1. Consider the class of power utility function

\[ U(x) = \frac{x^\gamma}{\gamma} \quad \text{for} \quad \gamma \leq 1. \]

This class includes the logarithm utility. (Hint: add \(-\frac{1}{\gamma}\) to \(U(x)\) and consider \(\gamma \to 0^+\)).

The log-optimal strategy has been shown to exhibit the property that the maximization of \(E[U(X_k)]\) with a fixed-proportions strategy only requires the maximization of the expected utility of single-period investment as given by \(E[U(X_1)]\). Check whether such property can be extended to the power utility function.

2. This exercise is related to the Dictionary Order. Consider the choice set

\[ B = \{(x, y) : x \in [0, \infty) \text{ and } y \in [0, \infty)\}. \]

Consider the following preference relation:

\[ (x_1, y_1) \in B \quad \text{and} \quad (x_2, y_2) \in B \]
\[ (x_1, y_1) \succeq (x_2, y_2) \quad \text{if and only if} \]
\[ x_1 > x_2 \quad \text{or} \quad x_1 = x_2 \text{ and } y_1 \geq y_2. \]

Show that \(\succeq\) satisfies the three axioms of Reflexivity, Comparability and Transitivity.

3. Recall the “Order Preserving” Axiom:

For any \(x, y \in B\), where \(x \succ y\) and \(\alpha, \beta \in [0, 1]\),

\[ [\alpha x + (1 - \alpha)y] > [\beta x + (1 - \beta)y] \quad \text{if and only if} \quad \alpha > \beta. \]

Show that the above Dictionary Order satisfies this Axiom.

4. It is known that the Dictionary Order does not satisfy the “Intermediate Value” Axiom. Show that the function

\[ U(x, y) = \ln(x + y) \]

cannot be an utility function representing the Dictionary Order.

\textit{Hint:} A utility function \(U : B \to R\) satisfies

(i) \(x \succ y\) if and only if \(U(x) > U(y)\).

(ii) \(x \sim y\) if and only if \(U(x) = U(y)\).

5. Consider the choices of a firm with 8 different input level \(\{\ell_1, \cdots, \ell_8\}\) and suppose that there are 3 states \(\{s_1, s_2, s_3\}\) which occur with equal probability. Assume that only 3 profit levels are possible \((\pi_A, \pi_B, \pi_C)\) which are ranked \(\pi_A < \pi_B < \pi_C\). The mapping from states and actions (input levels) to outcomes (profit levels) is given as follows:
Choosing between actions randomly induces further probability distribution over these outcomes.

(a) Suppose the choice between input levels $\ell_1$ and $\ell_2$ is made by tossing a fair coin, say, choosing $\ell_1$ if “head” comes up and $\ell_2$ if “tail” results, find the probability distribution over the profit levels $(\pi_A, \pi_B, \pi_C)$.

(b) Argue why one can obtain any probability distribution over the three outcomes by using an appropriate randomization over actions.

6. Suppose indifference curves $U(p_1, p_2, p_3) = k_i$ for levels $k_1 > k_2 > k_3$ are drawn in the $p_1-p_3$ plane as follows:

Here, $p_i = \text{Prob}\{s \in S|f(s,a) = c_i\}$. Explain how to deduce that $c_3$ is the most preferred outcome.