MATH685X – Mathematical Models in Financial Economics

Topic 1 – Utility theory and decision making under uncertainty

1.1 Maximum expected return criterion and St Peterburg's paradox

1.2 Preference orderings and utility functions

- Preference relations
- Theorem for the existence of utility function

1.3 Maximum expected utility approach

- Choices among lotteries and maximum expected utility

1.4 Choices of probability distributions over outcomes

- Ellsberg paradox
- Von Neumann – Morgenstern framework
- Independence axiom and Allais paradox

1.1 Maximum expected return criterion (for risky investments)

Identifies the investment with the highest expected return.

$$\begin{aligned} E_C(x) &= \frac{1}{4}(-5) + \frac{1}{2}(0) + \frac{1}{4}(40) = 8.75 \\ E_D(x) &= \frac{1}{5}(-10) + \frac{1}{5}(10) + \frac{2}{5}(20) + \frac{1}{5}(30) = 14. \end{aligned}$$

According to the maximum expected return criterion, D is preferred over C. However, some investors may prefer C on the ground that it has a smaller downside loss of -5 and a higher upside gain of 40.

Is such procedure well justified? How to include the risk appetite of an individual investor into the decision procedure?

St Petersburg paradox (failure of Maximum Expected Return Criterion)

Tossing of a fair coin until the first head shows up. The prize is 2^{k-1} where k is the number of tosses until the first head shows up (the game is then ended). There is a very small chance to receive a large sum of money. This occurs when x is large.

Expected prize of the game $= \sum_{k=1}^{\infty} \frac{1}{2^k} 2^{k-1} = \infty$.

- When people are faced with such a lottery in experimental trials, they refuse to pay more than a finite price (usually low). "Few people would pay even $25 to enter such a game."

- The decision criterion which takes only the expected value into account would recommend a course of action that no (real) rational person would be willing to take.

- Finite resources of the participants – one simply cannot buy that which is not sold. Sellers would not produce a lottery whose potential loss were unacceptable.

  If the total resources (or maximum jackpot) of the casino is W, then the expected value of the lottery is

$$
\begin{aligned}
E &= \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \min(2^k, W) \\
&= \sum_{k=0}^{L-1} \frac{1}{2^{k+1}} 2^k + \sum_{k=L}^{\infty} \frac{1}{2^{k+1}} W \\
&= \frac{L}{2} + \frac{W}{2^L}, \text{ where } L = 1 + floor(\ln_2 W).
\end{aligned}
$$

The following table shows the expected value E of the game with various potential backers and their bankroll W

Backer	*Bankroll*	*Expected value of lottery*
Friendly game	\$100	\$4.28
Millionaire	\$100,000,000	\$10.95
Billionaire	\$1,000,000,000	\$15.93
Bill Gates (2008)	\$58,000,000,000	\$18.84
U.S. GDP (2007)	\$13.8 trillion	\$22.79
World GDP (2007)	\$54.3 trillion	\$23.77
Googolaire	$\$10^{100}$	\$166.50

Notes: The estimated net worth of Bill Gates is from Forbes. The GDP data are as estimated for 2007 by the International Monetary Fund, where one trillion dollars equals $\$10^{12}$. A "googolaire" is a hypothetical person worth a googol dollars ($\$10^{100}$).

1.2 Preference orderings and utility functions

Preference relations

The building block is pairwise comparison. Given the set of alternatives B, how to determine which element in the choice set B that is preferred?

The individual first considers two arbitrary elements: $x_1, x_2 \in B$. He then picks the preferred element x_1 and discards the other. From the remaining elements, he picks the third one and compares with the winner. The process continues and the best choice among all alternatives is identified.

Choice set

Let the choice set B be a *convex* subset of the n-dimensional Euclidean space. The component x_i of the n-dimensional vector x may represent x_i units of commodity i. By convex, we mean that if $x_1, x_2 \in B$, then $\alpha x_1 + (1-\alpha)x_2 \in B$ for any $\alpha \in [0,1]$.

- Each individual is endowed with a *preference relation*, $\succeq$.

