Advanced Topics in Derivative Pricing Models

Topic 3 - Derivatives with averaging style payoffs

3.1 Pricing models of Asian options

- Partial differential approach for continuous models
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3.1 Pricing models of Asian options

- Asian options are averaging options whose terminal payoff depends on some form of averaging of the price of the underlying asset over a part or the whole of option's life. Many equity-linked variable annuities products have terminal payoff structures that are dependent on some averaging form of the corresponding underlying asset price process.

- There are frequent situations where traders may be interested to hedge against the average price of a commodity over a period rather than, say, end-of-period price.

- Averaging options are particularly useful for business involved in trading on thinly-traded commodities. The use of such financial instruments may avoid the price manipulation near the end of the period.

The most common averaging procedures are the discrete arithmetic averaging defined by

$$A_T = \frac{1}{n} \sum_{i=1}^{n} S_{t_i}$$

and the discrete geometric averaging defined by

$$G_T = \left[\prod_{i=1}^{n} S_{t_i} \right]^{1/n} .$$

Here, S_{t_i} is the asset price at discrete time $t_i, i = 1, 2, \cdots, n$.

In the limit $n \to \infty$, the discrete sampled averages become the continuous sampled averages. The continuous arithmetic average is given by

$$A_T = \frac{1}{T} \int_0^T S_t \, dt,$$

while the continuous geometric average is defined to be

$$G_T = \exp \left(\frac{1}{T} \int_0^T \ln S_t \, dt \right) .$$

Partial differential approach for continuous models

It is necessary to find the change of the path dependent variable with respect to time. It turns out $\frac{dA}{dt}$ (or $\frac{dG}{dt}$) is non-stochastic and it is dependent on S, t and A (or G).

Fir example, consider $I_t = \int_0^t \ln S_u \, du$, we have

$$dI = \lim_{\Delta t \to 0} \int_t^{t+\Delta t} \ln S \, du = \lim_{\Delta t \to 0} \ln S(u^*) \, dt = \ln S \, dt, \; t < u^* < t + \Delta t.$$

(i) Arithmetic averaging, $A_t = \frac{1}{t}\int_0^t S_u\ du$, then

$$f(S, A, t) = \frac{dA}{dt} = \frac{1}{t}(S - A).$$

(ii) Geometric averaging, $G_t = \exp\left(\frac{1}{t}\int_0^t \ln S_u\ du\right)$, then

$$f(S, G, t) = \frac{dG}{dt} = G\left(\frac{\ln S - \ln G}{t}\right).$$

Partial differential equation formulation

Consider a portfolio which contains one unit of the Asian option and $-\Delta$ units of the underlying asset. We then choose Δ such that the stochastic components associated with the option and the underlying asset cancel off each other.

Assume the asset price dynamics to be given by

$$dS = [\mu S - D(S,t)]\,dt + \sigma S\,dZ,$$

where Z is the standard Wiener process, $D(S,t)$ is the dividend yield on the asset, μ and σ are the expected rate of return and volatility of the asset price, respectively. Let $V(S,A,t)$ denote the value of the Asian option and let Π denote the value of the above portfolio. The portfolio value is given by

$$\Pi = V(S,A,t) - \Delta S,$$

and its differential is found to be

$$d\Pi = \frac{\partial V}{\partial t}\,dt + f(S,A,t)\frac{\partial V}{\partial A}\,dt + \frac{\partial V}{\partial S}\,dS + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2}\,dt - \Delta\,dS - \Delta D(S,t)\,dt.$$

- $f(S,A,t)\dfrac{\partial V}{\partial A}\,dt$ is the extra deterministic term added due to the new path dependent state variable A.

- The Δ units of stock lead to dividend amount $\Delta D(S,t)\,dt$ collected over dt.

As usual, we choose $\Delta = \frac{\partial V}{\partial S}$ so that the stochastic terms containing dS cancel. The absence of arbitrage dictates

$$d\Pi = r\Pi \; dt,$$

where r is the riskless interest rate. Putting the results together, we obtain

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + [rS - D(S,t)] \frac{\partial V}{\partial S} + f(S,t) \frac{\partial V}{\partial A} - rV = 0.$$

The equation is a degenerate diffusion equation since it contains diffusion term corresponding to S only but not for A. The auxiliary conditions in the pricing model depend on the specific details of the Asian option contract.

Continuously monitored geometric averaging options

- We take time zero to be the time of initiation of the averaging period, t is the current time and T denotes the expiration time.

- We define the continuously monitored geometric averaging of the asset price S_u over the time period $[0, t]$ by

$$G_t = \exp\left(\frac{1}{t}\int_0^t \ln S_u \, du\right).$$

  The terminal payoff of the fixed strike call option and floating strike call option are, respectively, given by

$$\begin{aligned} c_{fix}(S_T, G_T, T; X) &= \max(G_T - X, 0) \\ c_{f\ell}(S_T, G_T, T) &= \max(S_T - G_T, 0), \end{aligned}$$

  where X is the fixed strike price.

Pricing formula of continuously monitored fixed strike geometric averaging call option

We assume the existence of a risk neutral pricing measure Q under which discounted asset prices are martingales, implying the absence of arbitrage. Under the measure Q, the asset price follows

$$\frac{dS_t}{S_t} = (r-q)\,dt + \sigma\,dZ_t,$$

where Z_t is a Q-Brownian motion. For $0 < t < T$, the solution of the above stochastic differential equation is given by

$$\ln S_u = \ln S_t + \left(r - q - \frac{\sigma^2}{2}\right)(u-t) + \sigma(Z_u - Z_t),\ \ u > t.$$

Recall that

$$T \ln G_T - t \ln G_t = \int_t^T \ln S_u\,du.$$

By integrating $\ln S_u$ over $[t,T]$, we obtain

$$\begin{aligned}\ln G_T &= \frac{t}{T}\ln G_t + \frac{1}{T}\left[(T-t)\ln S_t + \left(r-q-\frac{\sigma^2}{2}\right)\frac{(T-t)^2}{2}\right] \\ &\quad + \frac{\sigma}{T}\int_t^T (Z_u - Z_t)\, du.\end{aligned}$$

The stochastic term $\frac{\sigma}{T}\int_t^T (Z_u - Z_t)\, du$ can be shown to be Gaussian with zero mean and variance $\frac{\sigma^2}{T^2}\frac{(T-t)^3}{3}$. By the risk neutral valuation principle, the value of the European fixed strike Asian call option is given by

$$c_{fix}(S_t, G_t, t) = e^{-r(T-t)}E[\max(G_T - X, 0)],$$

where E is the expectation under Q conditional on $S_t = S, G_t = G$.

For pricing the fixed strike geometric averaging call option, it suffices to specify the distribution of G_T.

We assume the current time t to be within the averaging period. By defining

$$\overline{\mu} = \left(r - q - \frac{\sigma^2}{2}\right)\frac{(T-t)^2}{2T} \quad \text{and} \quad \overline{\sigma} = \frac{\sigma}{T}\sqrt{\frac{(T-t)^3}{3}},$$

G_T can be written as

$$G_T = G_t^{t/T} S_t^{(T-t)/T} \exp(\overline{\mu} + \overline{\sigma}\widehat{Z}),$$

where $\widehat{Z}$ is the standard normal random variable.

