MATH685Z — Mathematical Models in Financial Economics

Topic 4 – Valuation of contingent claims

4.1 Single-period securities models

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4.3 Valuation of contingent claims and compute markets

4.1 Single-period securities models

- The initial prices of M risky securities, denoted by $S_1(0), \cdots, S_M(0)$, are positive scalars that are known at $t = 0$.

- Their values at $t = 1$ are random variables, which are defined with respect to a sample space $\Omega = \{\omega_1, \omega_2, \cdots, \omega_K\}$ of K possible outcomes (or states of the world).

- At $t = 0$, the investors know the list of all possible outcomes, but which outcome does occur is revealed only at the end of the investment period $t = 1$.

- A probability measure P satisfying $P(\omega) > 0$, for all $\omega \in \Omega$, is defined on Ω.

- We use $\boldsymbol{S}$ to denote the price process $\{\boldsymbol{S}(t) : t = 0, 1\}$, where $\boldsymbol{S}(t)$ is the row vector $\boldsymbol{S}(t) = (S_1(t)\ S_2(t) \cdots S_M(t))$.

Example

3 risky assets with initial time-0 price vector

$$\boldsymbol{S}(0) = (S_1(0) \quad S_2(0) \quad S_3(0)) = (1 \quad 2 \quad 3).$$

At time 1, there are 2 possible states of the world:

$$\begin{aligned} \omega_1 &= \text{Hang Seng index is at or above } 22,000 \\ \omega_2 &= \text{Hang Seng index falls below } 22,000. \end{aligned}$$

If ω_1 occurs, then

$$\boldsymbol{S}(1;\omega_1) = (1.2 \quad 2.1 \quad 3.4);$$

otherwise, ω_2 occurs and

$$\boldsymbol{S}(1;\omega_2) = (0.8 \quad 1.9 \quad 2.9).$$

- The possible values of the asset price process at $t = 1$ are listed in the following $K \times M$ matrix

$$S(1;\Omega) = \begin{pmatrix} S_1(1;\omega_1) & S_2(1;\omega_1) & \cdots & S_M(1;\omega_1) \\ S_1(1;\omega_2) & S_2(1;\omega_2) & \cdots & S_M(1;\omega_2) \\ \cdots & \cdots & \cdots & \cdots \\ S_1(1;\omega_K) & S_2(1;\omega_K) & \cdots & S_M(1;\omega_K) \end{pmatrix}.$$

- Since the assets are limited liability securities, the entries in $S(1;\Omega)$ are non-negative scalars.

- Existence of a strictly positive riskless security or bank account, whose value is denoted by S_0. Without loss of generality, we take $S_0(0) = 1$ and the value at time 1 to be $S_0(1) = 1 + r$, where $r \geq 0$ is the deterministic interest rate over one period.

- We define the discounted price process by

$$S^*(t) = S(t)/S_0(t), \quad t = 0, 1,$$

that is, we use the riskless security as the *numeraire* or *accounting unit*.

- The payoff matrix of the discounted price processes of the M risky assets and the riskless security can be expressed in the form

$$\widehat{S}^*(1;\Omega) = \begin{pmatrix} 1 & S_1^*(1;\omega_1) & \cdots & S_M^*(1;\omega_1) \\ 1 & S_1^*(1;\omega_2) & \cdots & S_M^*(1;\omega_2) \\ \cdots & \cdots & \cdots & \cdots \\ 1 & S_1^*(1;\omega_K) & \cdots & S_M^*(1;\omega_K) \end{pmatrix}.$$

Trading strategies

- An investor adopts a *trading strategy* by selecting a portfolio of the M assets at time 0.

- The number of units of asset m held in the portfolio from $t = 0$ to $t = 1$ is denoted by $h_m, m = 0, 1, \cdots, M$.

- The scalars h_m can be positive (long holding), negative (short selling) or zero (no holding).

- An investor is endowed with an initial endowment V_0 at time 0 to set up the trading portfolio. How to choose the portfolio holding h_m of the assets such that the expected portfolio value at time 1 is maximized?

Portfolio value process

- Let $V = \{V_t : t = 0, 1\}$ denote the value process that represents the total value of the portfolio over time. It is seen that

$$V_t = h_0 S_0(t) + \sum_{m=1}^{M} h_m S_m(t), \quad t = 0, 1.$$

- Let G be the random variable that denotes the total gain generated by investing in the portfolio. We then have

$$G = h_0 r + \sum_{m=1}^{M} h_m \Delta S_m, \quad \Delta S_m = S_m(1) - S_m(0).$$

Account balancing

- If there is no withdrawal or addition of funds within the investment horizon (self-financing trading strategy), then

$$V_1 = V_0 + G.$$

- Suppose we use the bank account as the numeraire, and define the discounted value process by $V_t^* = V_t / S_0(t)$ and discounted gain by $G^* = V_1^* - V_0^*$, we then have

$$\begin{aligned} V_t^* &= h_0 + \sum_{m=1}^{M} h_m S_m^*(t), \quad t = 0, 1; \\ G^* &= V_1^* - V_0^* = \sum_{m=1}^{M} h_m \Delta S_m^*. \end{aligned}$$

  There is no contribution from the riskfree asset to the discounted gain.

Dominant trading strategies

A trading strategy is characterized by the asset holding in the portfolio. A trading strategy $\mathcal{H}$ is said to be *dominant* if there exists another trading strategy $\widehat{\mathcal{H}}$ such that

$$V_0 = \widehat{V}_0 \qquad \text{and} \quad V_1(\omega) > \widehat{V}_1(\omega) \quad \text{for all } \omega \in \Omega.$$

- Suppose $\mathcal{H}$ dominates $\widehat{\mathcal{H}}$, we define a new trading strategy $\widetilde{\mathcal{H}} = \mathcal{H} - \widehat{\mathcal{H}}$. Let $\widetilde{V}_0$ and $\widetilde{V}_1$ denote the portfolio value of $\widetilde{\mathcal{H}}$ at $t = 0$ and $t = 1$, respectively. We then have $\widetilde{V}_0 = 0$ and $\widetilde{V}_1(\omega) > 0$ for all $\omega \in \Omega$.

- This trading strategy is dominant since it dominates the strategy which starts with zero value and does no investment at all.

Asset span

- Consider the two risky securities whose discounted payoff vectors are

$$\boldsymbol{S}_1^*(1) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \boldsymbol{S}_2^*(1) = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}.$$

- The payoff vectors are used to form the payoff matrix

$$S^*(1) = \begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 3 & 2 \end{pmatrix}.$$

- Let the current discounted prices be represented by the row vector $\boldsymbol{S}^*(0) = (1 \quad 2)$.

- We write $\boldsymbol{h}$ as the column vector whose entries are the weights of the securities in the portfolio. The trading strategy is characterized by specifying $\boldsymbol{h}$. The current portfolio value and the discounted portfolio payoff are given by $\boldsymbol{S}^*(0)\boldsymbol{h}$ and $S^*(1)\boldsymbol{h}$, respectively.

- The set of all portfolio payoffs via different holding of securities is called the *asset span* $\mathcal{S}$. The asset span is seen to be the column space of the payoff matrix $S^*(1)$, which is the subspace in $\mathbb{R}^K$ spanned by the columns of $S^*(1)$. Here, K is the number of possible states in the sample space Ω.

$$
\begin{aligned}
\text{asset span} &= \text{column space of } S^*(1) \\
&= \text{span}(\boldsymbol{S}_1^*(1) \cdots \boldsymbol{S}_M^*(1))
\end{aligned}
$$

Recall that

$$
\begin{aligned}
\text{column rank} &= \text{dimension of column space} \\
&= \text{number of independent columns.}
\end{aligned}
$$

It is well known that number of independent columns = number of independent rows, so column rank = row rank = rank $\leq \min(K, M)$.

- In the above numerical example, the asset span consists of all vectors of the form $h_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + h_2 \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$, where h_1 and h_2 are scalars.

Redundant security and complete model

- If the discounted terminal payoff vector of an added security lies inside $\mathcal{S}$, then its payoff can be expressed as a linear combination of $\boldsymbol{S}_1^*(1)$ and $\boldsymbol{S}_2^*(1)$. In this case, it is said to be a *redundant security*. The added security is said to be replicable by some combination of existing securities.

- A securities model is said to be *complete* if every payoff vector lies inside the asset span. That is, all new securities can be replicated by existing securities. This occurs if and only if the dimension of the asset span equals the number of possible states, that is, the asset span becomes the whole $\mathbb{R}^K$.

Given the securities model with 4 risky securities and 3 possible states of world:

$$S^*(1;\Omega) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 5 & 7 \\ 3 & 5 & 8 & 11 \end{pmatrix}, \quad \boldsymbol{S}^*(0) = (1 \quad 2 \quad 4 \quad 7).$$

asset span = span$(\boldsymbol{S}_1^*(1), \boldsymbol{S}_2^*(1))$, which has dimension $= 2 < 3 =$ number of possible states. Hence, the securities model is not complete! For example

$$\boldsymbol{S}_\beta^*(1;\Omega) = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

does not lie in the asset span of the securities model. There is no solution to

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 5 & 7 \\ 3 & 5 & 8 & 11 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}.$$

Pricing problem

Given a new security that is replicable by existing securities, its price with reference to a given securities model is given by the cost of setting up the replicating portfolio.