- Given any pair of elements x_1 and $x_2 \in B, x_1 \succeq x_2$ means either that x_1 is preferred to x_2 or that x_1 is indifferent to x_2.

Three axioms for $\succeq$

Reflexivity

For any $x_1 \in B, x_1 \succeq x_1$.

Comparability

For any $x_1, x_2 \in B$, either $x_1 \succeq x_2$ or $x_2 \succeq x_1$.

Transitivity

For $x_1, x_2, x_3 \in B$, given $x_1 \succeq x_2$ and $x_2 \succeq x_3$, then $x_1 \succeq x_3$.

Remarks

1. Without the comparability axiom, an individual could not determine an optimal choice. There would exist at least two elements of B between which the individual could not discriminate.
2. The transitivity axiom ensures that the choices are consistent.

Example 1 – Total quantity

Let $B = \{(x,y) : x \in [0,\infty) \text{ and } y \in [0,\infty)\}$ represent the set of alternatives. Let x represent ounces of orange soda and y represent ounces of grape soda. It is easily seen that B is a convex subset of $\mathbb{R}^2$.

Suppose the individual is concerned only with the total quantity of soda available, the more the better, then the individual is endowed with the following preference relation:

For $(x_1, y_1), (x_2, y_2) \in B$,

$$(x_1, y_1) \succeq (x_2, y_2) \text{ if and only if } x_1 + y_1 \geq x_2 + y_2.$$

Example 2 – Dictionary order

Let the choice set $B = \{(x, y) : x \in [0, \infty), y \in [0, \infty)\}$. The dictionary order $\succeq$ is defined as follows:

Suppose $(x_1, y_1) \in B$ and $(x_2, y_2) \in B$, then

$$(x_1, y_1) \succeq (x_2, y_2) \text{ if and only if}$$
$$[x_1 > x_2] \text{ or } [x_1 = x_2 \text{ and } y_1 \geq y_2].$$

It is easy to check that the dictionary order satisfies the three basic axioms of a preference relation.

Definition

Given $x, y \in B$ and a preference relation $\succeq$ satisfying the above three axioms.

1. x is indifferent to y, written as

$$x \sim y \quad \text{if and only if} \quad x \succeq y \text{ and } y \succeq x.$$

2. x is strictly preferred to y, written as

$$x \succ y \quad \text{if and only if} \quad x \succeq y \text{ and not } x \sim y.$$

Axiom 4 – Order Preserving

For any $x, y \in B$ where $x \succ y$ and $\alpha, \beta \in [0, 1]$,

$$[\alpha x + (1 - \alpha)y] \succ [\beta x + (1 - \beta)y] \quad \text{if and only if } \alpha > \beta.$$

Example 1 revisited – checking the Order Preserving Axiom

Recall the preference relation defined in Example 1, we take (x_1, y_1), $(x_2, y_2) \in B$ such that $(x_1, y_1) \succ (x_2, y_2)$ so that $x_1 + y_1 - x_2 - y_2 > 0$.

Take $\alpha, \beta \in [0, 1]$ such that $\alpha > \beta$, and observe

$$\alpha[(x_1 + y_1) - (x_2 + y_2)] > \beta[(x_1 + y_1) - (x_2 + y_2)].$$

Adding $x_2 + y_2$ to both sides, we obtain

$$\alpha(x_1 + y_1) + (1 - \alpha)(x_2 + y_2) > \beta(x_1 + y_1) + (1 - \beta)(x_2 + y_2).$$

Axiom 5 – Intermediate Value

For any $x, y, z \in B$, if $x \succ y \succ z$, then there exists a unique $\alpha \in (0, 1)$ such that

$$\alpha x + (1 - \alpha)z \sim y.$$

Remark

Given 3 alternatives with rankings of $x \succ y \succ z$, there exists a fractional combination of x and z that is indifferent to y. Trade-offs between the alternatives exist.