For convenience, we set

$$F = G^{t/T} S^{(T-t)/T}.$$

Recall

$$\begin{aligned} & E[\max(F\exp(\overline{\mu} + \overline{\sigma}\widehat{Z}) - X, 0] \\ = \quad & F e^{\overline{\mu} + \overline{\sigma}^2/2} N\left(\frac{\ln\frac{F}{X} + \overline{\mu} + \overline{\sigma}^2}{\overline{\sigma}}\right) - XN\left(\frac{\ln\frac{F}{X} + \overline{\mu}}{\overline{\sigma}}\right), \end{aligned}$$

we then deduce that

$$c_{fix}(S, G, t) = \ e^{-r(T-t)}\left[G^{t/T} S^{(T-t)/T} e^{\overline{\mu} + \overline{\sigma}^2/2} N(d_1) - XN(d_2)\right],$$

where

$$\begin{aligned} d_2 &= \left(\frac{t}{T}\ln G + \frac{T-t}{T}\ln S + \overline{\mu} - \ln X\right)\Big/\overline{\sigma}, \\ d_1 &= d_1 + \overline{\sigma}. \end{aligned}$$

European floating strike Asian call option (Geometric averaging)

Since the terminal payoff of the floating strike Asian call option involves S_T and G_T, pricing by the risk neutral expectation approach would require the *joint distribution* of S_T and G_T. For floating strike Asian options, the partial differential equation method provides the more effective approach to derive the price formula for $c_{f\ell}(S,G,t)$. This is because the *similarity reduction technique* can be applied to reduce the dimension of the differential equation.

When continuously monitored geometric averaging is adopted, the governing equation for $c_{f\ell}(S,G,t)$ can be expressed as

$$\frac{\partial c_{f\ell}}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 c_{f\ell}}{\partial S^2} + (r-q)S\frac{\partial c_{f\ell}}{\partial S} + \frac{G}{t}\ln\frac{S}{G}\frac{\partial c_{f\ell}}{\partial G} - rc_{f\ell} = 0, \quad 0 < t < T;$$

with terminal payoff: $c_{f\ell}(S,G,T) = \max(S_T - G_T, 0)$.

We define the similarity variables:

$$y = t \ln \frac{G}{S} \quad \text{and} \quad W(y,t) = \frac{c_{f\ell}(S,G,t)}{S}.$$

This is equivalent to choose S as the numeraire. The governing equation for $c_{f\ell}(S,G,t)$ becomes

$$\frac{\partial W}{\partial t} + \frac{\sigma^2 t^2}{2}\frac{\partial^2 W}{\partial y^2} - \left(r - q + \frac{\sigma^2}{2}\right) t \frac{\partial W}{\partial y} - qW = 0, \quad 0 < t < T,$$

with terminal condition: $W(y,T) = \max(1 - e^{y/T}, 0)$.

We write $\tau = T - t$ and let $F(y,\tau;\eta)$ denote the fundamental solution to the following parabolic equation with time dependent coefficients

$$\frac{\partial F}{\partial \tau} = \frac{\sigma^2 (T-\tau)^2}{2}\frac{\partial^2 F}{\partial y^2} - \left(r - q + \frac{\sigma^2}{2}\right)(T-\tau)\frac{\partial F}{\partial y}, \quad \tau > 0,$$

with initial condition at $\tau = 0$ (corresponding to $t = T$) given as

$$F(y,0;\eta) = \delta(y - \eta).$$

Though the differential equation has time dependent coefficients, the fundamental solution is readily found to be

$$F(y, \tau; \eta) = n\left(\frac{y - \eta - \left(r - q + \frac{\sigma^2}{2}\right)\int_0^\tau (T-u)\,du}{\sigma\sqrt{\int_0^\tau (T-u)^2\,du}}\right).$$

The solution to $W(y, \tau)$ is then given by

$$W(y, \tau) = e^{-q\tau}\int_{-\infty}^{\infty} \max(1 - e^{\eta/T}, 0) F(y, \tau; \eta)\,d\eta.$$

- For unrestricted Brownian motions, even with time dependent drift and volatility, the transition density function can be found readily. This is in contrast with a restricted Brownian motion with barrier (absorbing or reflecting).

The direct integration of the above integral gives

$$c_{f\ell}(S,G,t) = \; Se^{-q(T-t)}N(\widehat{d}_1) - G^{t/T}S^{(T-t)/T}e^{-q(T-t)}e^{-\widehat{Q}}N(\widehat{d}_2),$$

where

$$\begin{aligned}\widehat{d}_1 &= \frac{t\ln\frac{S}{G} + \left(r - q + \frac{\sigma^2}{2}\right)\frac{T^2-t^2}{2}}{\sigma\sqrt{\frac{T^3-t^3}{3}}}, \quad \widehat{d}_2 = \widehat{d}_1 - \frac{\sigma}{T}\sqrt{\frac{T^3-t^3}{3}},\\ \widehat{Q} &= \frac{r-q+\frac{\sigma^2}{2}}{2}\frac{T^2-t^2}{T} - \frac{\sigma^2}{6}\frac{T^3-t^3}{T^2}.\end{aligned}$$

Continuously monitored arithmetic averaging options

We consider a European fixed strike European Asian call based on continuously monitored arithmetic averaging. The terminal payoff is defined by

$$c_{fix}(S_T, A_T, T; X) = \max\left(A_T - X, 0\right).$$

To motivate the choice of variable transformation, we consider the following expectation representation of the price of the Asian call at time t

$$\begin{aligned}
c_{fix}(S_t, A_t, t) &= e^{-r(T-t)} E\left[\max\left(A_T - X, 0\right)\right] \\
&= e^{-r(T-t)} E\left[\max\left(\frac{1}{T}\int_0^t S_u\,du - X + \frac{1}{T}\int_t^T S_u\,du, 0\right)\right] \\
&= \frac{S_t}{T} e^{-r(T-t)} E\left[\max\left(x_t + \int_t^T \frac{S_u}{S_t}\,du, 0\right)\right],
\end{aligned}$$

where the state variable x_t is defined by

$$x_t = \frac{1}{S_t}(I_t - XT), \qquad I_t = \int_0^t S_u\,du = tA_t.$$

Recall that $\dfrac{S_u}{S_t} = e^{(r-q-\frac{\sigma^2}{2})(u-t)+\sigma Z_{u-t}}$, so the expectation of $\displaystyle\int_t^T \frac{S_u}{S_t}\, dt$ depends on the drift and volatility parameters, with no dependence on the state variables S_t and I_t.

