Advanced Topics in Derivative Pricing Models

Topic 4 - Variance products and volatility derivatives

4.1 Volatility trading and replication of variance swaps

4.2 Volatility swaps

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4.1 Volatility trading

Characteristics of volatility

- Likely to grow when uncertainty and risk increase.

- Volatilities appear to revert to the mean.

- Volatility is often negatively correlated with stock or index level, and tends to stay high after large downward moves.

- Stock options are impure: they provide exposure to both direction of the stock price and its volatility. If one hedges the options according to Black-Scholes prescription, then she can remove the exposure to the stock price.

Delta-hedging is at best inaccurate since volatility cannot be accurately estimated, stocks cannot be traded continuously, together with transaction costs and jumps.

Businesses that are implicitly short volatility (lose when volatility increases)

- Investors following active benchmarking strategies may require more frequent rebalancing and incur higher transaction expenses during volatile periods.

- Equity funds are probably short volatility due to the negative correlation between index level and volatility.

- Hedge funds that take positions on the spread between stocks of companies planning mergers will narrow. If volatility increases, the merger may become less likely and spread may widen.

Volatility swaps are forward contracts on future realized stock volatility; and similarly, variance swaps on future variance (square of future volatility). They provide pure exposure to volatility and variance, respectively.

Replication of variance swaps - continuous model

The fair strike of a variance swap (continuously monitored) is given by

$$K_{\text{var}} = E_0[V_R] = E_0\left(\frac{1}{T}\int_0^T \sigma_t^2 \; dt\right)$$

Suppose the asset price process S_t follows the following Brownian motion:

$$\frac{dS_t}{S_t} = r \; dt + \sigma_t \; dW_t,$$

where W_t is the standard Brownian motion and σ_t is non-stochastic (though may be state dependent). We may rewrite the dynamics equation as follows:

$$d \ln S_t = \left(r - \frac{\sigma_t^2}{2}\right) dt + \sigma_t \; dW_t.$$

Subtracting the two, we obtain

$$\frac{dS_t}{S_t} - d\ln S_t = \frac{\sigma_t^2}{2}\ dt$$

The measure of the continuous realized variance is then given by

$$V_R = \frac{1}{T}\int_0^T \sigma_t^2\ dt = \frac{2}{T}\left(\int_0^T \frac{dS_t}{S_t} - \ln\frac{S_t}{S_0}\right).$$

The formula dictates the strategy that can be adopted to replicate the realized variance.

We take $\frac{1}{S_t}$ units of stock at time t paying \$1, and enter a "static" short position at time 0 in a forward contract which at maturity has a payoff equals to the logarithm of the total return of on the stock $\ln\frac{S_T}{S_0}$, where $\frac{S_T}{S_0}$ is the total return over $[0, T]$.

This is a self-financing strategy

Suppose the stock price goes up, the investor sells $\frac{1}{S_t}$ units of stock and buys $\frac{1}{S_{t+dt}}$ units paying \$1. The net amount $\frac{S_{t+dt}}{S_t} - 1$ is invested in the riskfree asset. Over the same period, the forward value $F_t = E_t\left[\ln \frac{S_T}{S_0}\right]$ of the log contract increases in value. The short position in the log contract offsets the gain on the long stock position. Recall

$$\ln \frac{S_{t+dt}}{S_t} \approx \frac{S_{t+dt}}{S_t} - 1$$

since $\ln(1 + x) \approx x$.

The pricing issue is to find the fair strike of the variance swap.

$$\begin{aligned} K_{\text{var}} &= E_0[V_R] = \frac{2}{T}E_0\left[\int_0^T \frac{dS_t}{S_t} - \ln\frac{S_T}{S_0}\right] \\ &= \frac{2}{T}\left\{\underbrace{E_0\left[\int_0^T r\ dt\right]}_{rT} + \underbrace{E_0\left[\int_0^T \sigma_t\ dW_t\right]}_{\text{zero}} - E_0\left[\ln\frac{S_T}{S_0}\right]\right\}. \end{aligned}$$

The expectation of the long stock position gives rT since the dollar value of the stock position is always \$1. How to replicate the log contract using basic instruments of forward contracts, calls and puts?