Example 1 revisited – checking the Intermediate Value Axiom

Given $x_1 + y_1 > x_2 + y_2 > x_3 + y_3$, choose

$$\alpha = \frac{(x_2 + y_2) - (x_3 + y_3)}{(x_1 + y_1) - (x_3 + y_3)}.$$

Rearranging gives

$$\alpha(x_1 + y_1) + (1 - \alpha)(x_3 + y_3) = x_2 + y_2$$

so that

$$[\alpha(x_1, y_1) + (1 - \alpha)(x_3, y_3)] \sim (x_2, y_2).$$

Dictionary order does not satisfy the intermediate value axiom

Suppose $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in B$ such that $(x_1, y_1) \succ (x_2, y_2) \succ (x_3, y_3)$ and $x_1 > x_2 = x_3$ and $y_2 > y_3$. For any $\alpha \in (0, 1)$, we have

$$\begin{aligned} & \alpha(x_1, y_1) + (1-\alpha)(x_3, y_3) \\ = \; & \alpha(x_1, y_1) + (1-\alpha)(x_2, y_3) \\ = \; & (\alpha x_1 + (1-\alpha)x_2, \alpha y_1 + (1-\alpha)y_3). \end{aligned}$$

But for $\alpha > 0$, we have $\alpha x_1 + (1-\alpha)x_2 > x_2$ so

$$\alpha(x_1, y_1) + (1-\alpha)(x_3, y_3) \succ (x_2, y_2) \quad \text{for all} \quad \alpha \in (0, 1).$$

In other words, there is no $\alpha \in (0, 1)$ such that

$$\alpha x + (1-\alpha)z \sim y.$$

Axiom 6 – Boundedness

There exist $x^*, y^* \in B$ such that $x^* \succeq z \succeq y^*$ for all $z \in B$.

- This Axiom ensures the existence of a most preferred element $x^* \in B$ and a least preferred element $y^* \in B$.

Example 1 revisited – checking the Boundedness Axiom

Recall $B = \{(x, y) : x \in [0, \infty) \text{ and } y \in [0, \infty)\}$. Given any $(z_1, z_2) \in B$, we have

$$(z_1 + 1, z_2) \succeq (z_1, z_2) \text{ since } z_1 + z_2 + 1 > z_1 + z_2.$$

Therefore, a maximum does not exist.

Motivation for defining utility

Knowledge of the preference relation $\succeq$ effectively requires a complete listing of preferences over all pairs of elements from the choice set B. We define a *utility* function that assigns a numeric value to each element of the choice set such that a larger numeric value implies a higher preference.

- We establish the theorem on the existence of utility function.

- The optimal criterion for ranking alternative investments is based on the ranking of the expected utility values of the various investments.

Theorem – Existence of Utility Function

Let B denote the set of payoffs from a finite number of securities, also being a convex subset of $\mathbf{R}^n$. Let $\succeq$ denote a preference relation on B. Suppose $\succeq$ satisfies the following axioms

(i) $\forall x \in B, x \succeq x$.

(ii) $\forall x, y \in B, x \succeq y$ or $y \succeq x$.

(iii) For any $x, y, z \in B$, if $x \succeq y$ and $y \succeq z$, then $x \succeq z$.

(iv) For any $x, y \in B, x \succeq y$ and $\alpha, \beta \in [0, 1]$,

$$\alpha x + (1 - \alpha) y \succeq \beta x + (1 - \beta) y \quad \text{if and only if} \quad \alpha > \beta.$$

(v) For any $x, y, z \in B$, suppose $x \succ y \succ z$, then there exists a unique $\alpha \in (0, 1)$ such that $\alpha x + (1 - \alpha) z \sim y$.

(vi) There exist $x^*, y^* \in B$ such that $\forall z \in B, x^* \succeq z \succeq y^*$.

Then there exists a utility function $U : B \to \mathbf{R}$ such that

(a) $x \succ y$ iff $U(x) > U(y)$.

(b) $x \sim y$ iff $U(x) = U(y)$.

To show the existence of $U : B \to \mathbf{R}$, we write down one such function and show that it satisfies the stated conditions.