In subsequent exposition, it is more convenient to use I_t instead of A_t as the averaging state variable. Since $S_u/S_t, u > t$, is independent of the history of the asset price up to time t, one argues that the conditional expectation is a function of x_t only. We then deduce that

$$c_{fix}(S_t, I_t, t) = S_t f(x_t, t)$$

for some function of f. In other words, $f(x_t, t)$ is given by

$$f(x_t, t) = \frac{e^{-r(T-t)}}{T} E\left[\max\left(x_t + \int_t^T \frac{S_u}{S_t}\, du, 0\right)\right].$$

If we write the price function of the fixed strike call as $c_{fix}(S, I, t)$, then the governing equation for $c_{fix}(S, I, t)$ is given by

$$\frac{\partial c_{fix}}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 c_{fix}}{\partial S^2} + (r - q) S \frac{\partial c_{fix}}{\partial S} + S \frac{\partial c_{fix}}{\partial I} - r c_{fix} = 0.$$

Suppose we define the following transformation of variables:

$$x = \frac{1}{S}(I - XT) \quad \text{and} \quad f(x, t) = \frac{c_{fix}(S, I, t)}{S},$$

then the governing differential equation for $f(x, t)$ becomes

$$\frac{\partial f}{\partial t} + \frac{\sigma^2}{2} x^2 \frac{\partial^2 f}{\partial x^2} + [1 - (r - q)x] \frac{\partial f}{\partial x} - qf = 0, \quad -\infty < x < \infty, t > 0.$$

Note that $I - XT$ can be positive or negative, so x can assume values from $-\infty$ to ∞.

The terminal condition is given by

$$f(x, T) = \frac{1}{T} \max(x, 0).$$

The difficulty in finding closed form solution stems from the occurrence of the linear function $1 - (r - q)x$ in the coefficient of $\frac{\partial f}{\partial x}$.

When $x_t \geq 0$, which corresponds to $\frac{1}{T} \int_0^t S_u \, du \geq X$, it is possible to find closed form analytic solution to $f(x, t)$. Since x_t is an increasing function of t so that $x_T \geq 0$, the terminal condition $f(x, T)$ reduces to x/T. In this case, $f(x, t)$ admits solution of the affine form

$$f(x, t) = a(t)x + b(t).$$

By substituting the assumed form of solution into the governing equation, we obtain the following pair of governing equations for $a(t)$ and $b(t)$:

$$
\begin{aligned}
\frac{da(t)}{dt} - ra(t) = 0, &\qquad a(T) = \frac{1}{T}, \\
\frac{db(t)}{dt} - a(t) - qb(t) = 0, &\qquad b(T) = 0.
\end{aligned}
$$

When $r \neq q, a(t)$ and $b(t)$ are found to be

$$
a(t) = \frac{e^{-r(T-t)}}{T} \quad \text{and} \quad b(t) = \frac{e^{-q(T-t)} - e^{-r(T-t)}}{T(r-q)}.
$$

Hence, the option value for $I \geq XT$ is given by

$$
c_{fix}(S, I, t) = \left(\frac{I}{T} - X\right) e^{-r(T-t)} + \frac{e^{-q(T-t)} - e^{-r(T-t)}}{T(r-q)} S.
$$

Since the option payoff reduces to a forward payoff, it is not surprising that σ does not appear. The gamma is easily seen to be zero while the delta is a function of t and $T-t$ but not S or A.

- The first term corresponds to the sure payout $\frac{I}{T} - x$ to be paid at maturity, so its present value is $(\frac{I}{T} - x)e^{-r(T-t)}$. The second term corresponds to the number of units of shares n_t used to replicate the payoff $\int_t^T S_u \, du$. We write

$$n_t = b(t) = \frac{e^{-q(T-t)} - e^{-r(T-t)}}{T(r-q)},$$

  we observe $\frac{dn_t}{dt} - qn_t = a(t) = \frac{e^{-r(T-t)}}{T}$.

- For $I < XT$, there is no closed form analytic solution.

Replication of Asian forwards

Under a risk neutral measure Q, the dynamics of stock price S_t is governed by

$$\frac{dS_t}{S_t} = (r - q)\ dt + \sigma\ dZ_t.$$

Let X_t denote the value of the trading portfolio, consisting of the riskfree asset and n_t units of the risky stock. The trading account value evolves based on the following self-financing strategy:

$$\begin{aligned} dX_t &= n_t\ dS_t + r \underbrace{(X_t - n_t S_t)}_{\text{riskfree asset}} dt + \underbrace{n_t q S_t\ dt}_{\text{dividend amount}} \\ &= rX_t\ dt + n_t(dS_t - rS_t\ dt + qS_t\ dt). \end{aligned}$$

Recall: $d(e^{-rt}X_t) = e^{-rt}(dX_t - rX_t dt)$.

The portfolio is started with 100% stock. We swap between stock and riskfree asset so that we keep n_t units of stock consistently. At maturity T, the terminal portfolio value X_T replicates the Asian forward $\frac{1}{T}\int_0^T S_t \ dt$. Given $X_0 = n_0 S_0$, multiplying both sides of the stochastic differential equation by e^{-rt} and integrating from 0 to T, we obtain

$$e^{-rT} X_T - n_0 S_0 = \int_0^T e^{-rt} n_t (dS_t - rS_t \ dt + qS_t \ dt)$$

so that

$$X_T = e^{rT} n_0 S_0 + \int_0^T e^{r(T-t)} n_t (dS_t - rS_t \ dt + qS_t \ dt).$$

By virtue of the identity:

$$d\left[e^{r(T-t)} n_t S_t\right] = e^{r(T-t)} n_t \ dS_t - re^{r(T-t)} n_t S_t \ dt + e^{r(T-t)} S_t \ dn_t$$

we obtain

$$n_T S_T - e^{rT} n_0 S_0 = \int_0^T e^{r(T-t)} n_t (dS_t - rS_t \; dt) + \int_0^T e^{r(T-t)} S_t \; dn_t.$$

By setting $n_T = 0$ and combining other results, we have

$$X_T = e^{rT} n_0 S_0 + \underbrace{n_T S_T}_{\text{zero}} \; - \; e^{rT} \underbrace{n_0 S_0}_{X_0} + \int_0^T e^{r(T-t)} S_t (n_t q \; dt - n_t' \; dt).$$

Suppose we aim to replicate $\frac{1}{T} \int_0^T S_t \; dt$ by the time-T portfolio value X_T, then n_t should observe

$$\frac{dn_t}{dt} - qn_t = \frac{e^{-r(T-t)}}{T}, \; n(T) = 0.$$

The solution of the differential equation gives

$$n_t = \frac{e^{-q(T-t)} - e^{-r(T-t)}}{(r-q)T}.$$

By following the self-financing strategy where we start with n_0 units of stock and no riskfree asset, keeping n_t units of stock at time t, and zero unit ($n_T = 0$) of stock at maturity T, the terminal value of the portfolio (money market accounts only) becomes $\frac{1}{T}\int_0^T S_u \, du$. This is precisely the payoff of an Asian forward.

Therefore, we can replicate the arithmetic average of the stock price by a self-financing trading strategy on stock.

