MATH685X – Mathematical Models in Financial Economics

Homework Two

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1. The HARA (for hyperbolic absolute risk aversion) class of utility functions is defined by

$$U(x) = \frac{1-\gamma}{\gamma}\left(\frac{ax}{1-\gamma} + b\right)^{\gamma}, \qquad b > 0.$$

The functions are defined for those values of x where the term in parentheses is nonnegative. Show how the parameters γ, a and b can be chosen to obtain the following special cases (or an equivalent form).

   (a) Linear or risk neutral: $U(x) = x$

   (b) Quadratic: $U(x) = x - \frac{1}{2}cx^2$

   (c) Exponential: $U(x) = e^{-ax}$ [Try $\gamma = -\infty$.]

   (d) Power: $U(x) = cx^{\gamma}$

   (e) Logarithmic: $U(x) = \ln x$ [Try $U(x) = (1-\gamma)^{1-\gamma}(x^{\gamma} - 1)/\gamma$.]
   Show that the Arrow-Pratt risk aversion coefficient is of the form $1/(cx + d)$.

2. There is a useful approximation to the certainty equivalent that is easy to derive. A second-order expansion near $\overline{x} = E[x]$ gives

$$U(x) \approx U(\overline{x}) + U'(\overline{x})(x - \overline{x}) + \frac{1}{2}U''(\overline{x})(x - \overline{x})^2.$$

Hence,

$$E[U(x)] \approx U(\overline{x}) + \frac{1}{2}U''(\overline{x}) \text{ var}(x).$$

On the other hand, if we let c denote the certainty equivalent and assume that it is close to x, we can use the first-order expansion

$$U(c) \approx U(\overline{x}) + U'(\overline{x})(c - \overline{x}).$$

Using these approximations, show that

$$c \approx \overline{x} + \frac{U''(\overline{x})}{2U'(\overline{x})} \text{ var}(x).$$

3. The f-average of n positive numbers: $a_1, a_2, \cdots, a_n$, is defined by

$$M_f = f^{-1}\left(\frac{f(a_1) + f(a_2) + \cdots + f(a_n)}{n}\right).$$

   (a) Show that if we take f to be the natural logarithm function then the corresponding f-average is the geometric mean.

(b) Let f and g be twice-differentiable strictly increasing positive-valued function defined on $(0, \infty)$. For x and $y \in (0, \infty)$ and $p \in [0, 1]$, show that

$$f^{-1}(pf(x) + (1-p)f(y)) \leq g^{-1}(pg(x) + (1-p)g(y))$$

$\Leftrightarrow$

$$-\frac{g''(x)}{g'(x)} \leq -\frac{f''(x)}{f'(x)}.$$

Hint: Define $h = gof^{-1}$, show that

$$h''(x) = \frac{g''(f^{-1}(x))[f'(f^{-1}(x))]^{-1}f'(f^{-1}(x)) - g'(f^{-1}(x))f''(f^{-1}(x))[f'(f^{-1}(x))]^{-1}}{[f'(f^{-1}(x))]^2}.$$

4. Given two twice-differentiable, increasing and strictly concave utility functions $U_1(w)$ and $U_2(w)$, show that the following statements are equivalent:

   (i) $R_1^A(w) \geq R_2^A(w)$ for all $w \in R_+$, where $R_i^A(w)$ is the absolute risk aversion coefficient of $U_i(w), i = 1, 2$.

   (ii) There exists an increasing and concave function $g(\cdot)$ such that

   $$U_1(w) = g(U_2(w)) \quad \text{for all } w \in \mathbb{R}_+.$$

   (iii) $U_1(w)$ is more risk averse that $U_2(w)$, that is,

   $$\pi_1(w + \widetilde{\epsilon}) \geq \pi_2(w + \widetilde{\epsilon})$$

   for all $w \in \mathbb{R}_+$ and for any random variable $\widetilde{\epsilon}$ such that $E[\widetilde{\epsilon}] = 0$. Here, $\pi_i(\widetilde{w})$ is the risk premium of the gamble $\widetilde{w}$ under $U_i(\cdot), i = 1, 2$.

5. Consider the following utility function

   $$U(w) = \begin{cases} a_+(w - \overline{w}), & w \geq \overline{w} \\ a_-(w - \overline{w}), & w < \overline{w} \end{cases},$$

   where $a_- > a_+ > 0$. The function U is seen to be non-differentiable at $w = \overline{w}$. Suppose the wealth level happens to be $\overline{w}$, and consider the fair Bernuolli gamble where the gain and loss are both δ. Show that the risk premium is

   $$\pi = \frac{1}{2}(a_- - a_+)\delta.$$

   The variance of the random return of the gamble is δ^2. Recall that when U is twice differentiable, $\pi \approx R_A(w)\delta^2$, where $R_A(w)$ is the absolute risk aversion coefficient. Given your comments on the above observations.

6. Consider the following investments:

A		B		C	
probability	return (%)	probability	return (%)	probability	return (%)
0.4	3	0.1	5	0.1	5
0.3	4	0.2	6	0.1	7
0.1	6	0.1	8	0.2	8
0.1	7	0.2	9	0.2	9
0.1	9	0.4	10	0.4	11

(a) What can be said about the desirability of the investments using first-order and second-order stochastic dominance?

(b) Using geometric mean return as a criterion, which investment is preferred?

7. Assume that the utility function $u(x)$ satisfies (i) $u'(x) > 0$, (ii) $u''(x) < 0$ and $u'''(x) > 0$. The distribution F dominates G by the third order stochastic dominance if and only if

$$\int_C u(x)\, dF(x) \geq \int_C u(x)\, dG(x)$$

where C is the set of all possible outcomes. Show that $F(x)$ dominates G by the third order dominance if

(i) $\int_a^x \int_a^t [F(y) - G(y)]\, dydt \leq 0$ for all x and the strict inequality holds for some value, where t lies between a and b, and

(ii) $\int_a^b F(t)\, dt \leq \int_a^b G(t)\, dt.$

Consider the integration by parts of

$$\int_a^b u''(x) \int_a^x [F(y) - G(y)]\, dydx.$$