Consider a new security with discounted payoff at $t = 1$ as given by

$$\boldsymbol{S}^*_\alpha(1;\Omega) = \begin{pmatrix} 5 \\ 8 \\ 13 \end{pmatrix},$$

which is seen to be

$$\boldsymbol{S}^*_\alpha(1;\Omega) = \boldsymbol{S}^*_2(1;\Omega) + \boldsymbol{S}^*_3(1;\Omega) = \boldsymbol{S}^*_1(1;\Omega) + 2\boldsymbol{S}^*_2(1;\Omega).$$

This new security is redundant. Unfortunately, the price of this security can be either

$$\boldsymbol{S}^*_2(0) + \boldsymbol{S}^*_3(0) = 6 \quad \text{or} \quad \boldsymbol{S}^*_1(0) + 2\boldsymbol{S}^*_2(0) = 5.$$

There are two possible prices, so the law of one price does NOT hold.

Question

How to modify $\boldsymbol{S}^*(0)$ such that the law of one price hold?

Note that $S_3^*(1;\Omega) = S_1^*(1;\Omega) + S_2^*(1;\Omega)$ and $S_4^*(1;\Omega) = S_1^*(1;\Omega) + S_3^*(1;\Omega)$, both the third and fourth security are redundant securities. To achieve the law of one price, we modify $S_3^*(0)$ and $S_4^*(0)$ such that

$$S_3^*(0) = S_1^*(0) + S_2^*(0) = 3 \quad \text{and} \quad S_4^*(0) = 2S_1^*(0) + S_2^*(0) = 4.$$

Conjecture

If there are no redundant securities, then the law of one price holds. Mathematically, non-existence of redundant securities means $S^*(1;\Omega)$ has full column rank. That is, column rank = number of columns. This gives a sufficient condition for "law of one price".

Example

A gambler pays a bet of \$10 with the Jockey Club. The payoff of the bet is

$$\begin{cases} \$25 & \text{if Horse A wins in the first race} \\ \$40 & \text{if Horse B wins in the second race} \\ \$160 & \text{if both horses win in their respective race} \\ 0 & \text{if none of the above} \end{cases}.$$

This can be visualized as a security with $S(0) = 10$ and $\boldsymbol{S}(1) = (25 \quad 40 \quad 160)^T$.

Suppose the gambler places the same betting game with an illegal market maker (外圍馬坐莊者), the initial betting amount required is \$9.5 only (representing 5% discount). This is a violation of the law of one price.

How to take arbitrage? Buy the bet at \$9.5 and sell the bet at \$10? Can that be done? Why does 5% discount exist? What are the risks faced by the gambler when he deals with the illegal market maker.

Law of one price (pricing of securities that lie in the asset span)

1. The law of one price states that all portfolios with the same payoff have the same price.

2. Consider two portfolios with different portfolio weights $\boldsymbol{h}$ and $\boldsymbol{h}'$. Suppose these two portfolios have the same discounted payoff, that is, $S^*(1)\boldsymbol{h} = S^*(1)\boldsymbol{h}'$, then the law of one price infers that $\boldsymbol{S}^*(0)\boldsymbol{h} = \boldsymbol{S}^*(0)\boldsymbol{h}'$.

3. The trading strategy $\boldsymbol{h}$ is obtained by solving

$$S^*(1)\boldsymbol{h} = \boldsymbol{S}_\alpha^*(1).$$

Solution exists if $\boldsymbol{S}_\alpha^*(1)$ lies in the asset span. Uniqueness of solution is equivalent to null space of $S^*(1)$ having zero dimension. There is only one trading strategy that replicates the security with discounted terminal payoff $\boldsymbol{S}_\alpha^*(1)$. In this case, the law of one price always holds.

Law of one price and dominant trading strategy

If the law of one price fails, then it is possible to have two trading strategies $\boldsymbol{h}$ and $\boldsymbol{h}'$ such that $S^*(1)\boldsymbol{h} = S^*(1)\boldsymbol{h}'$ but $\boldsymbol{S}^*(0)\boldsymbol{h} > \boldsymbol{S}^*(0)\boldsymbol{h}'$.

Let $G^*(\omega)$ and $G^{*\prime}(\omega)$ denote the respective discounted gain corresponding to the trading strategies $\boldsymbol{h}$ and $\boldsymbol{h}'$. We then have $G^{*\prime}(\omega) > G^*(\omega)$ for all $\omega \in \Omega$, so there exists a dominant trading strategy. The corresponding dominant trading strategy is $\boldsymbol{h}' - \boldsymbol{h}$ so that $V_0 < 0$ but $V_1^*(\omega) = 0$ for all $\omega \in \Omega$.

Hence, the non-existence of dominant trading strategy implies the law of one price. However, the converse statement does not hold.

[See later numerical example.]

Pricing functional

- Given a discounted portfolio payoff $\boldsymbol{x}$ that lies inside the asset span, the payoff can be generated by some linear combination of the securities in the securities model. We have $\boldsymbol{x} = S^*(1)\boldsymbol{h}$ for some $\boldsymbol{h} \in \mathbb{R}^M$. Existence of the solution $\boldsymbol{h}$ is guaranteed since $\boldsymbol{x}$ lies in the asset span, or equivalently, $\boldsymbol{x}$ lies in the column space of $S^*(1)$.

- The current value of the portfolio is $\boldsymbol{S}^*(0)\boldsymbol{h}$, where $\boldsymbol{S}^*(0)$ is the initial price vector.

- We may consider $\boldsymbol{S}^*(0)\boldsymbol{h}$ as a pricing functional $F(\boldsymbol{x})$ on the payoff $\boldsymbol{x}$. If the law of one price holds, then the pricing functional is single-valued. Furthermore, it is a linear functional, that is,

$$F(\alpha_1\boldsymbol{x}_1 + \alpha_2\boldsymbol{x}_2) = \alpha_1 F(\boldsymbol{x}_1) + \alpha_2 F(\boldsymbol{x}_2)$$

  for any scalars α_1 and α_2 and payoffs $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$.

Arrow security and state price

- Let $\boldsymbol{e}_k$ denote the k^{th} coordinate vector in the vector space $\mathbb{R}^K$, where $\boldsymbol{e}_k$ assumes the value 1 in the k^{th} entry and zero in all other entries. The vector $\boldsymbol{e}_k$ can be considered as the discounted payoff vector of a security, and it is called the Arrow security of state k. This Arrow security has unit payoff when state k occurs and zero payoff otherwise.

- Suppose the securities model is complete (all Arrow securities lie in the asset span) and the law of one price holds, then the pricing functional F assigns unique value to each Arrow security. We write $s_k = F(\boldsymbol{e}_k)$, which is called the state price of state k. Note that state price must be non-negative. Take

$$\boldsymbol{S}_\alpha^*(1) = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_K \end{pmatrix} = \sum_{k=1}^K \alpha_k \boldsymbol{e}_k, \text{ then}$$

$$S_\alpha^*(0) = F(\boldsymbol{S}_\alpha^*(1)) = F\left(\sum_{k=1}^K \alpha_k \boldsymbol{e}_k\right) = \sum_{k=1}^K \alpha_k F(\boldsymbol{e}_k) = \sum_{k=1}^K \alpha_k s_k.$$

Summary

Given a securities model endowed with $S^*(1;\Omega)$ and $\boldsymbol{S}^*(0)$, can we find a trading strategy to form a portfolio that replicates a new security $\boldsymbol{S}^*_\alpha(1;\Omega)$ (also called a contingent claim) that is outside the universe of the M available risky securities in the securities model?

Replication means the terminal payoff of the replicating portfolio matches with that of the contingent claim under all scenarios of occurrence of the state of the world at $t = 1$.

1. Formation of the replicating portfolio is possible if we have *existence of solution* $\boldsymbol{h}$ to the following system

$$S^*(1;\Omega)\boldsymbol{h} = \boldsymbol{S}^*_\alpha(1;\Omega).$$

This is equivalent to the fact that "$\boldsymbol{S}^*_\alpha(1;\Omega)$ lies in the asset span (column space) of $S^*(1;\Omega)$". The solution $\boldsymbol{h}$ is the corresponding trading strategy. Note that $\boldsymbol{h}$ may not be unique.

Completeness of securities model

If all contingent claims are replicable, then the securities model is said to be *complete*. This is equivalent to

$$\text{dim(asset span)} = K = \text{number of possible states},$$

that is, asset span $= \mathbb{R}^K$. In this case, solution $\boldsymbol{h}$ always exists.

2. Uniqueness of trading strategy

If $\boldsymbol{h}$ is unique, then there is only one trading strategy that generates the replicating portfolio. This occurs when the columns of $S^*(1;\Omega)$ are independent. Equivalently, column rank $= M$ and all securities are non-redundant. Mathematically, this is equivalent to observe that the homogeneous system

$$S^*(1;\Omega)\boldsymbol{h} = \boldsymbol{0}$$

admits only the trivial zero solution. In other words, the dimension of the null space of $S^*(1;\Omega)$ is zero.