Technical result For any twice-differentiable function f: $\mathbb{R} \to \mathbb{R}$, and any $S_* \geq 0$, we have

$$\begin{aligned} f(S_T) &= f(S_*) + f'(S_*)(S_T - S*) + \int_0^{S_*} f''(K)(K - S_T)^+\ dK \\ &\quad + \int_{S_*}^{\infty} f''(K)(S_T - K)^+\ dK. \end{aligned}$$

Proof

$$
\begin{aligned}
f(S_T) &= \int_0^{S_*} f(K)\delta(S_T - K)\ dK + \int_{S_*}^{\infty} f(K)\delta(S_T - K)\ dK \\
&= f(K)\mathbf{1}_{\{S_T<K\}}\Big]_0^{S_*} - \int_0^{S_*} f'(K)\mathbf{1}_{\{S_T<K\}}\ dK \\
&\quad + f(K)\mathbf{1}_{\{S_T\geq K\}}\Big]_{S_*}^{\infty} - \int_{S_*}^{\infty} f'(K)\mathbf{1}_{\{S_T\geq K\}}\ dK \\
&= f(S_*)\mathbf{1}_{\{S_T<S_*\}} - \left[f'(K)(K - S_T)^+\right]_0^{S_*} + \int_0^{S_*} f''(K)(K - S_T)^+\ dK \\
&\quad + f(S_*)\mathbf{1}_{\{S_T\geq S_*\}} - \left[f'(K)(S_T - K)^+\right]_{S_*}^{\infty} + \int_{S_*}^{\infty} f''(K)(S_T - K)^+\ dK \\
&= f(S_*) + f'(S_*)[(S_T - S_*)^+ - (S_* - S_T)^+] \\
&\quad + \int_0^{S_*} f''(K)(K - S_T)^+\ dK + \int_{S_*}^{\infty} f''(K)(S_T - K)^+\ dK
\end{aligned}
$$

$$
\begin{aligned}
f(S_T) - f(S_*) &= f'(S_*)(S_T - S_*) + \int_0^{S_*} f''(K)(K - S_T)^+\ dK \\
&\quad + \int_{S_*}^{\infty} f''(K)(S_T - K)^+\ dK.
\end{aligned}
$$

The log payoff $\ln \dfrac{S_T}{S_0}$ can be rewritten as

$$\ln \frac{S_T}{S_0} = \ln \frac{S_T}{S_*} + \ln \frac{S_*}{S_0}, \;\; S_* \text{ is an arbitrage non-negative number.}$$

Applying the technical formula for $f(S_T) = \ln S_T$, we have

$$\ln S_T - \ln S_* = \frac{S_T - S_*}{S_*} - \int_0^{S_*} \frac{1}{K^2}(K - S_T)^+ \, dK - \int_{S_*}^{\infty} \frac{1}{K^2}(S_T - K)^+ \, dK.$$

- Hold a long position in $\dfrac{1}{S_*}$ forward contracts with forward price S_*;
- Short positions in $\dfrac{1}{K^2}$ put options with strike K, K from 0 to S_*; short positions in $\dfrac{1}{K^2}$ call options with strike K, K from S_* to ∞.

All contracts have the same maturity T.