Dimension reduction in partial differential equation formulation

We would like to derive the partial differential equation formulation of a floating strike arithmetic averaging option, whose terminal payoff is $\left(\overline{S}_T - KS_T\right)^+$, where $\overline{S}_T = \frac{1}{T}\int_0^T S_t \, dt$.

We adopt the share measure and define

$$Y_t = \frac{X_t}{e^{qt}S_t},$$

that is, the share price is used as the numeraire.

According to Ito's lemma, one can show that

$$dY_t = -\left(Y_t - e^{-qt}n_t\right)\sigma d\tilde{Z}_t,$$

where $\widehat{Z}_t = Z_t - \sigma t$ is a Brownian motion under the share measure Q_S.

Define

$$V(S_0, 0; K) = e^{-rT} E_Q\left[(X_T - KS_T)^+\right] = S_0 E_{Q_S}\left[(Y_T - K)^+\right] = S_0 u(Y_0, 0).$$

The governing equation of $u(y, t)$ is seen to be

$$\frac{\partial u}{\partial t} + \frac{\sigma^2}{2}\left(y - e^{-qt} n_t\right)^2 \frac{\partial^2 u}{\partial y^2} = 0,$$

with terminal condition: $u(y, T) = (y - K)^+$.

- There is no convective term in the pde. It is very straightforward to solve the pde numerically using finite difference methods.

- Similar result can be deemed for discretely monitored fixed strike Asian options.

Closed form pricing formulas for discretely monitored models

Fixed strike options with discrete geometric averaging

Consider the discrete geometric averaging of the asset prices at evenly distributed discrete times $t_i = i\Delta t, i = 1, 2, \cdots, n$, where Δt is the uniform time interval between fixings and $t_n = T$ is the time of expiration. Define the running geometric averaging by

$$G_k = \left[\prod_{i=1}^{k} S_{t_i}\right]^{1/k}, \quad k = 1, 2, \cdots, n.$$

The terminal payoff of a European average value call option with discrete geometric averaging is given by $\max(G_n - X, 0)$, where X is the strike price.

Suppose the asset price follows the Geometric Brownian process, then the asset price ratio $R_i = \dfrac{S_{t_i}}{S_{t_{i-1}}}, i = 1, 2, \cdots, n$ is lognormally distributed.

Assume that under the risk neutral measure Q

$$\ln R_i \sim N\left(\left(r - \frac{\sigma^2}{2}\right)\Delta t, \sigma^2 \Delta t\right), \quad i = 1, 2, \cdots, n,$$

where r is the riskless interest rate and $N(\mu, \sigma^2)$ represents a normal distribution with mean μ and variance σ^2.

European fixed strike call option

The price formula of the European fixed strike call option depends on whether the current time t is prior to or after time t_0. First, we consider $t < t_0$ and write

$$\frac{G_n}{S_t} = \frac{S_{t_0}}{S_t}\left\{\frac{S_{t_n}}{S_{t_{n-1}}}\left[\frac{S_{t_{n-1}}}{S_{t_{n-2}}}\right]^2 \cdots \left[\frac{S_{t_1}}{S_{t_0}}\right]^n\right\}^{1/n},$$

so that

$$\ln\frac{G_n}{S_t} = \ln\frac{S_{t_0}}{S_t} + \frac{1}{n}\left[\ln R_n + 2\ln R_{n-1} + \cdots + n\ln R_1\right], \quad t < t_0.$$

Since $\ln R_i, i = 1, 2, \cdots n$ and $\ln\dfrac{S_{t_0}}{S_t}$ represent independent Brownian increments over non-overlapping time intervals, they are normally distributed and independent.

Observe that $\ln \dfrac{G_n}{S_t}$ is a linear combination of these independent Brownian increments, so it remains to be normally distributed with mean

$$\begin{aligned}&\left(r-\frac{\sigma^2}{2}\right)(t_0-t)+\frac{1}{n}\left(r-\frac{\sigma^2}{2}\right)\Delta t\sum_{i=1}^{n} i\\ =&\left(r-\frac{\sigma^2}{2}\right)\left[(t_0-t)+\frac{n+1}{2n}(T-t_0)\right],\end{aligned}$$

and variance

$$\sigma^2(t_0-t)+\frac{1}{n^2}\sigma^2\Delta t\sum_{i=1}^{n} i^2=\sigma^2\left[(t_0-t)+\frac{(n+1)(2n+1)}{6n^2}(T-t_0)\right].$$

Let $\tau = T - t$, where τ is the time to expiry. Suppose we write

$$\begin{aligned}
\sigma_G^2 \tau &= \sigma^2 \left\{ \tau - \left[1 - \frac{(n+1)(2n+1)}{6n^2} \right] (T - t_0) \right\} \\
\left(\mu_G - \frac{\sigma_G^2}{2} \right) \tau &= \left(r - \frac{\sigma^2}{2} \right) \left[\tau - \frac{n-1}{2n} (T - t_0) \right],
\end{aligned}$$

then the transition density function of G_n at time T, given the asset price S_t at an earlier time $t < t_0$, can be expressed as

$$\psi(G_n; S_t) = \frac{1}{G_n \sqrt{2\pi\sigma_G^2 \tau}} \exp \left(- \frac{\left\{ \ln G_n - \left[\ln S_t + \left(\mu_G - \frac{\sigma_G^2}{2} \right) \tau \right] \right\}^2}{2\sigma_G^2 \tau} \right).$$

By the risk neutral valuation approach, the price of the European fixed strike call with discrete geometric averaging is given by

$$\begin{aligned} c_G(S_t, t) &= e^{-r\tau} E_Q[\max(G_n - X, 0)] \\ &= e^{-r\tau}\left[S_t e^{\mu_G \tau} N(d_1) - X N(d_2)\right], \quad t < t_0 \end{aligned}$$

where

$$d_1 = \frac{\ln \frac{S_t}{X} + \left(\mu_G + \frac{\sigma_G^2}{2}\right)\tau}{\sigma_G \sqrt{\tau}}, \quad d_2 = d_1 - \sigma_G \sqrt{\tau}.$$

When $n = 1, \sigma_G^2 \tau$ and $\left(\mu_G - \dfrac{\sigma_G^2}{2}\right)\tau$ reduce to $\sigma^2 \tau$ and $\left(r - \dfrac{\sigma^2}{2}\right)\tau$, respectively, so that the call price reduces to that of a European vanilla call option. We observe that $\sigma_G^2 \tau$ is a decreasing function of n, which is consistent with the intuition that the more frequent we take the averaging, the lower volatility is resulted.