When we have unique solution $\boldsymbol{h}$, the initial cost of setting up the replicating portfolio (price at time 0) as given by $\boldsymbol{S}^*(0)\boldsymbol{h}$ is unique. In this case, law of one price holds.

Matrix properties of $S^*(1)$ that are related to financial economics concepts

The securities model is endowed with

(i) discounted terminal payoff matrix $= \left(\boldsymbol{S}_1^*(1) \quad \cdots \quad \boldsymbol{S}_M^*(1) \right)$, and
(ii) initial price vector; $\boldsymbol{S}^*(0) = (S_1(0) \cdots S_M(0))$.

Recall that

$$\text{column rank } \leq \min(K, M)$$

where $K =$ number of possible states, $M =$ number of risky securities.

List of terms: redundant securities, complete model, replicating portfolio, asset holding, asset span, law of one price, dominant trading strategy

Given a risky security with the discounted terminal payoff $S_\alpha^*(1)$, we are interested to explore the existence and uniqueness of solution to

$$S^*(1)\boldsymbol{h} = \boldsymbol{S}_\alpha^*(1).$$

Here, $\boldsymbol{h}$ is the asset holding of the portfolio that replicates $\boldsymbol{S}_\alpha^*(1)$.

(i) column rank $= K$

asset span $= \mathbb{R}^K$, so the securities model is complete. Any risky securities is replicable. In this case, solution $\boldsymbol{h}$ always exists.

(ii) column rank $= M$ (all columns of $S^*(1)$ are independent)

All securities are non-redundant. In this case, $\boldsymbol{h}$ may or may not exist. However, if $\boldsymbol{h}$ exists, then it must be unique. The price of any replicable security is unique.

(iii) column rank $< K$

Solution $\boldsymbol{h}$ exists if and only if $\boldsymbol{S}_\alpha^*(1)$ lies in the asset span. However, there is no guarantee for the uniqueness of solution.

(iv) column rank $< M$

Existence of redundant securities, so the law of one price may fail.

To explore "law of one price" and "existence of dominant trading strategies", one has to consider the nature of the solution to the linear system of equations

$$\boldsymbol{S}^*(0) = \boldsymbol{x}S^*(1).$$

Linear pricing measure

We consider securities models with the inclusion of the riskfree security. A non-negative row vector $\boldsymbol{q} = (q(\omega_1) \cdots q(\omega_K))$ is said to be a linear pricing measure if for every trading strategy we have

$$V_0^* = \sum_{k=1}^{K} q(\omega_k) V_1^*(\omega_k).$$

Note that $\boldsymbol{q}$ may not be unique, but the same initial price V_0^* is always resulted as there is no dependence of V_0^* on the asset holding of the portfolio. Implicitly, this implies that the law of one price holds.

1. Suppose we take the holding amount of every risky security to be zero, thereby $h_1 = h_2 = \cdots = h_M = 0$, then

$$V_0^* = h_0 = \sum_{k=1}^{K} q(\omega_k) h_0$$

so that

$$\sum_{k=1}^{K} q(\omega_k) = 1.$$

2. By taking the security to be the k^{th} Arrow security, we obtain

$$s_k = q(\omega_k), \quad k = 1, 2, \cdots, K.$$

That is, the state price of the k^{th} state is simply $q(\omega_k)$. This result is valid given that the securities model includes the riskfree security.

- Since we have taken $q(\omega_k) \geq 0, k = 1, \cdots, K$, and their sum is one, we may interpret $q(\omega_k)$ as a probability measure on the sample space Ω.

- Note that $q(\omega_k)$ is not related to the actual probability of occurrence of the state k, though the current discounted security price is given by the expectation of the security payoff one period later under the linear pricing measure.

- By taking the portfolio weights to be zero except for the m^{th} security, we have

$$S_m^*(0) = \sum_{k=1}^{K} q(\omega_k) S_m^*(1; \omega_k), \quad m = 1, \cdots, M.$$

  In matrix form,

$$\widehat{\boldsymbol{S}}^*(0) = \boldsymbol{q} \widehat{S}^*(1; \Omega), \quad \boldsymbol{q} \geq \mathbf{0}.$$

Numerical example

Consider a securities model with 2 risky securities and the riskfree security, and there are 3 possible states. The current discounted price vector $\widehat{\boldsymbol{S}}^*(0)$ is $(1 \quad 4 \quad 2)$ and the discounted payoff matrix at $t = 1$ is $\widehat{S}^*(1) = \begin{pmatrix} 1 & 4 & 3 \\ 1 & 3 & 2 \\ 1 & 2 & 4 \end{pmatrix}$. Here, the law of one price holds since the only solution to $\widehat{S}^*(1)\boldsymbol{h} = \boldsymbol{0}$ is $\boldsymbol{h} = \boldsymbol{0}$. This is because the columns of $\widehat{S}^*(1)$ are independent so that the dimension of the nullspace of $\widehat{S}^*(1)$ is zero.

The linear pricing probabilities $q(\omega_1), q(\omega_2)$ and $q(\omega_3)$, if exist, should satisfy the following equations:

$$\begin{aligned} 1 &= q(\omega_1) + q(\omega_2) + q(\omega_3) \\ 4 &= 4q(\omega_1) + 3q(\omega_2) + 2q(\omega_3) \\ 2 &= 3q(\omega_1) + 2q(\omega_2) + 4q(\omega_3). \end{aligned}$$

Solving the above equations, we obtain $q(\omega_1) = q(\omega_2) = 2/3$ and $q(\omega_3) = -1/3$.

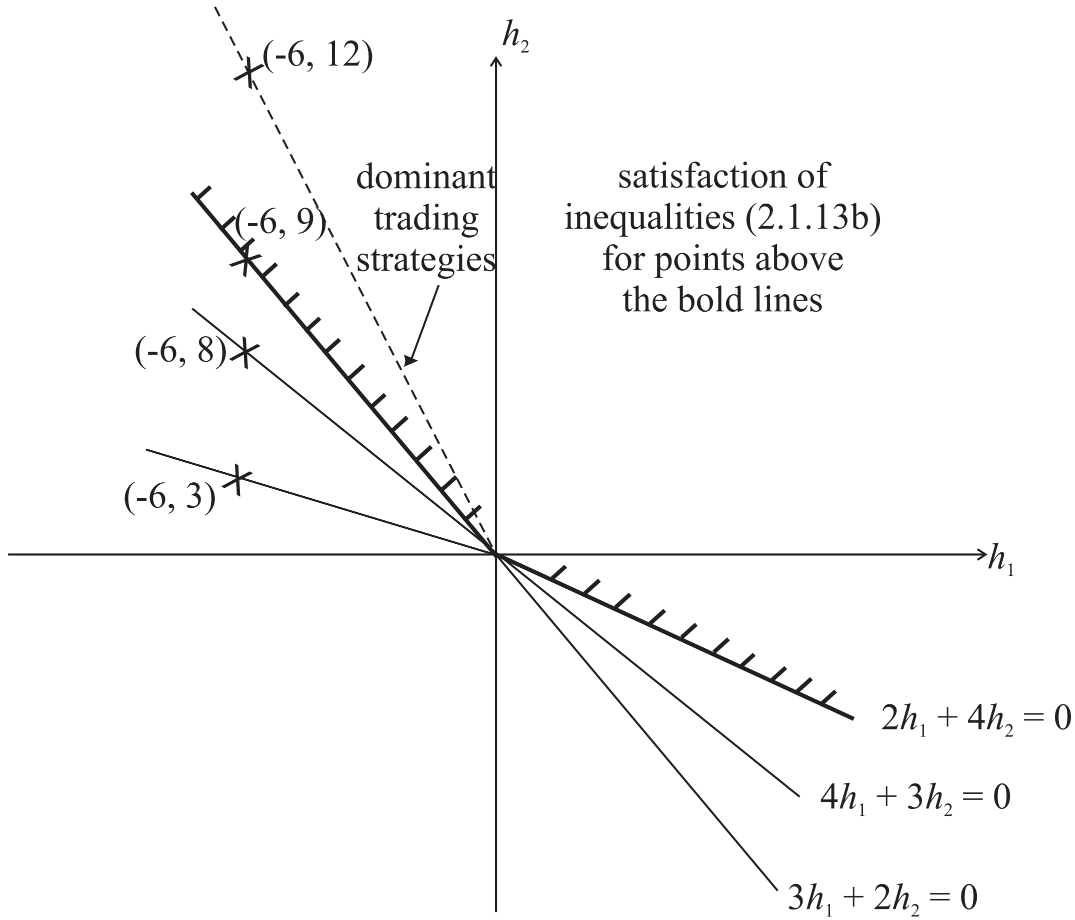
- Since not all the pricing probabilities are non-negative, the linear pricing measure does not exist for this securities model.