Valuation of fair strike

$$K_{\text{var}} = \frac{2}{T}\left\{rT - E_0\left[\ln\frac{S_*}{S_0} + \frac{S_T - S_*}{S_*} - \int_0^{S_*}\frac{1}{K^2}(K - S_T)^+ \, dK - \int_{S_*}^{\infty}\frac{1}{K^2}(S_T - K)^+ \, dK\right]\right\}.$$

Note that

$$S_0 = e^{-rT}E_0[S_T], \; C_0(K) = e^{-rT}E_0[(S_T - K)^+],$$
$$P_0(K) = e^{-rT}E_0[(K - S_T)^+].$$

We then have

$$\begin{aligned} K_{\text{var}} \;=\; & \frac{2}{T}\left[rT - \left(\frac{S_0}{S_*}e^{rT} - 1\right) - \ln\frac{S_*}{S_0}\right. \\ & \left. + \, e^{rT}\int_0^{S_*}\frac{1}{K^2}P_0(K)\, dK + e^{rT}\int_{S_*}^{\infty}\frac{1}{K^2}C_0(K)\, dK\right]. \end{aligned}$$

The formula requires an infinite number of strikes in order to be exact, while the market provides only a finite number of options.

Profit and loss (P&L) if one hedges at the wrong volatility

- Assume that the futures price process is continuous and that the true vol is given by some unknown stochastic process σ_t:

$$\frac{dF_t}{F_t} = \mu_t \, dt + \sigma_t \, dW_t, \qquad t \in [0, T].$$

- Assume that a claim on the futures price is sold for an initial implied vol of σ_i and that delta-hedging is conducted using the Black model delta evaluated at a constant hedge vol σ_h. Applying Ito's lemma to $V(F,t)e^{r(T-t)}$:

$$\begin{aligned} V(F_T, T) &= V(F_0, 0)e^{rT} + \int_0^T e^{r(T-t)} \frac{\partial V}{\partial F}(F_t, t) \, dF_t \\ &+ \int_0^T e^{r(T-t)} \frac{1}{2} \frac{\partial^2 V}{\partial F^2}(F_t, t) \, (dF_t)^2 \\ &+ \int_0^T e^{r(T-t)} \left[\frac{\partial V}{\partial t}(F_t, t) - rV(F_t, t) \right] dt. \end{aligned}$$

- Since the futures has stochastic volatility, $(dF_t)^2 = \sigma_t^2 F_t^2 \; dt$ and so

$$\begin{aligned} V(F_T,T) &= V(F_0,0)e^{rT} + \int_0^T e^{r(T-t)} \frac{\partial V}{\partial F}(F_t,t) \; dF_t \\ &+ \int_0^T e^{r(T-t)} \left[\frac{\sigma_t^2 F_t^2}{2} \frac{\partial^2 V}{\partial F^2}(F_t,t) + \frac{\partial V}{\partial t}(F_t,t) - rV(F_t,t) \right] dt. \end{aligned}$$

- Suppose we have chosen our function $V(F,t)$ to be the function $V(F,t;\sigma_h)$ which solves the Black PDE:

$$\frac{\sigma_h^2 F^2}{2} \frac{\partial^2 V}{\partial F^2}(F,t;\sigma_h) + \frac{\partial V}{\partial t}(F,t;\sigma_h) - rV(F,t;\sigma_h) = 0,$$

and the terminal condition: $V(F,T;\sigma_h) = f(F)$.

- Substitution gives

$$\begin{aligned} f(F_T) &= V(F_0,0;\sigma_h)e^{rT} + \int_0^T e^{r(T-t)} \frac{\partial V}{\partial F}(F_t,t;\sigma_h) \; dF_t \\ &+ \int_0^T e^{r(T-t)} [\sigma_t^2 - \sigma_h^2] \frac{F_t^2}{2} \frac{\partial^2 V}{\partial F^2}(F_t,t;\sigma_h) \; dt. \end{aligned}$$

- Adding $V(F_0, 0; \sigma_i)e^{rT}$ to both sides and re-arranging, we obtain

$$\begin{aligned} P\&L_T &= [V(F_0, 0; \sigma_t) - V(F_0, 0; \sigma_h)]e^{rT} \\ &+ \int_0^T e^{r(T-t)} \frac{F_t^2}{2} \frac{\partial^2 V}{\partial F^2}(F_t, t; \sigma_h)(\sigma_h^2 - \sigma_t^2)\ dt. \end{aligned}$$

- In words, when we sell the claim for an implied vol of σ_i initially, the total P&L from delta-hedging with constant vol σ_h over $(0, T)$ is the future value of the difference in Black Scholes valuations plus half the accumulated dollar gamma weighted average of the difference between the hedge variance and the true variance.