When $n \to \infty$, $\sigma_G^2\tau$ and $\left(\mu_G - \dfrac{\sigma_G^2}{2}\right)\tau$ tend to $\sigma^2\left[\tau - \dfrac{2}{3}(T - t_0)\right]$ and $\left(r - \dfrac{\sigma^2}{2}\right)\left(\tau - \dfrac{T - t_0}{2}\right)$, respectively. Correspondingly, discrete geometric averaging becomes its continuous analog. In particular, the price of a European fixed strike call with continuous geometric averaging at $t = t_0$ is found to be

$$c_G(S_{t_0}, t_0) = S_{t_0} e^{-\frac{1}{2}\left(r+\frac{\sigma^2}{6}\right)(T-t_0)} N(\widehat{d_1}) - X e^{-r(T-T_0)} N(\widehat{d_2}),$$

where

$$\widehat{d_1} = \frac{\ln \frac{S_{t_0}}{X} + \frac{1}{2}\left(r + \frac{\sigma^2}{6}\right)(T - t_0)}{\sigma\sqrt{\frac{T-t_0}{3}}}, \quad \widehat{d_2} = \widehat{d_1} - \sigma\sqrt{\frac{T - t_0}{3}}.$$

Next, we consider the in-progress option where the current time t is within the averaging period, that is, $t \geq t_0$. Here, $t = t_k + \xi \Delta t$ for some integer $k, 0 \leq k \leq n-1$ and $0 \leq \xi < 1$. Now, $S_{t_1}, S_{t_2} \cdots S_{t_k}, S_t$ are known quantities while the price ratios $\frac{S_{t_{k+1}}}{S_t}, \frac{S_{t_{k+2}}}{S_{t_{k+1}}}, \cdots, \frac{S_{t_n}}{S_{t_{n-1}}}$ are independent lognormal random variables. We may write

$$
\begin{aligned}
G_n &= \left[S_{t_1} \cdots S_{t_k}\right]^{1/n} S_t^{(n-k)/n} \\
&\quad \left\{ \frac{S_{t_n}}{S_{t_{n-1}}} \left[\frac{S_{t_{n-1}}}{S_{t_{n-2}}}\right]^2 \cdots \left[\frac{S_{t_{k+1}}}{S_t}\right]^{n-k} \right\}^{1/n}
\end{aligned}
$$

so that

$$
\ln \frac{G_n}{\widetilde{S}_t} = \frac{1}{n} \left[\ln R_n + 2 \ln R_{n-1} + \cdots + (n-k-1) \ln R_{k+2} + (n-k) \ln R_t\right]
$$

where

$$
\widetilde{S}_t = [S_{t_1} \cdots S_{t_k}]^{1/n} S_t^{(n-k)/n} = G_k^{k/n} S_t^{(n-k)/n} \text{ and } R_t = S_{t_{k+1}}/S_t.
$$

Let the variance and mean of $\ln \dfrac{G_n}{\widetilde{S}_t}$ be denoted by $\widetilde{\sigma}_G^2 \tau$ and $\left(\widetilde{\mu}_G - \dfrac{\widetilde{\sigma}_G^2}{2}\right)\tau$, respectively. They are found to be

$$\widetilde{\sigma}_G^2 \tau = \sigma^2 \Delta t \left[\frac{(n-k)^2}{n^2}(1-\xi) + \frac{(n-k-1)(n-k)(2n-2k-1)}{6n^2}\right]$$

and

$$\left(\widetilde{\mu}_G - \frac{\widetilde{\sigma}_G^2}{2}\right)\tau = \left(r - \frac{\sigma^2}{2}\right)\Delta t \left[\frac{n-k}{n}(1-\xi) + \frac{(n-k-1)(n-k)}{2n}\right].$$

The price formula of the in-progress European fixed strike call option takes the form

$$c_G(S_t, \tau) = e^{-r\tau}\left[\widetilde{S}_t e^{\widetilde{\mu}_G \tau} N(\widetilde{d_1}) - X N(\widetilde{d_2})\right], \quad t \geq t_0,$$

where

$$\widetilde{d_1} = \frac{\ln \frac{\widetilde{S}_t}{X} + \left(\widetilde{\mu}_G + \frac{\widetilde{\sigma}_G^2}{2}\right)\tau}{\widetilde{\sigma}_G \sqrt{\tau}}, \quad \widetilde{d_2} = \widetilde{d_1} - \widetilde{\sigma}_G \sqrt{\tau}.$$

Again, by taking the limit $n \to \infty$, the limiting values of $\tilde{\sigma}_G^2, \tilde{\mu}_G - \dfrac{\tilde{\sigma}_G^2}{2}$ and $\tilde{S}(t)$ become

$$\lim_{n\to\infty} \widetilde{\sigma}_G^2 = \left(\frac{T-t}{T-t_0}\right)^2 \frac{\sigma^2}{3}, \quad \lim_{n\to\infty} \widetilde{\mu}_G - \frac{\widetilde{\sigma}_G^2}{2} = \left(r - \frac{\sigma^2}{2}\right) \frac{T-t}{2(T-t_0)},$$

and

$$\lim_{n\to\infty} \widetilde{S}_t = S_t^{\frac{T-t}{T-t_0}} \widetilde{G}_t \text{ where } \widetilde{G}_t = \exp\left(\frac{1}{T-t_0}\int_{t_0}^{t} \ln S_u \, du\right).$$

The price of the corresponding continuous geometric averaging call option can be obtained by substituting these limiting values into the price formula of the continuous counterpart.

European fixed strike put option

Using a similar derivation procedure, the price of the corresponding European fixed strike put option with discrete geometric averaging can be found to be

$$p_G(S,\tau) = \begin{cases} e^{-r\tau}\left[XN(-d_2) - Se^{\mu_G\tau}N(-d_1)\right], & t < t_0 \\ e^{-r\tau}\left[XN(-\widetilde{d_2}) - \widetilde{S}e^{\widetilde{\mu}_G\tau}N(-\widetilde{d_1})\right], & t \geq t_0, \end{cases}$$

where d_1 and d_2 are given on p.34, and $\tilde{d}_1$ and $\tilde{d}_2$ are given on p.37. The put-call parity relation for the European fixed strike Asian options with discrete geometric averaging can be deduced to be

$$c_G(S,\tau) - p_G(S,\tau) = \begin{cases} e^{-r\tau}Se^{\mu_G\tau} - Xe^{-r\tau}, & t < t_0 \\ e^{-r\tau}\widetilde{S}e^{\widetilde{\mu}_G\tau} - Xe^{-r\tau}, & t \geq t_0. \end{cases}$$

3.2 Put-call parity relations and fixed-floating symmetry relations

Fixed strike arithmetic averaging options

Let $c_{fix}(S, I, t)$ and $p_{fix}(S, I, t)$ denote the price function of the fixed strike arithmetic averaging Asian call option and put option, respectively. Their terminal payoff functions are given by

$$\begin{aligned} c_{fix}(S, I, T) &= \max\left(\frac{I}{T} - X, 0\right) \\ p_{fix}(S, I, T) &= \max\left(X - \frac{I}{T}, 0\right), \end{aligned}$$

where $I = \int_0^T S_u \, du$. Let $D(S, I, t)$ denote the difference of c_{fix} and p_{fix}. Since both c_{fix} and p_{fix} are governed by the same equation so does $D(S, I, t)$. The terminal condition of $D(S, I, t)$ is given by

$$D(S, I, T) = \max\left(\frac{I}{T} - X, 0\right) - \max\left(X - \frac{I}{T}, 0\right) = \frac{I}{T} - X.$$

The terminal condition $D(S, I, T)$ is the same as that of the continuously monitored arithmetic averaging option with $I \geq XT$. Hence, when $r \neq q$, the put-call parity relation between the prices of fixed strike Asian options under continuously monitored arithmetic averaging is given by

$$\begin{aligned} & c_{fix}(S, I, t) - p_{fix}(S, I, t) \\ = & \left(\frac{I}{T} - X\right) e^{-r(T-t)} + \frac{e^{-q(T-t)} - e^{-r(T-t)}}{T(r-q)} S. \end{aligned}$$

- Note that $\left(\frac{I}{T} - X\right)$ is the sure amount to be paid at T.