Existence of dominant trading strategies

- Can we find a trading strategy $(h_1 \;\; h_2)$ such that $V_0^* = 4h_1 + 2h_2 = 0$ but $V_1^*(\omega_k) > 0, k = 1, 2, 3$? This is equivalent to ask whether there exist h_1 and h_2 such that $4h_1 + 2h_2 = 0$ and

$$\begin{aligned} 4h_1 + 3h_2 &> 0 \\ 3h_1 + 2h_2 &> 0 \\ 2h_1 + 4h_2 &> 0. \end{aligned} \qquad (A)$$

- The region is found to be lying on the top right sides above the two bold lines: (i) $3h_1 + 2h_2 = 0, h_1 < 0$ and (ii) $2h_1 + 4h_2 = 0, h_1 > 0$. It is seen that all the points on the dotted half line: $4h_1 + 2h_2 = 0, h_1 < 0$ represent dominant trading strategies that start with zero wealth but end with positive wealth with certainty.



The region above the two bold lines represents trading strategies that satisfy inequalities (A). The trading strategies that lie on the dotted line: $4h_1 + 2h_2 = 0, h_1 < 0$ are dominant trading strategies.

- Suppose the initial discounted price vector is changed from $(4 \quad 2)$ to $(3 \quad 3)$, the new set of linear pricing probabilities will be determined by

$$\begin{aligned}
1 &= q(\omega_1) + q(\omega_2) + q(\omega_3) \\
3 &= 4q(\omega_1) + 3q(\omega_2) + 2q(\omega_3) \\
3 &= 3q(\omega_1) + 2q(\omega_2) + 4q(\omega_3),
\end{aligned}$$

  which is seen to have the solution: $q(\omega_1) = q(\omega_2) = q(\omega_3) = 1/3$. Now, all the pricing probabilities have non-negative values, the row vector $\boldsymbol{q} = (1/3 \quad 1/3 \quad 1/3)$ represents a linear pricing measure.

- The line $3h_1 + 3h_2 = 0$ always lies outside the region above the two bold lines.

- We cannot find $(h_1 \quad h_2)$ such that $3h_1 + 3h_2 = 0$ together with h_1 and h_2 satisfying all these inequalities.

Theorem

There exists a linear pricing measure if and only if there are no dominant trading strategies.

The above linear pricing measure theorem can be seen to be a direct consequence of the Farkas Lemma.

Farkas Lemma

There does not exist $\boldsymbol{h} \in \mathbb{R}^M$ such that

$$\widehat{S}^*(1;\Omega)\boldsymbol{h} > \mathbf{0} \quad \text{and} \quad \widehat{\boldsymbol{S}}^*(0)\boldsymbol{h} = 0$$

if and only if there exists $\boldsymbol{q} \in \mathbb{R}^K$ such that

$$\widehat{\boldsymbol{S}}^*(0) = \boldsymbol{q}\widehat{S}^*(1;\Omega) \quad \text{and} \quad \boldsymbol{q} \geq \mathbf{0}.$$

4.2 Fundamental Theorem of Asset Pricing

- An *arbitrage opportunity* is some trading strategy that has the following properties: (i) $V_0^* = 0$, (ii) $V_1^*(\omega) \geq 0$ and $EV_1^*(\omega) > 0$, where E is the expectation under the actual probability measure P.

- The existence of a dominant strategy requires a portfolio with initial zero wealth to end up with a *strictly* positive wealth in all states.

- The existence of a dominant trading strategy implies the existence of an arbitrage opportunity, but the converse is not necessarily true.

Risk neutral probability measure

A probability measure Q on Ω is a risk neutral probability measure if it satisfies

(i) $Q(\omega) > 0$ for all $\omega \in \Omega$, and

(ii) $E_Q[\Delta S_m^*] = 0, m = 1, \cdots, M$, where E_Q denotes the expectation under Q.

Note that $E_Q[\Delta S_m^*] = 0$ is equivalent to $S_m^*(0) = \sum_{k=1}^{K} Q(\omega_k) S_m^*(1; \omega_k)$.

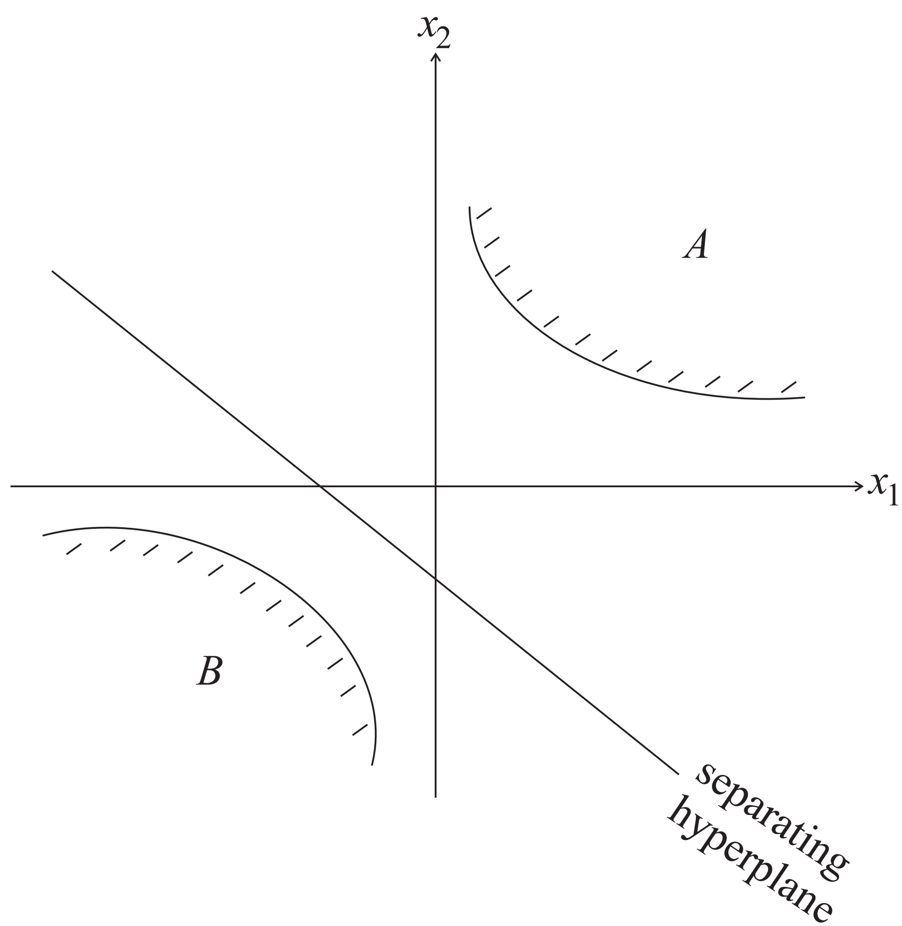
- In financial markets with no arbitrage opportunities, every investor should use such risk neutral probability measure (though not necessarily unique) to find the fair value of a portfolio, irrespective to the risk preference of the investor (independent of the assessment of the probabilities of occurrence of different states).

Fundamental Theorem of Asset Pricing

No arbitrage opportunities exist if and only if there exists a risk neutral probability measure Q.

- The proof of the Theorem requires the Separating Hyperplane Theorem.

- The Separating Hyperplane Theorem states that if A and B are two non-empty disjoint convex sets in a vector space V, then they can be separated by a hyperplane.



The hyperplane (represented by a line in $\mathbb{R}^2$) separates the two convex sets A and B in $\mathbb{R}^2$. A set C is convex if any convex combination $\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}, 0 \leq \lambda \leq 1$, of a pair of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in C also lies in C.

The hyperplane $[\boldsymbol{f}, \alpha]$ separates the sets A and B in $\mathbb{R}^n$ if there exists α such that $\boldsymbol{f} \cdot \boldsymbol{x} \geq \alpha$ for all $\boldsymbol{x} \in A$ and $\boldsymbol{f} \cdot \boldsymbol{y} < \alpha$ for all $\boldsymbol{y} \in B$.

For example, the hyperplane $\left[\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, 0\right]$ separates the two disjoint convex sets $A = \left\{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\right\}$

and $B = \left\{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 < 0, x_2 < 0, x_3 < 0\right\}$ in $\mathbb{R}^3$.

Proof of Theorem

"$\Leftarrow$ part".

Assume that a risk neutral probability measure Q exists, that is, $\widehat{\boldsymbol{S}}^*(0) = \boldsymbol{\pi}\widehat{S}^*(1;\Omega)$, where $\boldsymbol{\pi} = (Q(\omega_1) \cdots Q(\omega_K))$. Consider a trading strategy $\boldsymbol{h} = (h_0\ h_1 \quad \cdots \quad h_M)^T \in \mathbb{R}^{M+1}$ such that $\widehat{S}^*(1;\Omega)\boldsymbol{h} \geq 0$ in all $\omega \in \Omega$ and with strict inequality in at least one state.