Based on Axiom 6, we choose $x^*, y^* \in B$ such that

$$x^* \succeq z \succeq y^* \quad \text{for all} \quad z \in B.$$

Without loss of generality, let $x^* \succ y^*$. [Otherwise, $x^* \sim z \sim y^*$ for all $z \in B$. In this case, $U(z) = 0$ for all $z \in B$, which is a trivial utility function that satisfies conditions (a) and (b).]

Consider an arbitrary $z \in B$. There are 3 possibilities:

1. $\quad z \sim x^*$; $\quad$ 2. $\quad x^* \succ z \succ y^*$; $\quad$ 3. $\quad z \sim y^*$.

We define U by giving its value under all 3 cases:

1. $U(z) = 1$
2. By Axiom 5, there exists a unique $\alpha \in (0, 1)$ such that
$$[\alpha x^* + (1 - \alpha) y^*] \sim z.$$
Define $U(z) = \alpha$.
3. $U(z) = 0$.

Such U satisfies properties (a) and (b).

Proof of property (a)

Necessity Suppose $z_1, z_2 \in B$ are such that $z_1 \succ z_2$, we show

$$U(z_1) > U(z_2).$$

Consider the four possible cases.

1. $z_1 \sim x^* \succ z_2 \succ y^*$
2. $z_1 \sim x^* \succ z_2 \sim y^*$
3. $x^* \succ z_1 \succ z_2 \succ y^*$
4. $x^* \succ z_1 \succ z_2 \sim y^*$.

Case 1 By definition, $U(z_1) = 1$ and $U(z_2) = \alpha$, where $\alpha \in (0, 1)$ uniquely satisfies

$$\alpha x^* + (1 - \alpha)y^* \sim z_2.$$

Now, $U(z_1) = 1 > \alpha = U(z_2)$.

Case 2 By definition, $U(z_1) = 1 > 0 = U(z_2)$.

Case 3 By defintion, $U(z_i) = \alpha_i$, where $\alpha_i \in (0,1)$ uniquely satisfies

$$\alpha_i x^* + (1-\alpha_i)y^* \sim z_i,$$

so that

$$z_1 \sim [\alpha_1 x^* + (1-\alpha_1)y^*] \text{ and } [\alpha_2 x^* + (1-\alpha_2)y^*] \sim z_2.$$

We claim $\alpha_1 > \alpha_2$. Assume not, then $\alpha_1 \leq \alpha_2$. By Axiom 4,

$$[\alpha_2 x^* + (1-\alpha_2)y^*] \succeq [\alpha_1 x^* + (1-\alpha_1)y^*].$$

This is a contradiction. Hence, $\alpha_1 > \alpha_2$ is true and

$$U(z_1) = \alpha_1 > U(z_2) = \alpha_2.$$

Case 4 By definition, $U(z_1) = \alpha_1$, where $\alpha_1 \in (0,1)$ uniquely satisfies

$$\alpha_1 x^* + (1-\alpha_1)y^* \sim y_1 \quad \text{and} \quad U(z_2) = 0.$$

We have $U(z_1) = \alpha_1 > 0 = U(z_2)$.

Sufficiency

Suppose, given $z_1, z_2 \in B$, that $U(z_1) > U(z_2)$, we would like to show $z_1 \succ z_2$. Consider the following 4 cases

1. $U(z_1) = 1$ and $U(z_2) = \alpha_2$, where $\alpha_2 \in (0,1)$ uniquely satisfies
$$[\alpha_2 x^* + (1-\alpha_2)y^*] \sim z_2.$$
2. $U(z_1) = 1$, where $z_1 \sim x^*$ and $U(z_2) = 0$, where $z_2 \sim y^*$.
3. $U(z_i) = \alpha_i$, where $\alpha_i \in (0,1)$ uniquely satisfies
$$[\alpha_i x^* + (1-\alpha_i)y^*] \sim z_i.$$
4. $U(z_1) = \alpha_1$ and $U(z_2) = 0$, where $z_2 \sim y^*$.

Case 1 $z_1 \sim x^* \sim [1 \cdot x^* + 0 \cdot y^*]$ and $z_2 \sim [\alpha_2 x^* + (1 - \alpha_2) y^*]$.