- Without optionality in the terminal payoff, we are able to express the price function as an affine function in the state variable. Such analytic tractability persists when the coefficient functions in the governing pde remain to be affine.

Floating strike geometric averaging options

Define $J_t = e^{\frac{1}{t}\int_0^t \ln S_u \, du}$; $x = \ln S$, $y = \dfrac{t \ln J + (T-t)\ln S}{T}$.

Let $C(S, J, t)$ and $P(S, J, t)$ be the price function of floating strike geometric averaging call and put, respectively. Let

$$W(S, J, t) = C(S, J, t) - P(S, J, t).$$

Within the domain $\{0 \le S < \infty,\ 0 \le J < \infty,\ 0 < t < T\}$, $W(S, J, t)$ satisfies

$$\frac{\partial W}{\partial t} + J\frac{\ln S - \ln J}{t}\frac{\partial W}{\partial J} + \frac{\sigma^2}{2}S^2\frac{\partial^2 W}{\partial S^2} + (r-q)S\frac{\partial W}{\partial S} - rW = 0,$$
$$W|_{t=T} = (S-J)^+ - (J-S)^+ = S - J.$$

Set $W(S, J, t) \equiv W(x, y, t)$, within the domain $\{(x, y) \in \mathbb{R}^2,\ 0 < t < T\}$, $W(x, y, t)$ satisfies

$$\frac{\partial W}{\partial t} + \frac{\sigma^2}{2}\left(\frac{T-t}{T}\right)^2 \frac{\partial^2 W}{\partial y^2} + \sigma^2 \left(\frac{T-t}{T}\right) \frac{\partial^2 W}{\partial x \partial y} + \frac{\sigma^2}{2}\frac{\partial^2 W}{\partial x^2} + \left(r - q - \frac{\sigma^2}{2}\right)\left[\frac{T-t}{T}\frac{\partial W}{\partial y} + \frac{\partial W}{\partial x}\right] - rW = 0,$$

terminal condition: $W|_{t=T} = e^x - e^y$.

We try solution of the form:

$$W(x, y, t) = a(t)e^x - b(t)e^y;\ a(t) \text{ and } b(t) \text{ to be determined.}$$

Substituting into the governing differential equation, we obtain

$$a'(t) - qa(t) = 0,$$
$$b'(t) + \frac{\sigma^2}{2}\left(\frac{T-t}{T}\right)^2 b(t) + \left(r - q - \frac{\sigma^2}{2}\right)\left(\frac{T-t}{T}\right) b(t) - rb(t) = 0,$$

with terminal conditions: $a(T) = 1$, $b(T) = 1$.

The solution of $a(t)$ and $b(t)$ are given by

$$\begin{aligned} a(t) &= e^{-q(T-t)}, \\ b(t) &= e^{\frac{\sigma^2(T-t)^3}{6T^2} + \frac{\left(r-q-\frac{\sigma^2}{2}\right)(T-t)^2}{2T} - r(T-t)}. \end{aligned}$$

Finally, the put-call parity relation is given by

$$C(S,J,t) - P(S,J,t) = Se^{-q(T-t)} - J^{\frac{t}{T}} S^{\frac{T-t}{T}} e^{\frac{\sigma^2(T-t)^3}{6T^2} + \frac{\left(r-q-\frac{\sigma^2}{2}\right)(T-t)^2}{2T} - r(T-t)}.$$

Fixed-floating symmetry relations

By applying a change of measure and identifying the time-reversal of a Brownian motion, it is possible to establish the symmetry relations between the prices of floating strike and fixed strike arithmetic averaging Asian options at the start of the averaging period.

Suppose we write the price functions of various continuously monitored arithmetic averaging option at the start of the averaging period (taken to be time zero) as

$$
\begin{aligned}
c_{f\ell}(S_0, \lambda, r, q, T) &= e^{-rT} E_Q[\max(\lambda S_T - A_T, 0)] \\
p_{f\ell}(S_0, \lambda, r, q, T) &= e^{-rT} E_Q[\max(A_T - \lambda S_T, 0)] \\
c_{fix}(X, S_0, r, q, T) &= e^{-rT} E_Q[\max(A_T - X, 0)] \\
p_{fix}(X, S_0, r, q, T) &= e^{-rT} E_Q[\max(X - A_T, 0)].
\end{aligned}
$$

Under the risk neutral measure Q, the asset price S_t follows the Geometric Brownian motion

$$\frac{dS_t}{S_t} = (r - q)\ dt + \sigma\ dZ_t.$$

Here, Z_t is a Q-Brownian motion. Suppose the asset price is used as the numeraire, then

$$\begin{aligned} c^*_{f\ell} = \frac{c_{f\ell}}{S_0} &= \frac{e^{-rT}}{S_0} E_Q[\max(\lambda S_T - A_T, 0)] \\ &= E_Q\left[\frac{S_T e^{-rT}}{S_0} \frac{\max(\lambda S_T - A_T, 0)}{S_T}\right]. \end{aligned}$$

To effect the change of numeraire, we define the measure Q^* by

$$\left.\frac{dQ^*}{dQ}\right|_{\mathcal{F}_T} = e^{-\frac{\sigma^2}{2}T+\sigma Z_T} = \frac{S_T e^{-rT}}{S_0 e^{-qT}}.$$

By virtue of the Girsanov Theorem, $Z_T^* = Z_T - \sigma T$ is Q^*-Brownian.

If we write $A_T^* = A_T/S_T$, then

$$c_{f\ell}^* = e^{-qT} E_{Q^*}[\max(\lambda - A_T^*, 0)],$$

where E_{Q^*} denotes the expectation under Q^*.

We consider

$$A_T^* = \frac{1}{T}\int_0^T \frac{S_u}{S_T}\ du = \frac{1}{T}\int_0^T S_u^*(T)\ du,$$

where

$$S_u^*(T) = \exp\left(-\left(r - q - \frac{\sigma^2}{2}\right)(T-u) - \sigma(Z_T - Z_u)\right).$$

In terms of the Q^*-Brownian motion Z_t^*, where

$$Z_T - Z_u = \sigma(T-u) + Z_T^* - Z_u^*,$$

we can write

$$S_u^*(T) = \exp\left(\left(r - q + \frac{\sigma^2}{2}\right)(u-T) + \sigma(Z_u^* - Z_T^*)\right).$$

We define a reflected Q^*-Brownian motion $\widehat{Z}_t$ such that

$$\widehat{Z}_t = -Z_t^* \text{ for all } t,$$

then $\widehat{Z}_{T-u}$ equals in law to $-(Z_T^* - Z_u^*) = Z_u^* - Z_T^*$ due to the statonary increment property of a Brownian motion.

