[Z_1] \ ds\right) \prod_{k=1}^{N_t} (1 + Z_k), \ \ t \in \mathcal{R}_+,$$

which solves

$$dS_t = \mu_t S_t \ dt + S_{t^-}(dY_t - \lambda E[Z_1] dt).$$

Risk neutral measures

Consider the asset price process modeled by

$$dS_t = \mu_t S_t \, dt + \sigma S_t \, dW_t + S_{t^-} \, dY_t,$$

where Y_t, $t \in \mathcal{R}_+$, is a compound Poisson process. The solution is given by

$$S_t = S_0 \exp\left(\mu t + \sigma W_t - \frac{\sigma^2 t}{2}\right) \prod_{k=1}^{N_t} (1 + Z_k), \quad t \in \mathcal{R}_+.$$

We would like to determine a risk neutral probability measure under which the discounted process $e^{-rt} S_t$, $t \in \mathcal{R}_+$, is a martingale. Cosider

$$\begin{aligned} d(e^{-rt} S_t) &= -re^{-rt} S_t \, dt + e^{-rt} \, dS_t \\ &= (\mu - r) e^{-rt} S_t \, dt + \sigma e^{-rt} S_t \, dW_t + e^{-rt} S_{t^-} \, dY_t \\ &= (\mu - r + \lambda E_\nu[Z_1]) e^{-rt} S_t \, dt + \sigma e^{-rt} S_t \, dW_t \\ &\quad + e^{-rt} S_{t^-} (dY_t - \lambda E_\nu[Z_1] \, dt), \end{aligned}$$

which yields a martingale provided that $\mu - r + \lambda E_\nu[Z_1] = 0$.

In order for the discounted process $e^{-rt}S_t$ to be a martingale, we choose $u \in \mathcal{R}$, $\widetilde{\lambda} > 0$ and the measure $\widetilde{\nu}$ such that

$$\mu - r = \sigma u - \widetilde{\lambda} E_{\widetilde{\nu}}[Z_1].$$

The Girsanov Theorem for jump processes shows that

$$dW_t + u\ du + dY_t - \widetilde{\lambda} E_{\widetilde{\nu}}[Z_1]dt$$

is a martingale under $P_{u,\widetilde{\lambda},\widetilde{\nu}}$. The discounted asset price process becomes

$$\begin{aligned} d(e^{-rt}S_t) &= (\mu - r)e^{-rt}S_t\ dt + \sigma e^{-rt}S_t\ dW_t + e^{-rt}S_{t^-}\ dY_t \\ &= \sigma e^{-rt}S_t(dW_t + u\ dt) + e^{-rt}S_{t^-}(dY_t - \widetilde{\lambda} E_{\widetilde{\nu}}[Z_1]dt), \end{aligned}$$

so that $e^{-rt}S_t$ is a martingale under $P_{u,\widetilde{\lambda},\widetilde{\nu}}$.

The non-uniqueness of the risk neutral measure is apparent since higher degrees of freedom are involved in the choices of u, λ and $\widetilde{\nu}$. In the non-jump case, the choice of $u = \dfrac{\mu - r}{\sigma}$ is unique.