- Note that the P&L vanishes if $\sigma_t = \sigma_h = \sigma_i$. If $\frac{\partial^2 V}{\partial F^2}(F_t, t; \sigma_h) \geq 0$ as is true for options, and if $\sigma_i = \sigma_h < \sigma_t$ for all $t \in [0, T]$, then you sold the claim for too low a vol and a loss results, regardless of the path. Conversely, if $\sigma_i > \sigma_t$ for all $t \in [0, T]$, then delta-hedging at $\sigma_h = \sigma_i$ guarantees a positive P&L.

In terms of stock price S_0, the P&L is given by

$$\begin{aligned} P\&L &= [V(S_0, 0; \sigma_i) - V(S_0, 0; \sigma_h)]e^{rT} \\ &+ \int_0^T e^{r(T-t)}(\sigma_h^2 - \sigma_t^2)\frac{S_t^2}{2}\frac{\partial^2}{\partial S^2}V(S_t, t; \sigma_h)\ dt. \end{aligned}$$

Let us linearize the first two terms in σ^2 around σ_h. The P&L can be rewritten as

$$\begin{aligned} P\&L &= Te^{rT}(\sigma_t^2 - \sigma_h^2)\frac{S_0^2}{2}\frac{\partial^2}{\partial S^2}V(S_0, t; \sigma_h) \\ &+ \int_0^T e^{r(T-t)}(\sigma_h^2 - \sigma_t^2)\frac{S_t^2}{2}\frac{\partial^2}{\partial S^2}V(S_t, t; \sigma_h)\ dt. \end{aligned}$$

Define the volatility over the period 0 to T by $\widehat{\sigma}$, where $\widehat{\sigma}$ is given by $\widehat{\sigma}^2 = \frac{1}{T}\int_0^T \sigma_t^2\ dt$.

The P&L may be expressed as

$$
\begin{aligned}
& P\&L \\
= \; & (\sigma_t^2 - \widehat{\sigma}^2) T e^{rT} \frac{S_0^2}{2} \frac{\partial^2}{\partial S^2} V(S_0, T; \sigma_h) \\
& + \; (\widehat{\sigma}^2 - \sigma_h^2) T \left[e^{rT} \frac{S_0^2}{2} \frac{\partial^2}{\partial S^2} V(S_0, T; \sigma_h) - \frac{1}{T} \int_0^T e^{r(T-t)} \frac{S_t^2}{2} \frac{\partial^2}{\partial S^2} V(S_t, T-t; \sigma_h) \; dt \right] \\
& + \; \int_0^T (\widehat{\sigma}^2 - \sigma_t^2) e^{r(T-t)} \frac{S_t^2}{2} \frac{\partial^2}{\partial S^2} V(S_t, T-t; \sigma_h) \; dt.
\end{aligned}
$$

Defining g_t as

$$
g_t = e^{r(T-t)} \frac{S_t^2}{2} \frac{\partial^2}{\partial S^2} V(S_t, T-t; \sigma_h) = \frac{e^{r(T-t)}}{2\sigma_h} \frac{\partial}{\partial \sigma} V(S_t, T-t; \sigma_h),
$$

then the P&L is thus equal to

$$
P\&L = (\sigma_t^2 - \widehat{\sigma}^2) T g_0 + (\widehat{\sigma}^2 - \sigma_h^2) T \left[g_0 - \frac{1}{T} \int_0^T g_t \; dt \right] + \int_0^T (\widehat{\sigma}^2 - \sigma_t^2) g_t \; dt.
$$

The P&L can be broken up into three components:

$$P\&L = \underbrace{(\sigma_t^2 - \widehat{\sigma}^2)Tg_0}_{1} + \underbrace{(\widehat{\sigma}^2 - \sigma_h^2)T\left[g_0 - \frac{1}{T}\int_0^T g_t \; dt\right]}_{2} + \underbrace{\int_0^T (\widehat{\sigma}^2 - \sigma_t^2)g_t \; dt}_{3}.$$

1. The "variance risk" component or a variance swap exposure for a notional amount equal to Tg_0 and a variance strike equal to the square of the option implied volatility.