Now consider $\widehat{\boldsymbol{S}}^*(0)\boldsymbol{h} = \boldsymbol{\pi}\widehat{S}^*(1;\Omega)\boldsymbol{h}$, it is seen that $\widehat{\boldsymbol{S}}^*(0)\boldsymbol{h} > 0$ since all entries in $\boldsymbol{\pi}$ are strictly positive and entries in $\widehat{S}^*(1;\Omega)\boldsymbol{h}$ are either zero or strictly positive. It is then impossible to have $\widehat{\boldsymbol{S}}(0)\boldsymbol{h} = 0$ and $S^*(1;\Omega)\boldsymbol{h} \geq 0$ in all $\omega \in \Omega$, with strict inequality in at least one state. Hence, no arbitrage opportunities exist.

"$\Rightarrow$ part".

First, we define the subset U in $\mathbb{R}^{K+1}$ which consists of vectors of the form $\begin{pmatrix} -\widehat{\boldsymbol{S}}^*(0)\boldsymbol{h} \\ \widehat{\boldsymbol{S}}^*(1;\omega_1)\boldsymbol{h} \\ \vdots \\ \widehat{\boldsymbol{S}}^*(1;\omega_K)\boldsymbol{h} \end{pmatrix}$, where $\widehat{\boldsymbol{S}}^*(1;\omega_k)$ is the k^{th} row in $\widehat{S}^*(1;\Omega)$ and $\boldsymbol{h} \in \mathbb{R}^{M+1}$ represents a trading strategy. This subset is seen to be a subspace. The convexity property of U is obvious.

Consider another subset $\mathbb{R}_+^{K+1}$ defined by

$$\mathbb{R}_+^{K+1} = \{\boldsymbol{x} = (x_0\ x_1 \cdots x_K)^T \in \mathbb{R}^{K+1} : x_i \geq 0 \quad \text{for all} \quad 0 \leq i \leq K\},$$

which is a convex set in $\mathbb{R}^{K+1}$.

We claim that the non-existence of arbitrage opportunities implies that U and $\mathbb{R}_+^{K+1}$ can only have the zero vector in common.

Assume the contrary, suppose there exists a non-zero vector $\boldsymbol{x} \in U \cap \mathbb{R}_+^{K+1}$. Since there is a trading strategy vector $\boldsymbol{h}$ associated with every vector in U, it suffices to show that the trading strategy $\boldsymbol{h}$ associated with $\boldsymbol{x}$ always represents an arbitrage opportunity.

We consider the following two cases: $-\widehat{\boldsymbol{S}}^*(0)\boldsymbol{h} = 0$ or $-\widehat{\boldsymbol{S}}^*(0)\boldsymbol{h} > 0$.

(i) When $\widehat{\boldsymbol{S}}^*(0)\boldsymbol{h} = 0$, since $\boldsymbol{x} \neq \boldsymbol{0}$ and $\boldsymbol{x} \in R_+^{K+1}$, then the entries $\widehat{\boldsymbol{S}}(1;\omega_k)\boldsymbol{h}, k = 1, 2, \cdots K$, must be all greater than or equal to zero, with at least one strict inequality. In this case, $\boldsymbol{h}$ is seen to represent an arbitrage opportunity.

(ii) When $\widehat{\boldsymbol{S}}^*(0)\boldsymbol{h} < 0$, all the entries $\widehat{\boldsymbol{S}}(1;\omega_k)\boldsymbol{h}, k = 1, 2, \cdots, K$ must be all non-negative. Correspondingly, $\boldsymbol{h}$ represents a dominant trading strategy and in turns $\boldsymbol{h}$ is an arbitrage opportunity.

Since $U \cap R_{+}^{K+1} = \{\mathbf{0}\}$, by the Separating Hyperplane Theorem, there exists a hyperplane that separates the pair of disjoint convex sets: $\mathbb{R}_{+}^{K+1} \backslash \{\mathbf{0}\}$ and U. This hyperplane must go through the origin, so its equation is of the form $[f, 0]$. Let $\boldsymbol{f} \in \mathbb{R}^{K+1}$ be the normal to this hyperplane, then we have $\boldsymbol{f} \cdot \boldsymbol{x} > \boldsymbol{f} \cdot \boldsymbol{y}$, for all $\boldsymbol{x} \in \mathbb{R}_{+}^{K+1} \backslash \{\mathbf{0}\}$ and $\boldsymbol{y} \in U$.

[*Remark*: We may have $\boldsymbol{f} \cdot \boldsymbol{x} < \boldsymbol{f} \cdot \boldsymbol{y}$, depending on the orientation of the normal vector $\boldsymbol{f}$. However, the final conclusion remains unchanged.]

Since U is a linear subspace so that a negative multiple of $\boldsymbol{y} \in U$ also belongs to U. Note that $\boldsymbol{f} \cdot \boldsymbol{x} > \boldsymbol{f} \cdot \boldsymbol{y}$ and $\boldsymbol{f} \cdot \boldsymbol{x} > \boldsymbol{f} \cdot (-\boldsymbol{y})$ both holds only if $\boldsymbol{f} \cdot \boldsymbol{y} = 0$ for all $\boldsymbol{y} \in U$.

We have $\boldsymbol{f} \cdot \boldsymbol{x} > 0$ for all $\boldsymbol{x}$ in $\mathbb{R}_{+}^{K+1} \backslash \{\boldsymbol{0}\}$. This requires all entries in $\boldsymbol{f}$ to be strictly positive. Note that if at least one of the components (say, the i^{th} component) of $\boldsymbol{f}$ is zero or negative, then we choose $\boldsymbol{x}$ to be the i^{th} coordinate vector. This gives $\boldsymbol{f} \cdot \boldsymbol{x} \leq 0$, a violation of $\boldsymbol{f} \cdot \boldsymbol{x} > 0$.

From $\boldsymbol{f} \cdot \boldsymbol{y} = 0$, we have

$$-f_0 \widehat{\boldsymbol{S}}^*(0)\boldsymbol{h} + \sum_{k=1}^{K} f_k \widehat{\boldsymbol{S}}^*(1;\omega_k)\boldsymbol{h} = 0$$

for all $\boldsymbol{h} \in \mathbb{R}^{M+1}$, where $f_j, j = 0, 1, \cdots, K$ are the entries of $\boldsymbol{f}$. We then deduce that

$$\widehat{\boldsymbol{S}}^*(0) = \sum_{k=1}^{K} Q(\omega_k)\widehat{\boldsymbol{S}}^*(1;\omega_k), \text{ where } Q(\omega_k) = f_k/f_0.$$

Consider the first component in the vectors on both sides of the above equation. They both correspond to the current price and discounted payoff of the riskless security, and all are equal to one. We then obtain

$$1 = \sum_{k=1}^{K} Q(\omega_k).$$

We obtain the risk neutral probabilities $Q(\omega_k), k = 1, \cdots, K$, whose sum is equal to one and they are all strictly positive since $f_j > 0, j = 0, 1, \cdots, K$.

Remark

Corresponding to each risky asset,

$$S_m^*(0) = \sum_{k=1}^{K} Q(\omega_k) S_m^*(1; \omega_k), \quad m = 1, 2, \cdots, M.$$

Hence, the current price of any one of risky securities in the securities model is given by the expectation of the discounted payoff under the risk neutral measure Q.

Equivalent martingale measure

- The risk neutral probability measure Q is commonly called the equivalent martingale measure. "Equivalent" refers to the equivalence between the physical measure P and martingale measure Q [observing $P(\omega) > 0 \Leftrightarrow Q(\omega) > 0$ for all $\omega \in \Omega$]*. The linear pricing measure falls short of this equivalence property since $q(\omega)$ can be zero.

*P and Q may not agree on the assignment of probability values to individual events, but they always agree as to which events are possible or impossible.

- Martingale property is defined for adapted stochastic processes*. In the context of one-period model, given the information on the set of possible outcomes at $t = 0$,

$$S_m^*(0) = E_Q[S_m^*(1;\Omega)] = \sum_{k=1}^{K} S_m^*(1;\omega_k)Q(\omega_k). \tag{1}$$

The discounted security price $S_m^*(t)$ is said to be a martingale† under Q.

*A stochastic process is adapted to a filtration with respect to a measure. Say S_m^* is adapted to $\mathbb{F} = \{\mathcal{F}_t; t = 0, 1, \cdots, T\}$, then $S_m^*(t)$ is $\mathcal{F}_t$-measurable.

†Martingale property with respect to Q and $\mathbb{F}$:

$$S_m^*(t) = E_Q[S_m^*(s+t)|\mathcal{F}_t] \text{ for all } t \geq 0, s \geq 0.$$

Martingale property of discounted portfolio value (assuming the existence of Q)

- Let $V_1^*(\Omega)$ denote the discounted payoff of a replicating portfolio. Since $V_1^*(\Omega) = \widehat{S}^*(1;\Omega)\boldsymbol{h}$ for some trading strategy $\boldsymbol{h}$, by Eq. (1),

$$
\begin{aligned}
V_0^* &= (S_0^*(0) \cdots S_M^*(0))\boldsymbol{h} \\
&= (E_Q[S_0^*(1;\Omega)] \cdots E_Q[S_M^*(1;\Omega)])\boldsymbol{h} \\
&= \sum_{m=0}^{M} \left[\sum_{k=1}^{K} S_m^*(1;\omega_k) Q(\omega_k) \right] h_m \\
&= \sum_{k=1}^{K} \left[\sum_{m=0}^{M} S_m^*(1;\omega_k) Q(\omega_k) h_m \right] = E_Q[V_1^*(\Omega)].
\end{aligned}
$$

- The equivalent martingale measure Q is not necessarily unique. Since "absence of arbitrage opportunities " implies "law of one price", the expectation value $E_Q[V_1^*(\Omega)]$ is single-valued under all equivalent martingale measures.