Hence, we establish

$$A_T^* \stackrel{\text{law}}{=} \widehat{A}_T = \frac{1}{T}\int_0^T e^{\sigma \widehat{Z}_{T-u} + \left(r-q+\frac{\sigma^2}{2}\right)(u-T)}\, du,$$

and via time-reversal of $\widehat{Z}_{T-u}$, we obtain

$$S_0 \widehat{A}_T = \frac{1}{T}\int_0^T S_0 e^{\sigma \widehat{Z}_\xi + \left(q-r-\frac{\sigma^2}{2}\right)\xi}\, d\xi.$$

Note that $\widehat{A}_T S_0$ is the arithmetic average of the price process with drift rate $q-r$. Summing the results together, we have

$$c_{f\ell} = S_0 c_{f\ell}^* = e^{-qT} E_{Q^*}[\max(\lambda S_0 - \widehat{A}_T S_0, 0)],$$

and from which we deduce the following fixed-floating symmetry relation

$$c_{f\ell}(S_0, \lambda, r, q, T) = p_{fix}(\lambda S_0, S_0, q, r, T).$$

Note that we swap r and q in the floating strike and fixed strike pricing formulas.

3.3 Guaranteed minimum withdrawal benefit

Product Nature

- Variable annuities — deferred annuities that are fund-linked.
- The single lump sum paid by the policyholder at initiation is invested in a portfolio of funds chosen by the policyholder — equity participation.
- The GMWB allows the policyholder to withdraw funds on an annual or semi-annual basis until the entire principal is returned.
- In 2004, 69% of all variable annuity contracts sold in the US included the GMWB option.

Numerical example

- Let the initial fund value be $\$100,000$ and the withdrawal rate be fixed at 7% per annum. Suppose the investment account earns 10% in the first two years but earns returns of -60% in each of the next three years.

Year	Rate of return during the year	Fund value before withdrawals	Amount withdrawn	Fund value after withdrawals	Guaranteed withdrawals remaining balance
1	10%	$110,000$	$7,000$	$103,000$	$93,000$
2	10%	$113,300$	$7,000$	$106,300$	$86,000$
3	-60%	$42,520$	$7,000$	$35,520$	$79,000$
4	-60%	$14,208$	$7,000$	$7,208$	$72,000$
5	-60%	$2,883$	$7,000$	0	$65,000$

- At the end of year five before any withdrawal, the fund value $\$2,883$ is not enough to cover the annual withdrawal payment of $\$7,000$.

The guarantee kicks in when the fund is non-performing

The value of the fund is set to be zero and the policyholder's 10 remaining withdrawal payments are financed under the writer's guarantee. The policyholder's income stream of annual withdrawals is protected irrespective of the market performance. The investment account balance will have shrunk to zero before the principal is repaid and will remain there.

Good performance of the fund

If the market does well, then there will be funds left at policy's maturity.

Numerical example revisited

Suppose the initial lump sum investment of $\$100,000$ is used to purchase 100 units of the mutual fund, so each unit worths $\$1,000$.

- After the first year, the rate of return is 10% so each unit is $\$1,100$. The annual guaranteed withdrawal of $\$7,000$ represents $\$7,000/\$1,100 = 6.364$ units. The remaining number of units of the mutual fund is $100 - 6.364 = 93.636$ units.

- After the second year, there is another rate of return of 10%, so each unit of the mutual fund worths $\$1,210$. The withdrawal of $\$7,000$ represents $\$7,000/\$1,210 = 5.785$ units, so the remaining number of units $= 87.851$.

- There is a negative rate of return of 60% in the third year, so each unit of the mutual fund worths \$484. The withdrawal of $\$7,000$ represents $\$7,000/\$484 = 14.463$ units, so the remaining number of units $= 73.388$.

- Depending on the performance of the mutual fund, there may be certain number of units remaining if the fund is performing or perhaps no unit is left if it comes to the worst senario.

  - In the former case, the holder receives the guaranteed total withdrawal amount of $\$100,000$ (neglecting time value) plus the remaining units of mutual funds held at maturity.

  - If the mutual fund is non-performing, then the total withdrawal amount of $\$100,000$ over the whole policy life is guaranteed.

How is the benefit funded?

- Proportional fee on the investment account value
  - for a contract with a 7% withdrawal allowance, a typical charge is around 40 to 50 basis points of proportional fee on the investment account value.
- GMWB can also be seen as a guaranteed stream of 7% per annum plus a call option on the terminal investment account value W_T, $W_T \geq 0$. The strike price of the call is zero.

Static withdrawal model – continuous version

- The withdrawal rate G (dollar per annum) is fixed throughout the life of the policy.

- When the investment account value W_t ever reaches 0, it stays at this value thereafter (absorbing barrier).

  $\tau = \inf\{t : W_t = 0\}$, $\quad \tau$ is the first passage time of hitting 0.

  Under the risk neutral measure Q, the dynamics of W_t is governed by

  $$\begin{aligned} dW_t &= (r - \alpha)W_t\,dt + \sigma W_t\,dB_t - G\,dt, && t < \tau \\ W_t &= 0, && t \geq \tau \\ W_0 &= w_0, \end{aligned}$$

  where α is the proportional annual fee charge on the investment account as the withdrawal allowance.

  policy value $= E_Q\left[\int_0^T Ge^{-ru}\,du\right] + E_Q[e^{-rT}W_T]$.

Surrogate unrestricted process

To enhance analytic tractability, the restricted account value process W_t is replaced by a surrogate unrestricted process $\widetilde{W}_t$ at the expense of introducing optionality in the terminal payoff (zero strike call payoff). Consider the modified unrestricted stochastic process:

$$\begin{aligned} d\widetilde{W}_t &= (r-\alpha)\widetilde{W}_t\,dt - G\,dt + \widetilde{W}_t\,dB_t, \quad t>0, \\ \widetilde{W}_0 &= w_0. \end{aligned}$$

Solving for $\widetilde{W}_t$, we obtain

$$\widetilde{W}_t = X_t\left(w_0 - G\int_0^t \frac{1}{X_u}\,du\right)$$

where

$$X_t = e^{\left(r-\alpha+\frac{\sigma^2}{2}\right)t+\sigma B_t}.$$

The solution is the unit exponential Brownian motion X_t multiplied by the number of units remaining after depletion by withdrawals.

