MATH685X – Mathematical Models in Financial Economics

Homework One

Course instructor: *Prof. Y.K. Kwok*

---

1. Consider the class of power utility function

$$U(x) = \frac{x^\gamma}{\gamma} \quad \text{for} \quad \gamma \leq 1.$$

This class includes the logarithm utility. (Hint: add $-\dfrac{1}{\gamma}$ to $U(x)$ and consider $\gamma \to 0^+$). The log-optimal strategy has been shown to exhibit the property that the maximization of $E[U(X_k)]$ with a fixed-proportions strategy only requires the maximization of the expected utility of single-period investment as given by $E[U(X_1)]$. Check whether such property can be extended to the power utility function.

2. This exercise is related to the *Dictionary Order*. Consider the choice set

$$B = \{(x, y) : x \in [0, \infty) \quad \text{and} \quad y \in [0, \infty)\}.$$

Consider the following preference relation:

$$\begin{array}{l} (x_1, y_1) \in B \quad \text{and} \quad (x_2, y_2) \in B \\ (x_1, y_1) \succeq (x_2, y_2) \text{ if and only if} \\ [x_1 > x_2] \quad \text{or} \quad [x_1 = x_2 \text{ and } y_1 \geq y_2]. \end{array}$$

Show that $\succeq$ satisfies the three axioms of Reflexivity, Comparability and Transitivity.

3. Recall the "Order Preserving" Axiom:

For any $x, y \in B$, where $x \succ y$ and $\alpha, \beta \in [0, 1]$,

$[\alpha x + (1 - \alpha)y] \succ [\beta x + (1 - \beta)y]$ if and only if $\alpha > \beta$. Show that the above Dictionary Order satisfies this Axiom.

4. It is known that the Dictionary Order does not satisfy the "Intermediate Value" Axiom. Show that the function

$$U(x, y) = \ln(x + y)$$

cannot be an utility function representing the Dictionary Order.

Hint: A utility function $U : B \to R$ satisfies

(i) $x \succ y$ if and only if $U(x) > U(y)$.

(ii) $x \sim y$ if and only if $U(x) = U(y)$.

5. Consider the choices of a firm with 8 different input level $\{\ell_1, \cdots, \ell_8\}$ and suppose that there are 3 states $\{s_1, s_2, s_3\}$ which occur with equal probability. Assume that only 3 profit levels are possible (π_A, π_B, π_C) which are ranked $\pi_A < \pi_B < \pi_C$. The mapping from states and actions (input levels) to outcomes (profit levels) is given as follows:

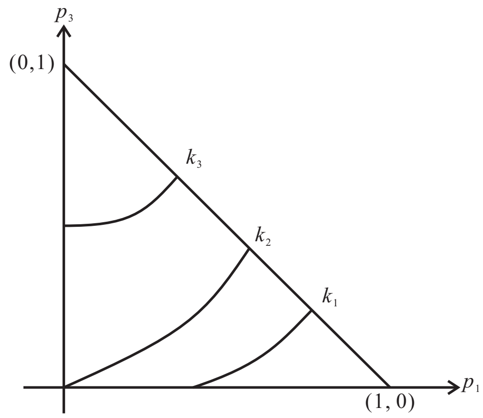
		Action							
		ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7	ℓ_8
	s_1	π_A	π_A	π_A	π_A	π_A	π_A	π_B	π_C
States	s_2	π_B	π_A	π_C	π_B	π_A	π_A	π_B	π_C
	s_3	π_C	π_C	π_C	π_B	π_B	π_A	π_B	π_C

Choosing between actions randomly induces further probability distribution over these outcomes.

(a) Suppose the choice between input levels ℓ_1 and ℓ_2 is made by tossing a fair coin, say, choosing ℓ_1 if "head" comes up and ℓ_2 if "tail" results, find the probability distribution over the profit levels (π_A, π_B, π_C).

(b) Argue why one can obtain any probability distribution over the three outcomes by using an appropriate randomization over actions.

6. Suppose indifference curves $U(p_1, p_2, p_3) = k_i$ for levels $k_1 > k_2 > k_3$ are drawn in the p_1-p_3 plane as follows:



Here, $p_i = Prob[\{s \in S | f(s, a) = c_i\}]$. Explain how to deduce that c_3 is the most preferred outcome.