Calculation of the risk neutral measures

Consider the earlier securities model with the riskfree security and only one risky security, where $\widehat{S}(1;\Omega) = \begin{pmatrix} 1 & 4 \\ 1 & 3 \\ 1 & 2 \end{pmatrix}$ and $\widehat{\boldsymbol{S}}(0) = (1 \quad 3)$. The risk neutral probability measure

$$\boldsymbol{\pi} = (Q(\omega_1) \quad Q(\omega_2) \quad Q(\omega_3)),$$

if exists, will be determined by the following system of equations

$$(Q(\omega_1) \quad Q(\omega_2) \quad Q(\omega_3)) \begin{pmatrix} 1 & 4 \\ 1 & 3 \\ 1 & 2 \end{pmatrix} = (1 \quad 3).$$

Since there are more unknowns than the number of equations, the solution is not unique. The solution is found to be $\boldsymbol{\pi} = (\lambda \quad 1 - 2\lambda \quad \lambda)$, where λ is a free parameter. In order that all risk neutral probabilities are all strictly positive, we must have $0 < \lambda < 1/2$.

Under market completeness, if the set of risk neutral measures is non-empty, then it must be a singleton.

Under market completeness, column rank of $\widehat{S}(1;\Omega)$ equals the number of states. Since column rank $=$ row rank, then all rows of $\widehat{S}(1;\Omega)$ are independent. If solution exists for

$$\boldsymbol{q}\widehat{S}^*(1;\Omega) = \boldsymbol{S}^*(0),$$

then it must be unique.

Suppose we add the second risky security with discounted payoff $\boldsymbol{S}_2^*(1) = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$ and current discounted value $S_2^*(0) = 3$. With this new addition, the securities model becomes complete.

With the new equation $3Q(\omega_1)+2Q(\omega_2)+4Q(\omega_3) = 3$ added to the system, this new securities model is seen to have the unique risk neutral measure $(1/3 \quad 1/3 \quad 1/3)$.

Subspace of discounted gains

Let W be a subspace in $\mathbb{R}^K$ which consists of discounted gains corresponding to some trading strategy $\boldsymbol{h}$. Note that W is spanned by the set of vectors representing discounted gains of the risky securities.

In the above securities model, the discounted gains of the first and second risky securities are $\begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$, respectively.

The discounted gain subspace is given by

$$W = \left\{ h_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + h_2 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix},\ \text{where } h_1 \text{ and } h_2 \text{ are scalars} \right\}.$$

For any risk neutral probability measure Q, we have

$$\begin{aligned} E_Q G^* &= \sum_{k=1}^{K} Q(\omega_k) \left[\sum_{m=1}^{M} h_m \Delta S_m^*(\omega_k) \right] \\ &= \sum_{m=1}^{M} h_m E_Q[\Delta S_m^*] = 0. \end{aligned}$$

For any $\boldsymbol{G}^* = (G(\omega_1) \cdots G(\omega_K))^T \in W$, we have

$$\boldsymbol{\pi}^T \boldsymbol{G}^* = 0, \text{ where } \boldsymbol{\pi} = (Q(\omega_1) \cdots Q(\omega_K))^T.$$

The risk neutral probability vector $\boldsymbol{\pi}$ must lie in the orthogonal complement $W^\perp$.

Characterization of the set of neutral measures

Since the sum of risk neutral probabilities must be one and all probability values must be positive, the risk neutral probability vector $\boldsymbol{\pi}$ must lie in the following subset

$$P^+ = \{\boldsymbol{y} \in \mathbb{R}^K : y_1 + y_2 + \cdots + y_K = 1 \quad \text{and} \quad y_k > 0, k = 1, \cdots K\}.$$

Let R denote the set of all risk neutral measures, then $R = P^+ \cap W^\perp$.

In the above numerical example, $W^\perp$ is the line through the origin in $\mathbb{R}^3$ which is perpendicular to $(1 \quad 0 \quad -1)$ and $(0 \quad -1 \quad 1)$. The line should assume the form $\lambda(1 \quad 1 \quad 1)$ for some scalar λ. We obtain the risk neutral probability vector $\boldsymbol{\pi} = (1/3 \quad 1/3 \quad 1/3)$.

4.3 Valuation of contingent claims and complete markets

- A contingent claim can be considered as a random variable Y that represents a terminal payoff whose value depends on the occurrence of a particular state ω_k, where $\omega_k \in \Omega$.

- Suppose the holder of the contingent claim is promised to receive the preset contingent payoff, how much should the writer of such contingent claim charge at $t = 0$ so that the price is *fair* to both parties.

- Consider the securities model with the riskfree security whose values at $t = 0$ and $t = 1$ are $S_0(0) = 1$ and $S_0(1) = 1.1$, respectively, and a risky security with $S_1(0) = 3$ and $\boldsymbol{S}_1(1) = \begin{pmatrix} 4.4 \\ 3.3 \\ 2.2 \end{pmatrix}$.

The set of $t = 1$ payoffs that can be generated by certain trading strategy is given by $h_0 \begin{pmatrix} 1.1 \\ 1.1 \\ 1.1 \end{pmatrix} + h_1 \begin{pmatrix} 4.4 \\ 3.3 \\ 2.2 \end{pmatrix}$ for some scalars h_0 and h_1.

For example, the contingent claim $\begin{pmatrix} 5.5 \\ 4.4 \\ 3.3 \end{pmatrix}$ can be generated by the trading strategy: $h_0 = 1$ and $h_1 = 1$, while the other contingent claim $\begin{pmatrix} 5.5 \\ 4.0 \\ 3.3 \end{pmatrix}$ cannot be generated by any trading strategy associated with the given securities model.

A contingent claim Y is said to be *attainable* if there exists some trading strategy $\boldsymbol{h}$, called the *replicating portfolio*, such that $V_1 = Y$ for all possible states occurring at $t = 1$.

The price at $t = 0$ of the replicating portfolio is given by

$$V_0 = h_0 S_0(0) + h_1 S_1(0) = 1 \times 1 + 1 \times 3 = 4.$$

Suppose there are no arbitrage opportunities (equivalent to the existence of a risk neutral probability measure), then the law of one price holds and so V_0 is unique.

- Consider a given attainable contingent claim Y which is generated by certain trading strategy. The associated discounted gain G^* of the trading strategy is given by $G^* = \sum_{m=1}^{M} h_m \Delta S_m^*$. Now, suppose a risk neutral probability measure Q associated with the securities model exists, we have

$$V_0 = E_Q V_0^* = E_Q[V_1^* - G^*].$$

Since $E_Q[G^*] = 0$ and $V_1^* = Y/S_0(1)$, we obtain

$$V_0 = E_Q[Y/S_0(1)].$$

- Recall that the existence of the risk neutral probability measure implies the law of one price. Does $E_Q[Y/S_0(1)]$ assume the same value for every risk neutral probability measure Q?

This must be true by virtue of the law of one price since we cannot have two different values for V_0 corresponding to the same contingent claim Y.

Risk neutral valuation principle:

The price at $t = 0$ of an attainable claim Y is given by the expectation under any risk neutral measure Q of the discounted value of the contingent claim.

Actually, there exists a stronger result: If $E_Q[Y/S_0(1)]$ takes the same value for every Q, then the contingent claim Y is attainable.

Corollary

If the risk neutral measure is unique, then for any contingent claim $Y, E_Q[Y^*]$ takes the same value for any Q (actually single Q). Hence, any contingent claim is attainable so the market is complete.

State prices

Suppose we take Y to be the following contingent claim: $Y^* = Y/S_0(1)$ equals one if $\omega = \omega_k$ for some $\omega_k \in \Omega$ and zero otherwise. This is just the Arrow security $\boldsymbol{e}_k$ corresponding to the state ω_k. We then have

$$E_Q[Y/S_0(1)] = \boldsymbol{\pi}\boldsymbol{e}_k = Q(\omega_k).$$

The price of the Arrow security with discounted payoff $\boldsymbol{e}_k$ is called the state price for state $\omega_k \in \Omega$. The state price for ω_k is equal to the risk neutral probability for the same state.

Any contingent claim Y can be written as a linear combination of these basic Arrow securities. Suppose $Y^* = Y/S_0(1) = \sum_{k=1}^{K} \alpha_k \boldsymbol{e}_k$, then the price at $t = 0$ of the contingent claim is equal to $\sum_{k=1}^{K} \alpha_k Q(\omega_k)$.