2. The "vega risk" factor, which stems from the fact that the option is hedged at the implied volatility σ_h instead of at the realized volatility $\widehat{\sigma}$. This term is indeed null if the trader is able to hedge at the realized but unknown volatility.

3. The "Volatility path dependency risk" or "model risk" factor that depends on the historical behavior of realized volatility. Under the Black & Scholes assumption, the instantaneous volatility σ_t is constant and thus equal to the realized volatility between time t and T. However, should the volatility vary over time, $(\widehat{\sigma}^2 - \sigma_t^2)$ will no longer be zero. As g_t is a decreasing function of time to maturity, this term will be positive if instantaneous, or intraday, volatility rises during the life of the option. The term also depends on the true distribution of stock returns.

- Based on realistic simulations, the variance risk only represents 52% of the total P&L resulting from delta-hedging an option.

- As a result, delta-hedging options does not provide pure exposure to volatility, given that the P&L not only depends on variance risk but also on a vega risk which itself results from the face that risk cannot be hedged at the unknown future realized volatility and that volatility may not be constant over time.

Delta-hedging options yields further risk sources

The above analysis was done without taking into account dividends and by assuming a constant interest rate. In practice, traders face the risk of unknown dividends being paid during the life of the option, while with interest rates liable to vary over time, the option vega may change if the interest rate changes.

Furthermore, option's delta-hedging is impacted by transaction costs and liquidity issues. The above analysis does not take into account the fact that trading stocks is costly and that certain stocks and indices may lack liquidity.

We need new derivatives instruments that enable investors to take a view on volatility without bearing any other risks.

4.2 Volatility swaps

Even though variance swaps can be priced and replicated easily, they are still less actively traded compared to volatility swaps.