By Axiom 4, $1 > \alpha_2$ so that $z_1 \succ z_2$.

Case 2 $z_1 \sim x^* \succ y^* \sim z_2$.

Case 3 $z_1 \sim [\alpha_1 x^* + (1 - \alpha_1) y^*]$

$z_2 \sim [\alpha_2 x^* + (1 - \alpha_2) y^*]$

Since $\alpha_1 > \alpha_2$, by Axiom 4, $z_1 \succ z_2$.

Case 4 $z_1 \sim [\alpha_1 x^* + (1 - \alpha_1) y^*]$ and

$z_2 \sim y^* \sim [0x^* + (1 - 0) y^*]$.

By Axiom 4 and Axiom 3, since $\alpha_1 > 0, z_1 \succ z_2$.

Proof of Property (b)

Necessity

Suppose $z_1 \sim z_2$ but $U(z_1) \neq U(z_2)$, then

$$U(z_1) > U(z_2) \quad \text{or} \quad U(z_2) > U(z_1).$$

By property (a), this implies $z_1 \succ z_2$ or $z_2 \succ z_1$, a contradiction. Hence,

$$U(z_1) = U(z_2).$$

Sufficiency

Suppose $U(z_1) = U(z_2)$, but $z_1 \succ z_2$ or $z_1 \prec z_2$. By property (a), this implies $U(z_1) > U(z_2)$ or $U(z_2) > U(z_1)$, a contradiction. Hence, $z_1 \sim z_2$.

Certainty equivalent

What is the certain amount that one would be willing to accept so that it is indifferent between playing the game for free or receiving this certain sum? This certain amount is called the *certainty equivalent* of the game. In simple language, every game or lottery has a price.

- Let $U(x)$ be the utility of the player, which measures the sense of satisfaction for a given wealth level x. Based on the expected utility criterion, the certainty equivalent c is given by

  $$U(c) = E[U(X)],$$

  where X is the random wealth at the end of the game. Certainty equivalent of the game of St. Peterburg paradox under log utility

  $$\ln c = E[\ln X] = \sum_{x=1}^{\infty} \frac{1}{2^x} \ln 2^{x-1} = \ln 2 \sum_{x=1}^{\infty} \frac{x-1}{2^x} = \ln 2$$

  so that $c = 2$ is the certainty equivalent.

1.3 Maximum expected utility criterion (MEUC)

How do we make a choice between the following two lotteries:

$$
\begin{aligned}
L_1 &= \{p_1, A_1; p_2, A_2; \cdots; p_n, A_n\} \\
L_2 &= \{q_1, A_1; q_2, A_2; \cdots; q_n, A_n\}?
\end{aligned}
$$

The monetary outcomes are $A_1, \cdots, A_n$; p_i and q_i are the probabilities of occurrence of A_i in L_1 and L_2, respectively. These outcomes are mutually exclusive and only one outcome can be realized under each investment. We are not limited to lotteries with the same set of outcomes. Suppose outcome A_i will not occur in Lottery L_1, we can simply set $p_i = 0$.

Comparability

When faced by two monetary outcomes A_i and A_j, the investor must say $A_i \succ A_j, A_j \succ A_i$ or $A_i \sim A_j$.

Continuity

If $A_3 \succeq A_2$ and $A_2 \succeq A_1$, then there exists $U(A_2)$ $[0 \leq U(A_2) \leq 1]$ such that

$$L = \{[1 - U(A_2)], A_1; U(A_2), A_3\} \sim A_2.$$

For a given set of outcomes A_1, A_2 and A_3, these probabilities are a function of A_2, hence the notation $U(A_2)$.

Why is it called the continuity axiom? When $U(A_2) = 1$, we obtain $L = A_3 \succeq A_2$; when $U(A_2) = 0$, we obtain $L = A_1 \preceq A_2$. If we increase $U(A_2)$ continuously from 0 to 1, we hit a value $U(A_2)$ such that $L \sim A_2$.

Remark
Though $U(A_2)$ is a probability value, we will see that it is also the investor's utility function.