Example

Suppose

$$Y^* = \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} \quad \text{and} \quad \widehat{S}^*(1;\Omega) = \begin{pmatrix} 1 & 4 \\ 1 & 3 \\ 1 & 2 \end{pmatrix},$$

Y^* is seen to be attainable. We have seen that the risk neutral probability is given by

$$\boldsymbol{\pi} = (\lambda \quad 1-2\lambda \quad \lambda), \text{ where } 0 < \lambda < 1/2.$$

The price at $t = 0$ of the contingent claim is given by

$$V_0 = 5\lambda + 4(1-2\lambda) + 3\lambda = 4,$$

which is independent of λ. This verifies the earlier claim that $E_Q[Y/S_0(1)]$ assumes the same value for any risk neutral measure Q.

Complete markets

Recall that a securities model is complete if every contingent claim Y lies in the asset span, that is, Y can be generated by some trading strategy.

Consider the augmented terminal payoff matrix

$$\widehat{S}(1;\Omega) = \begin{pmatrix} S_0(1;\omega_1) & S_1(1;\omega_1) & \cdots & S_M(1;\omega_1) \\ \vdots & \vdots & & \vdots \\ S_0(1;\omega_K) & S_1(1;\omega_K) & \cdots & S_M(1;\omega_K) \end{pmatrix},$$

Y always lies in the asset span if and only if the column space of $\widehat{S}(1;\Omega)$ is equal to $\mathbb{R}^K$.

- Since the dimension of the column space of $\widehat{S}(1;\Omega)$ cannot be greater than $M+1$, a necessary condition for market completeness is that $M+1 \geq K$.

- When $\widehat{S}(1;\Omega)$ has independent columns and the asset span is the whole $\mathbb{R}^K$, then $M+1=K$. Now, the trading strategy that generates Y must be unique since there are no redundant securities. In this case, any contingent claim is replicable and its price is unique.

- When the asset span is the whole $\mathbb{R}^K$ but some securities are redundant, the trading strategy that generates Y would not be unique.

- However, the price at $t=0$ of the contingent claim is unique under arbitrage pricing, independent of the chosen trading strategy. This is a consequence of the law of one price, which holds since a risk neutral measure exists.

- Non-existence of redundant securities is a sufficient but not necessary condition for the law of one price.

Non-attainable contingent claim

A *non-attainable contingent claim* cannot be priced using arbitrage pricing theory. However, we may specify an interval $(V_-(Y), V_+(Y))$ where a reasonable price at $t = 0$ of the contingent claim should lie. The lower and upper bounds are given by

$$\begin{aligned} V_+(Y) &= \inf\{E_Q[\widetilde{Y}/S_0(1)] : \widetilde{Y} \geq Y \text{ and } \widetilde{Y} \text{ is attainable}\} \\ V_-(Y) &= \sup\{E_Q[\widetilde{Y}/S_0(1)] : \widetilde{Y} \leq Y \text{ and } \widetilde{Y} \text{ is attainable}\}. \end{aligned}$$

Here, $V_+(Y)$ is the minimum value among all prices of attainable contingent claims that dominate the non-attainable claim Y, while $V_-(Y)$ is the maximum value among all prices of attainable contingent claims that are dominated by Y.

Suppose $V(Y) > V_+(Y)$, then an arbitrageur can lock in riskless profit by selling the contingent claim to receive $V(Y)$ and use $V_+(Y)$ to construct the replicating portfolio that generates the attainable $\widetilde{Y}$. The upfront positive gain is $V(Y) - V_+(Y)$.

How to solve for $V_+(Y)$? Let $R = W^\perp \cap P^+$ denote the set of risk neutral measures, with dim $W^\perp = J$. Let $\{\boldsymbol{Q}_1, \cdots, \boldsymbol{Q}_J\}$ denote the set of independent basis vectors of R. Suppose $\widetilde{Y}$ is attainable, and write $\lambda = E_Q[\widetilde{Y}^*]$, where $\widetilde{Y}^* = Y/S_0(1)$, then

$$\lambda = \boldsymbol{Q}_j^T \widetilde{\boldsymbol{Y}}^*, \quad j = 1, 2, \cdots, J.$$

Procedure

Minimize $\lambda \in \mathbb{R}$ such that $\widetilde{Y}^* \geq Y^*$ and $\lambda = \boldsymbol{Q}_j^T \widetilde{Y}^*, j = 1, \cdots, J$.

Remark: Since $\boldsymbol{Q}_j^T \widetilde{Y}^*$ assumes the same value for all j, by a previous theorem, $\widetilde{Y}$ is attainable.

Example

Take $\widehat{S}^*(1;\Omega) = \begin{pmatrix} 1 & 2 & 6 & 9 \\ 1 & 3 & 3 & 7 \\ 1 & 6 & 12 & 19 \end{pmatrix}$, the sum of the first 3 columns gives the fourth column. The first column corresponds to the discounted terminal payoff of the riskfree security under the 3 possible states of the world. The third risky security is a redundant security.

Let $\boldsymbol{S}^*(0) = (1 \quad 2 \quad 3 \quad k)$. We observe that solution to

$$(1 \quad 2 \quad 3 \quad k) = (\pi_1 \quad \pi_2 \quad \pi_3) \begin{pmatrix} 1 & 2 & 6 & 9 \\ 1 & 3 & 3 & 7 \\ 1 & 6 & 12 & 19 \end{pmatrix} \tag{1}$$

exists if and only if $k = 6$. That is, $S_3(0) = S_0(0) + S_1(0) + S_2(0)$.

When $k \neq 6$, the law of one price does not hold. The last equation: $9\pi_1 + 7\pi_2 + 19\pi_3 = k \neq 6$ is inconsistent with the first 3 equations.

We consider the linear system

$$\boldsymbol{S}^*(0) = \boldsymbol{\pi}\widehat{S}^*(1;\Omega),$$

solution exists if and only if $\boldsymbol{S}^*(0)$ lies in the row space of $\widehat{S}^*(1;\Omega)$. Uniqueness follows if the rows of $\widehat{S}^*(1;\Omega)$ are independent.

Since

$$\boldsymbol{S}_3^*(1;\Omega) = \boldsymbol{S}_0^*(1;\Omega) + \boldsymbol{S}_1^*(1;\Omega) + \boldsymbol{S}_2^*(1;\Omega),$$

the third risky security is replicable by holding one unit of each of the riskfree security and the first two risky securities. The initial price must observe the same relation in order that the law of one price holds.

Here, we have redundant securities. Actually, one may show that the law of one price holds if and only if we have existence of solution to the linear system. In this example, when $k = 6$, we obtain

$$\boldsymbol{\pi} = \begin{pmatrix} \frac{1}{2} & \frac{2}{3} & -\frac{1}{6} \end{pmatrix}.$$

This is *not* a risk neutral measure nor a linear pricing measure.

Consider another example

$$(1 \quad 2 \quad 3 \quad 6) = (\pi_1 \quad \pi_2 \quad \pi_3) \begin{pmatrix} 1 & 2 & 3 & 6 \\ 1 & 3 & 4 & 8 \\ 1 & 6 & 6 & 14 \end{pmatrix},$$

where the number of non-redundant securities is only 2. Note that

$$\boldsymbol{S}_2^*(1;\Omega) = \boldsymbol{S}_0^*(1;\Omega) + \boldsymbol{S}_1^*(1;\Omega) \quad \text{and}$$

$$\boldsymbol{S}_3^*(1;\Omega) = \boldsymbol{S}_0^*(1;\Omega) + \boldsymbol{S}_1^*(1;\Omega) + \boldsymbol{S}_2^*(1;\Omega),$$

and the initial prices have been set such that

$$S_2^*(0) = S_0^*(0) + S_1^*(0) \quad \text{and} \quad S_3^*(0) = S_0^*(0) + S_1^*(0) + S_2^*(0),$$

so we expect to have the existence of solution.

However, since $2 =$ number of non-redundant securities $<$ number of states $= 3$, we do not have uniqueness of solution. Indeed, we obtain

$$(\pi_1 \quad \pi_2 \quad \pi_3) = (1 \quad 0 \quad 0) + t(3 \quad -4 \quad 1), \quad t \text{ any value.}$$

In terms of linear algebra, we have existence of solution if the equations are consistent. Consider the present example, we have

$$\begin{aligned}
\pi_1 + \pi_2 + \pi_3 &= 1 \\
2\pi_1 + 3\pi_2 + 6\pi_3 &= 2 \\
3\pi_1 + 4\pi_2 + 7\pi_3 &= 3 \\
6\pi_1 + 8\pi_2 + 14\pi_3 &= 6
\end{aligned}$$

Note that the last two redundant equations are consistent.

Alternatively, we can interpret that the row vector $\boldsymbol{S}^*(0) = (1 \quad 2 \quad 3 \quad 6)$ lies in the row space of $\widehat{S}(1;\Omega)$, which is spanned by $\{(1 \quad 2 \quad 3 \quad 6)$, $(0 \quad 1 \quad 1 \quad 2)\}$.