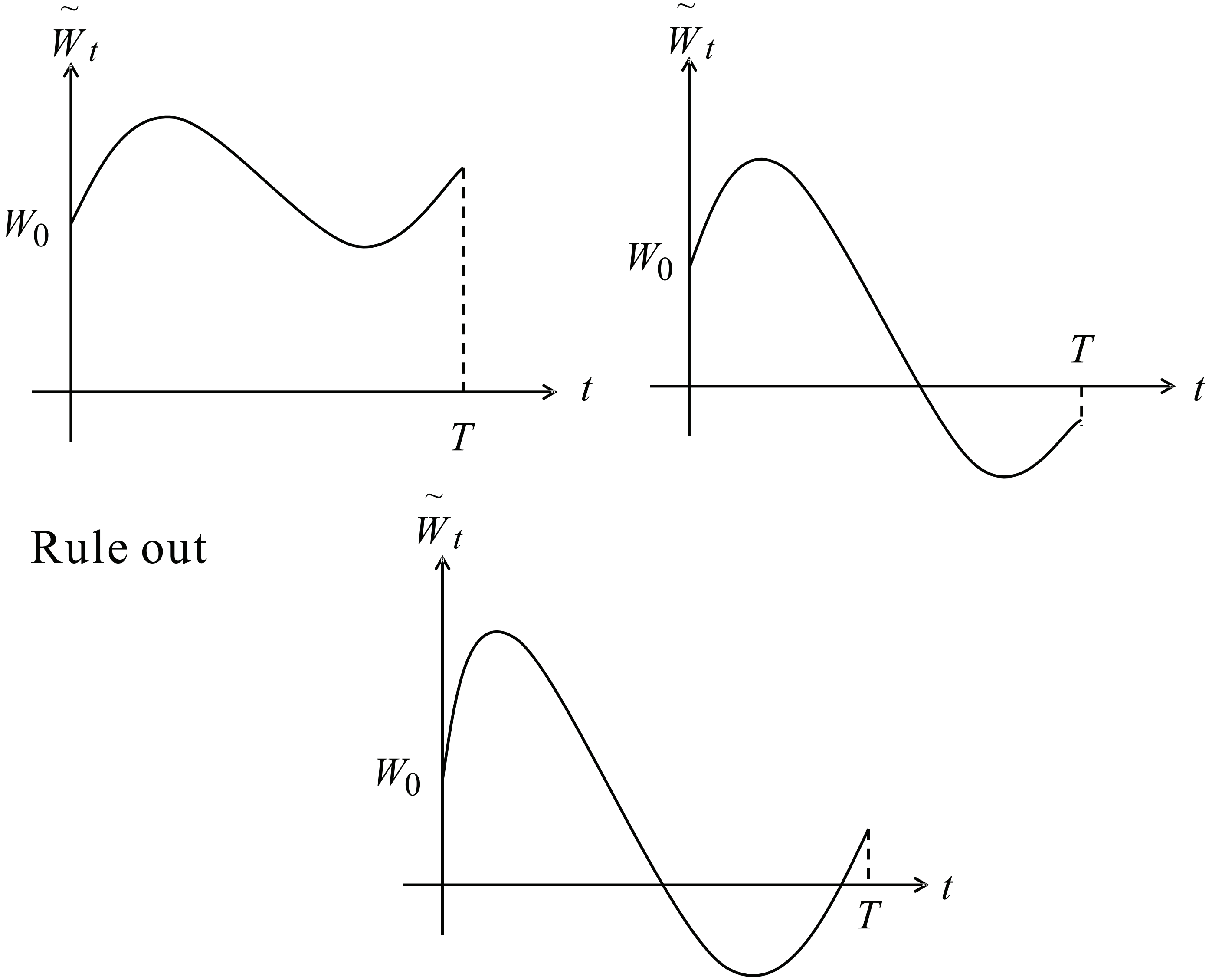
Financial interpretation

Take the initial value of one unit of the fund to be unity for convenience. Here, X_t represents the corresponding fund value process with $X_0 = 1$.

- The number of units acquired at initiation is w_0. The total number of units withdrawn over $(0, t]$ is given by $G \int_0^t \frac{1}{X_u} \, du$.

- Under the unrestricted process assumption, $\widetilde{W}_t$ may become negative when the number of units withdrawn exceeds w_0. However, in the actual case, W_t stays at the absorbing state of zero value once the number of unit withdrawn hits w_0.

Either $\widetilde{W}_t > 0$ for $t \leq T$ or $\widetilde{W}_T$ remains negative once W_t reaches the negative region at some earlier time prior to T.



Lemma $\tau_0 > T$ if and only if $\widetilde{W}_T > 0$.

$\Longrightarrow$ part. Suppose $\tau_0 > T$, then by the definition of the first passage time, we have $\widetilde{W}_T > 0$.

$\Longleftarrow$ part. Recall that

$$\widetilde{W}_t = X_t \left(w_0 - \int_0^t \frac{G}{X_u} \, du \right)$$

so that

$$\widetilde{W}_t > 0 \quad \text{if and only if} \quad \int_0^t \frac{G}{X_u} \, du < w_0.$$

Suppose $\widetilde{W}_T > 0$, this implies that the number of units withdrawn by time $T = \int_0^T \frac{G}{X_u} \, du < w_0$. Since $X_u \geq 0$, for any $t < T$, we have

$$\text{number of units withdrawn by time } t = \int_0^t \frac{G}{X_u} \, du \leq \int_0^T \frac{G}{X_u} \, du < w_0.$$

Hence, if $\widetilde{W}_T > 0$, then $\widetilde{W}_t > 0$ for any $t < T$.

Intuition of the dynamics

Once the process $\widetilde{W}_t$ becomes negative, it will never return to the positive region. This is because when $\widetilde{W}_t$ increases from below back to the zero level, only the drift term $-G\,dt$ survives. This always pulls $\widetilde{W}_t$ back into the negative region.

Relation between W_T and $\widetilde{W}_T$

Note that $W_T = 0$ if and only if $\tau \le T$. We then have

$$W_T = \widetilde{W}_T \mathbf{1}_{\{\tau > T\}} = \widetilde{W}_T \mathbf{1}_{\{\widetilde{W}_T > 0\}} = \max(\widetilde{W}_T, 0).$$

Optionality in the terminal payoff

The terminal payoff from the investment account becomes

$$\max(\widetilde{W}_T, 0) = G X_T \left(\frac{w_0}{G} - \int_0^T \frac{1}{X_u}\, dx \right)^+, \quad x^+ = \max(x, 0).$$

Defining $U_t = \dfrac{G}{w_0}\displaystyle\int_0^t \dfrac{1}{X_u}\,du$ and observing $T = \dfrac{w_0}{G}$, we obtain

$$E_Q[e^{-rT}\widetilde{W}_T^+] = w_0 E_Q[e^{-rT} X_T (1 - U_T)^+].$$

Here, U_t represents the fraction of units withdrawn up to time t, which captures the path dependence of the depletion process of the investment account due to the continuous withdrawal process. Lastly, we have

$$\text{policy value} = E_Q\left[\int_0^T G e^{-ru} du\right] + w_0 E_Q\left[e^{-rT} X_T (1 - U_T)^+\right].$$

The pricing issue is to find the fair value for the participating fee rate α such that the initial policy value equals the lump sum paid upfront by the policyholder so that the policy contract is fair to both counterparties.