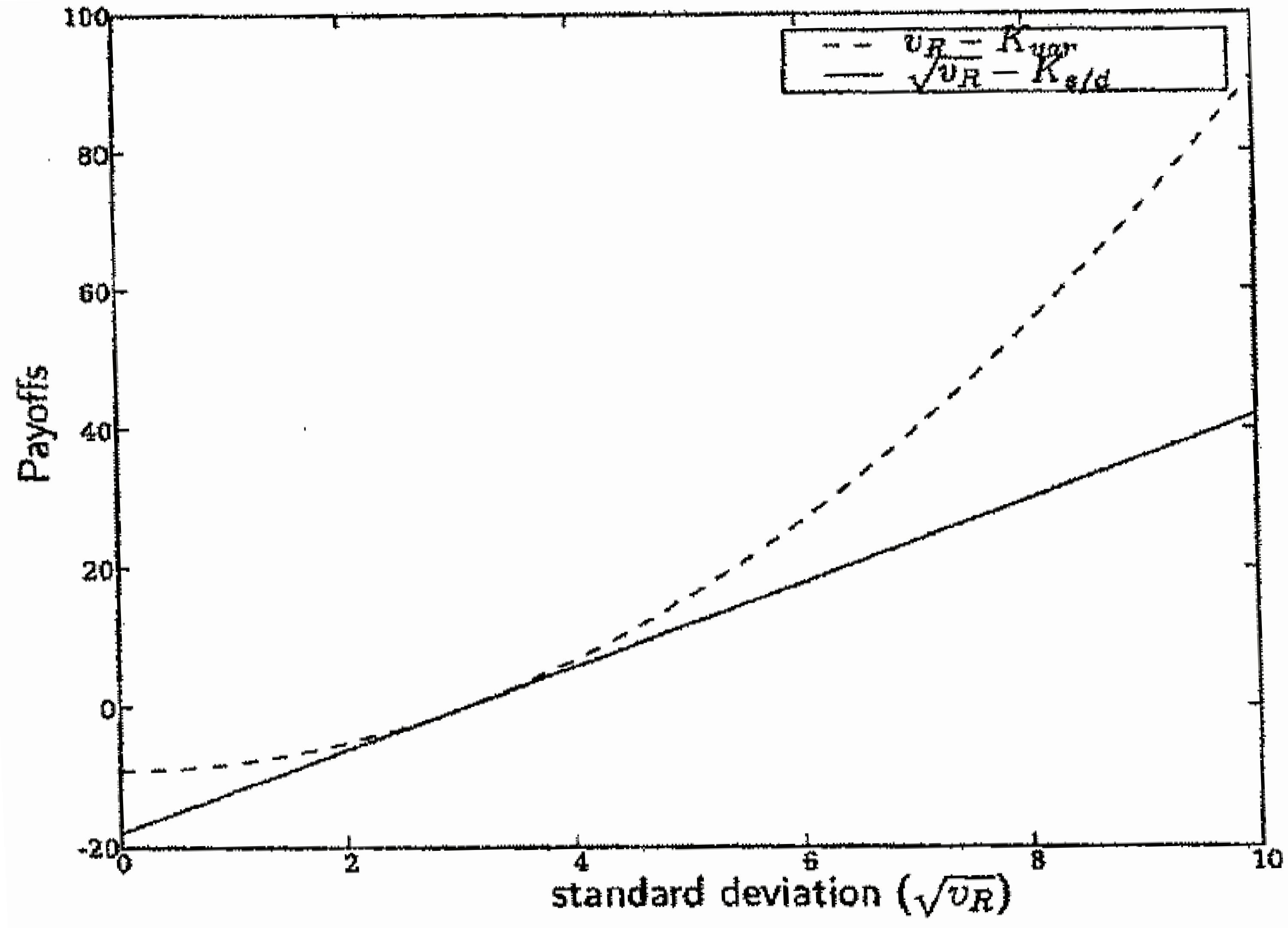
First order approximation:

$$(\sqrt{V_R})^2 - (K_{s/d})^2 \approx 2K_{s/d}(\sqrt{V_R} - K_{s/d})$$

$$\text{or} \qquad \sqrt{V_R} - K_{s/d} \approx \frac{1}{2K_{s/d}}\left[V_R - (K_{s/d})^2\right], \; K_{s/d} = \sqrt{K_{\text{var}}}.$$

To make $K_{s/d} < \sqrt{K_{\text{var}}}$, consider the second order Taylor expansion of $g(V_R) = \sqrt{V_R}$ around $K_{\text{var}} = E_0[V_R]$, we have

$$\sqrt{V_R} \approx \sqrt{K_{\text{var}}} + \frac{1}{2\sqrt{K_{\text{var}}}}(V_R - K_{\text{var}}) - \frac{1}{8(K_{\text{var}})^{3/2}}(V_R - K_{\text{var}})^2.$$



Variance and Standard deviation Payoffs

Taking the expected values on both sides, we obtain

$$K_{s/d} = E_0[\sqrt{V_R}] \approx \sqrt{K_{\text{var}}} - \underbrace{\frac{1}{8(K_{\text{var}})^{3/2}} \underbrace{E_0[(V_R - K_{\text{var}})^2]}_{\text{var}_0(V_R)}}_{\text{convexity correction}}$$

The convexity correction represents the mismatch between $K_{s/d}$ and $\sqrt{K_{\text{var}}}$. Under this approximation, we achieve $K_{s/d} < \sqrt{K_{\text{var}}}$.

- The above formula does not give a straightforward formula for $K_{s/d}$ since the conditional variance of the realized variance has to be estimated.

- Broadie and Jain (2008) show that this convexity correction formula to approximate fair volatility strikes may not provide good estimates in jump-diffusion models.