Pricing of attainable contingent claims

Let $V_1^*(1;\Omega)$ denote the value of the attainable contingent claim. Suppose the associated trading strategy to generate the replicating portfolio is $\boldsymbol{h}$, then

$$V_1^* = \widehat{S}^*(1;\Omega)\boldsymbol{h}.$$

The initial cost of setting up the replicating portfolio is

$$V_0^* = \boldsymbol{S}(0)\boldsymbol{h}.$$

Assuming $\boldsymbol{\pi}$ exists, where $\boldsymbol{S}(0) = \boldsymbol{\pi}\widehat{S}(1;\Omega)$ so that

$$\begin{aligned} V_0^* &= \boldsymbol{\pi}\widehat{S}(1;\Omega)\boldsymbol{h} = \boldsymbol{\pi}V_1^*(1;\Omega) \\ &= \sum_{k=1}^{K} \pi_k V_1^*(1;\omega_k), \text{ independent of } \boldsymbol{h}. \end{aligned}$$

Even $\boldsymbol{\pi}$ is not a risk neutral measure or linear pricing measure, the above pricing relation remains valid. Suppose $\boldsymbol{\pi}$ is not unique, do we have different values for V_0^*?

Using the same $\boldsymbol{S}(0)$ and $\widehat{S}(1;\Omega)$ as shown in equation (1), we consider the contingent claim $\begin{pmatrix} 5 \\ 7 \\ 13 \end{pmatrix}$. The claim is attainable by holding one unit of the first risky security and second risky security. Its price is seen to be

$$S_1^*(0) + S_2^*(0) = 2 + 3 = 5.$$

Applying the formula:

$$\begin{aligned} V^*(0) &= \sum_{k=1}^{3} \pi_k V(1;\omega_k) \\ &= 5(1+3t) + 7(-4t) + 13t = 5, \text{ independent of } t. \end{aligned}$$

This is not surprising. This is consistent with the law of one price.

How to relate the existence of solution to

$$\boldsymbol{\pi} S^*(1) = \boldsymbol{S}(0) \tag{A}$$

and the law of one price?

1. Suppose solution to (A) exists, let $\boldsymbol{h}$ and $\boldsymbol{h'}$ be two trading strategies such that their discounted terminal payoffs are the same. That is,

   $$S^*(1)\boldsymbol{h} = \boldsymbol{V} = \boldsymbol{V}' = S^*(1)\boldsymbol{h'}.$$

   We then have

   $$\begin{aligned} & \boldsymbol{\pi} S^*(1)(\boldsymbol{h} - \boldsymbol{h'}) = 0 \\ \Rightarrow \quad & \boldsymbol{S}(0)(\boldsymbol{h} - \boldsymbol{h'}) = 0 \qquad \Rightarrow V_0 = V_0'. \end{aligned}$$

2. Suppose solution to (A) does not exist for some $\boldsymbol{S}(0)$, that is, there exists $\boldsymbol{S}(0)$ that does not lie in the row space of $S^*(1)$. This implies dim(row space of $S^*(1)) < M$, where M is the number of securities = number of columns in $S^*(1)$.

   Recall that

$$\text{dim(null space of } S^*(1)) + \text{rank}(S^*(1)) = M$$

   so that dim(null space of $S^*(1)) > 0$.

   Hence, there exists non-zero solution $\boldsymbol{h}$ to

$$S^*(1)\boldsymbol{h} = \boldsymbol{0}.$$

Furthermore, since row space = orthogonal complement of null space, any of these non-zero solution $\boldsymbol{h}$ cannot be orthogonal to $\boldsymbol{S}(0)$. Otherwise, this leads to contradiction as $\boldsymbol{S}(0)$ does not lie in the row space.

Let $\boldsymbol{h} = \boldsymbol{h}_1 - \boldsymbol{h}_2$, where $\boldsymbol{h}_1 \neq \boldsymbol{h}_2$, then there exist two distinct trading strategies such that

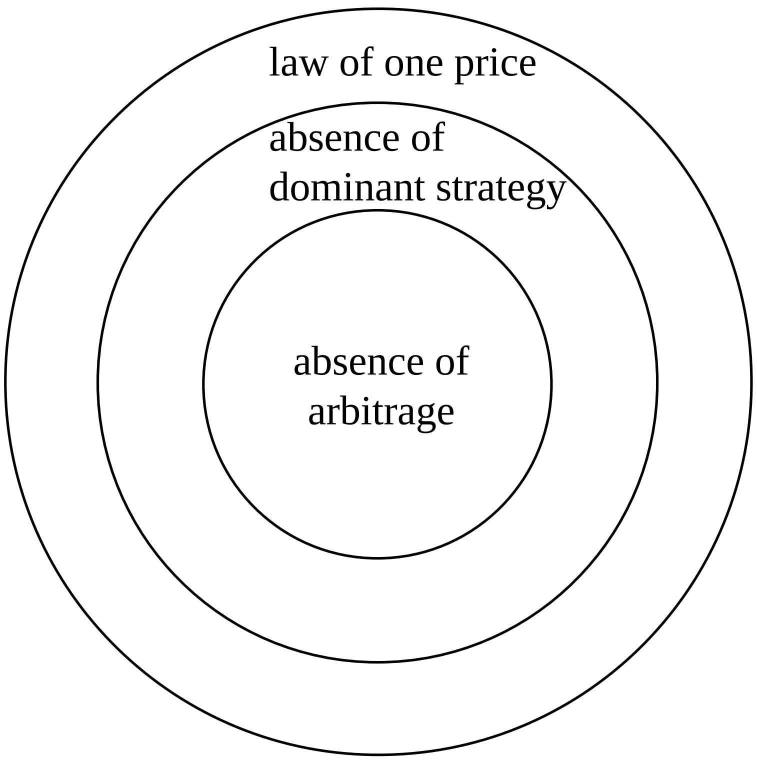
$$S^*(1)\boldsymbol{h}_1 = S^*(1)\boldsymbol{h}_2.$$

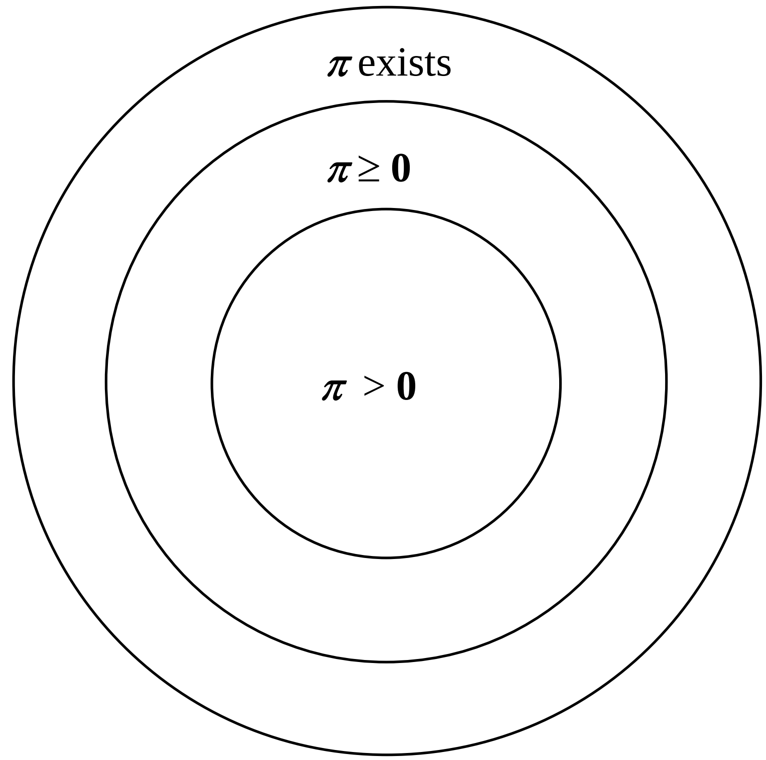
The two strategies have the same discounted terminal payoff under all states of the world. However, their initial prices are unequal since

$$\boldsymbol{S}(0)\boldsymbol{h}_1 \neq \boldsymbol{S}(0)\boldsymbol{h}_2,$$

by virtue of the property: $\boldsymbol{S}(0)\boldsymbol{h} \neq 0$. Hence, the law of one price does not hold.

Financial Economics *Linear Algebra*: $\boldsymbol{\pi} S^*(1) = \boldsymbol{S}(0)$





Remark The securities model contains the riskfree asset.

Summary *Arbitrage opportunity* 無風險套利機會

A self-financing trading strategy is requiring no initial investment, having no probability of negative value at expiration, and yet having some possibility of a positive terminal portfolio value.

- Commonly it is assumed that there are no arbitrage opportunities in well functioning and competitive financial markets.

1.
$$
\begin{aligned}
& \text{absence of arbitrage opportunities} \\
\Rightarrow\ & \text{absence of dominant trading strategies} \\
\Rightarrow\ & \text{law of one price}
\end{aligned}
$$

2. absence of arbitrage opportunities $\Leftrightarrow$ existence of risk neutral measure

   absence of dominant trading strategies $\Leftrightarrow$ existence of linear pricing measure.

3. The state prices are non-negative when a linear pricing measure exists and they become strictly positive when a risk neutral measure exists.

4. Under the absence of arbitrage opportunities, the risk neutral valuation principle can be applied to find the fair price of an attainable contingent claim.