MATH685X – Mathematical Models in Financial Economics

Homework Three

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1. Suppose we use the following quadratic concave function

$$u(x) = -(\alpha - x)^2, \quad x < \alpha.$$

Show that the demand function of the optimal portfolio of one riskfree asset and one risky asset endowed with an initial wealth W_0 is given by

$$a^* = \frac{(\alpha - W_0 r_f)[E(\widetilde{r}) - r_f]}{E[(\widetilde{r} - r_f)^2] + \{E[\widetilde{r}] - r_f\}^2}.$$

Assuming a positive risk premium of the risky asset, show that a^* is a decreasing function of W_0 and $E[(\widetilde{r} - r_f)^2]$. Give the economic intuition of the above results.

2. Let a^* denote the number of units demanded for the risky asset in the optimal portfolio of one risky asset and the riskfree asset. Let W_0 be the initial wealth. Write R_f and r_f as the return and rate of return of the riskfree asset, respectively, where $R_f = 1 + r_f$. Let $\widetilde{x}$ denote the random risk premium of the risky asset defined as $\widetilde{r} - r_f$. In this problem, we assume $\widetilde{x}$ to be fixed and examine the impact of changing r_f on a^*. Show that

$$\left.\frac{da^*}{dr_f}\right|_{\widetilde{x}} = \frac{E[R_A(W_0R_f + a^*\widetilde{x})u'(W_0R_f + a^*\widetilde{x})\widetilde{x}]}{E[u''(W_0R_f + a^*\widetilde{x})\widetilde{x}^2]}.$$

Give the financial interpretation of the above results.

3. This problem examines the sensitivity of the demand function a^* with respect to the expected return of the risky asset. We write the new random return of the risky asset as

$$\widetilde{R} = R_f + \mu + \widetilde{x},$$

where $\widetilde{x}$ is the original random risk premium and μ is the increment in the expected return. Show that

$$\left.\frac{da^*}{d\mu}\right|_{\mu=0} = \frac{E[R_A(W_0R_f + a^*\widetilde{x})u'(W_0R_f + a^*\widetilde{x})a^*\widetilde{x} - u'(W_0R_f + a^*\widetilde{x})]}{E[u''(W_0R_f + a^*\widetilde{x})\widetilde{x}^2]}.$$

Argue why the demand for the risky asset increases when the expected rate of return of the risky asset increases and the absolute risk aversion is decreasing with wealth.

4. Suppose there exist n risky assets, $n > 1$, and short selling is allowed. Show that the necessary and sufficient condition for an investor to choose holding long position of at least one risky asset is that the risk premium is positive for at least one risky asset.

5. Given N risky assets with random rates of return $\widetilde{r}_j, j = 1, 2, \cdots, N$; and the riskfree asset with the riskfree rate of return r_f. Let u be a strictly increasing and concave utility function. Let $\widetilde{r}^*$ denote the optimal portfolio rate of return, where $\widetilde{W}^* = r^*W_0, W_0$ is the initial wealth and $\widetilde{W}^*$ is the random terminal wealth of the optimal portfolio.

(a) Show that

$$-\{E[\widetilde{r}_j]-r_f\}=\frac{\operatorname{cov}(u'(W_0\widetilde{r}^*),\widetilde{r}_j)}{E[u'(W_0\widetilde{r}^*)]},\quad j=1,2,\cdots,N.$$

(b) Assuming small risks for all risky assets, show that

$$E[\widetilde{r}_j]-r_f\approx-\frac{u''(E[\widetilde{W}^*])}{u'(E[\widetilde{W}^*])}\operatorname{cov}(\widetilde{r}_j,\widetilde{W}^*),\quad j=1,2,\cdots,N.$$

6. In a betting game with m possible outcomes, the return from a unit bet on i if outcome j occurs is given by

$$r_{ij}=\begin{cases} d_i & \text{if } j=i \\ -1 & \text{if } j\neq i \end{cases},\quad j=1,2,\cdots,m,$$

where $d_i>0$, for all i. Assuming

$$\sum_{i=1}^{m}\frac{1}{1+d_i}<1,$$

show that the betting strategy

$$\alpha_i=\frac{\frac{1}{1+d_i}}{1-\sum_{i=1}^{m}\frac{1}{1+d_i}},\quad i=1,2,\cdots,m,$$

always yields a gain of exactly 1.

7. The random return vector of three securities achieves the following values: $(4\ \ 2\ \ 3)$ and $(2\ \ 4\ \ 3)$ with equal probabilities. Show that the optimal strategy based on the logarithm utility criterion is not unique. Find two such optimal strategies.

8. Consider a wheel with n sectors. If the wheel pointer lands on sector i, the payoff obtained is r_i for every unit bet on that sector. The chance of landing on sector i is $p_i, i=1,2,\cdots,n$. Let α_i be the fraction of one's capital bet on sector i. We require $\sum_{i=1}^{n}\alpha_i\leq 1$ and $\alpha_i\geq 0, i=1,2,\cdots,n$.

(a) Assuming $\alpha_i>0, i=1,2,\cdots,n$, show that the solution to α_k based on logarithm utility criterion must satisfy

$$\frac{p_k r_k}{r_k\alpha_k+1-\sum_{i=1}^{n}\alpha_i}-\sum_{j=1}^{n}\frac{p_j}{r_j\alpha_j+1-\sum_{i=1}^{n}\alpha_i}=0,\quad k=1,2,\cdots,n.$$

(b) Assuming $\sum_{i=1}^{n}\frac{1}{r_i}=1$, show that a possible solution is $\alpha_i=p_i, i=1,2,\cdots,n$.

(c) Suppose the sectors are ordered in such a way that $p_n r_n<p_i r_i$ for all $i=1,2,\cdots,n-1$, that is, sector n is the worst sector. Find a solution with $\alpha_n=0$ and all other α_i's positive.