4.3 Pricing of discrete variance swaps

- A discrete variance swap is a forward contract on the discrete realized variance of the price dynamics of the underlying security. The floating leg of the discrete variance swap is the discrete realized variance and is calculated using the second moment of log returns of the underlying asset.

$$R_t = \ln\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right), \quad i = 1, 2, \ldots, n.$$

  Here, $0 = t_0 < t_1 < \cdots < t_n = T$ is a partition of the time interval $[0, T]$ into n equal segments of length Δt, i.e., $t_i = iT/n$ for each $i = 0, 1, \ldots, n$.

The discrete realized variance, $V_d(0, n, T)$ can be written as:

$$V_d(0, n, T) = \frac{1}{(n-1)\Delta t} \sum_{i=1}^{n} R_i^2 = \frac{\sum_{i=1}^{n} \left(\ln \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2}{(n-1)\Delta t}.$$

The floating leg of the discretely sampled realized variance, $V_c(0, T)$, in the limit $n \to \infty$. That is,

$$V_c(0, T) \equiv \lim_{n \to \infty} V_d(0, n, T) = \lim_{n \to \infty} \frac{n}{(n-1)T} \sum_{i=1}^{n} R_i^2.$$

Heston stochastic volatility model

In this section, we present an analysis of the linear rate of convergence of discrete variance strikes to continuous variance strikes with number of sampling dates under the Heston stochastic volatility (SV) model. The Heston model is given by

$$dS_t = rS_t dt + \sqrt{v_t} S_t(\rho dW_t^1 + \sqrt{1-\rho^2} dW_t^2)$$
$$dv_t = \kappa(\theta - v_t)dt + \sigma_v \sqrt{v_t} dW_t^1.$$

The first equation gives the dynamics of the stock price S_t and $\sqrt{v_t}$ is the volatility. The second equation gives the evolution of the variance which follows a square root process: θ is the long run mean variance, κ represents the speed of mean reversion, and σ_v is a parameter which determines the volatility of the variance process. The process W_t^1 and W_t^2 are independent standard Brownian motions under risk-neutral measure Q, and ρ represents the instantaneous correlation between the return process and the volatility process.

SV model: Continuous variance strike

In the Heston stochastic volatility model, the fair continuous variance strike $K^*_{\text{var}} = E[V_c(0,T)]$ is given by

$$E\left(\frac{1}{T}\int_0^T v_s\ ds\right) = \theta + \frac{v_0 - \theta}{\kappa T}(1 - e^{-\kappa T}).$$

The fair continuous variance strike in the Heston stochastic volatility model is independent of the volatility of variance σ_v. Similarly, the variance of the continuous realized variance, $\text{Var}(V_c(0,T))$, can be obtained by calculating the second moment of the Laplace transform.

$$\begin{aligned}
&\text{var}\left(\frac{1}{T}\int_t^T v_s\ ds\right)\\
= \ &\frac{\sigma_v^2 e^{-2\kappa(T-t)}}{2\kappa^3 T^2}\{2[e^{2\kappa(T-t)} - 2e^{\kappa(T-t)}\kappa(T-t) - 1](v_t - \theta)\\
&+ [4e^{\kappa(T-t)} - 3e^{2\kappa(T-t)} + 2e^{2\kappa(T-t)}\kappa(T-t) - 1]\theta\}.
\end{aligned}$$

The variance of the continuous realized variance depends on the volatility of variance.

SV model: Discrete variance strike

In the Heston stochastic volatility model, the expectation of the discrete realized variance and its continuous counterpart are related by

$$E_0(V_d(0,n,T)) = E_0(V_c(0,T)) + g(r,\rho,\sigma_v,\kappa,\theta,n),$$

where

$$\begin{aligned} g(r,\rho,\sigma_v,\kappa,\theta,n) &= \frac{r^2T}{n-1} + \frac{1}{T}E\left(\frac{1}{T}\int_0^T v_t\ dt\right)\left[\frac{1}{n-1} - \frac{rT}{n-1} + \frac{\rho\kappa\theta T}{(n-1)\sigma_v}\right] \\ &+ \frac{\sum_{i=0}^{n-1} E(\int_{t_{-i}}^{t_{i+1}}\int_{t_i}^{t_{i+1}} v_t v_s\ dtds)}{(n-1)\Delta t}\left(\frac{1}{4} - \frac{\rho\kappa}{\sigma_v}\right) \\ &- \frac{\sum_{i=0}^{n-1} E[\rho(\int_{t_i}^{t_{i+1}} vt\ dt)(v_{t_{i+1}} - v_{t_{-i}})]}{\sigma_v(n-1)\Delta t}. \end{aligned}$$

Since $g(r,\rho,\sigma_v,\kappa,\theta,n) = O\left(\frac{1}{n}\right)$, the expectation of discrete realized variance converges to the expected continuous realized variance linearly with the sampling size ($n = T/\Delta t$). That is,

$$K^*_{\text{var}}(n) = K^*_{\text{var}} + g(r,\rho,\sigma_v,\kappa,\theta,n)$$

and the discrete variance strike converges to the continuous variance strike linearly with Δt, where $\Delta t = T/n$.

4.6 Timer options

A standard timer call option can be viewed as a call option with random maturity which depends on the time needed for a pre-specified variance budget to be fully consumed.

A variance budget is calculated as the target volatility squared, multiplied by the target maturity.

Mathematical definition

Suppose the asset price is monitored at $t_j = j\Delta t$, $j = 0, 1, \ldots, n$. The annualized realized variance for the period $[0, T]$ is defined as

$$\widehat{\sigma}_T^2 = \frac{1}{(n-1)\Delta t} \sum_{i=0}^{n-1} \left(\ln \frac{S_{i+1}}{S_i} \right)^2, \; T = n\Delta t.$$

The realized variance over time period $[0, T]$ is

$$V_T = n\Delta t \widehat{\sigma}_T^2 \approx \sum_{i=0}^{n-1} \left(\ln \frac{S_{i+1}}{S_i} \right)^2 .$$

The investor specifies a variance budget

$$B = \sigma_0^2 T_0,$$

where T_0 is the estimated investment horizon and σ_0 is the forecast volatility during the investing period.

The timer call with random maturity pays $\max(S_{t_j} - k, 0)$ at the first time t_j when the realized variance exceeds B. That is,

$$t_j = \min \left\{ k > 0, \sum_{i=0}^{k-1} \left(\ln \frac{S_{i+1}}{S_i} \right)^2 \geq B \right\} .$$

Option strategy without paying the volatility risk premium

The price of a vanilla call option is determined by the level of implied volatility quoted in the market (as well as maturity and strike price). The level of impled volatility is often higher than the realized volatility (risk premium due to the uncertainty of future market direction).

"High implied volatility means call options are often overpriced. In the timer option, the investor only pays the real cost of the call and does not suffer from high implied volatility."

- The "volatility target" may be set to lower the cost of hedging, as the implied realized risk premia may be "removed".

The first trade was in April 2007 on HSBC with a June expiry.

- The implied volatility on the plain vanilla call was slightly above 15%, but the client sets a target volatility level of 12%, a little higher than the prevailing realized volatility level of around 10%.

- The premium of the timer call has 20% discount compared to the vanilla call counterpart.

- The realized volatility has been around 9.5% since the inception of the trade. The maturity of the timer call is 60% longer than the original vanilla call.

Bullish view of the market

Long a timer call, short a vanilla call. Usually, stock price and volatility are negatively correlated.

- The implied volatility in the market is too high currently, and subsequent realized volatility will be less than that implied in the market.

By setting the volatility target to be below the current implied volatility level (cost of timer call would be less than the comparative vanilla call).

If the stock shifts higher over the period, the tenor of the Timer Option would be longer; and a net credit would be received that captures the value due to the difference in Time Value of the two options.