Interchangeability

Given $L_1 = \{p_1, A_1; p_2, A_2; p_3, A_3\}$ and $A_2 \sim A = \{q, A_1; (1-q), A_3\}$, the investor is indifferent between L_1 and $L_2 = \{p_1, A_1; p_2, A; p_3, A_3\}$.

Transitivity

Given $L_1 \succ L_2$ and $L_2 \succ L_3$, then $L_1 \succ L_3$.

Also, if $L_1 \sim L_2$ and $L_2 \sim L_3$, then $L_1 \sim L_3$.

Decomposability

A complex lottery has lotteries as prizes. A simple lottery has monetary values A_1, A_2, etc as prizes.

Consider a complex lottery $L^* = (q, L_1; (1-q), L_2)$, where

$$L_1 = \{p_1, A_1; (1-p_1), A_2\} \quad \text{and} \quad L_2 = \{p_2, A_1; (1-p_2), A_2\},$$

L^* can be decomposed into a simple lottery $L = \{p^*, A_1; (1-p^*), A_2\}$, with A_1 and A_2 as prizes where $p^* = qp_1 + (1-q)p_2$.

Monotonicity

(a) For monetary outcomes, $A_2 > A_1 \Longrightarrow A_2 \succ A_1$.

(b) For lotteries

(i) Let $L_1 = \{p, A_1; (1-p), A_2\}$ and $L_2 = \{p, A_1; (1-p), A_3\}$. If $A_3 > A_2$, then $A_3 \succ A_2$; and $L_2 \succ L_1$.

(ii) Let $L_1 = \{p, A_1; (1-p), A_2\}$ and $L_2 = \{q, A_1; (1-q), A_2\}$, also $A_2 > A_1$ (hence $A_2 \succ A_1$). If $p < q$, then $L_1 \succ L_2$.

Theorem

The optimal criterion for ranking alternative investments is the *expected utility* of the various investments.

Proof

How do we make a choice between L_1 and L_2

$$\begin{aligned} L_1 &= \{p_1, A_1; p_2, A_2; \cdots; p_n, A_n\} \\ L_2 &= \{q_1, A_1; q_2, A_2; \cdots; q_n, A_n\} \end{aligned}$$

$A_1 < A_2 < \cdots < A_n$, where A_i are various monetary outcomes?

- ⋆ By comparability axiom, we can compare A_i. Further, by monotonicity axiom, we determine that

$$A_1 < A_2 < \cdots < A_n \text{ implies } A_1 \prec A_2 \prec \cdots \prec A_n.$$

- ⋆ Define $A_i^* = \{[1 - U(A_i)], A_1; U(A_i), A_n\}$ where $0 \leq U(A_i) \leq 1$.

By the continuity axiom, for every A_i, there exists $U(A_i)$ such that $A_i \sim A_i^*$.

For $A_1, U(A_1) = 0$, hence $A_1^* \sim A_1$; for $A_n, U(A_n) = 1$. For other $A_i, 0 < U(A_i) < 1$. By the monotonicity and transitivity axioms, $U(A_i)$ increases from zero to one as A_i increases from A_1 to A_n.

- $\star$ Substitute A_i by A_i^* in L_1 successively and by the interchangeability axiom,

$$L_1 \sim \widetilde{L}_1 = \{p_1, A_1^*; p_2, A_2^*; \cdots ; p_n, A_n^*\}.$$

By the decomposability axiom, we observe

$$L_1 \sim \widetilde{L}_1 \sim L_1^* = \{\Sigma p_i[1 - U(A_i)], A_1; \Sigma p_i U(A_i), A_n\}.$$

Similarly

$$L_2 \sim L_2^* = \{\Sigma q_i[1 - U(A_i)], A_1; \Sigma q_i U(A_i), A_n\}.$$

By the monotonicity axiom, $L_1^* \succ L_2^*$ if

$$\Sigma p_i U(A_i) > \Sigma q_i U(A_i).$$

This is precisely the *expected utility criterion*. The same conclusion applies to $L_1 \succ L_2$, due to transitivity.

Remarks

Recall $A_i \sim A_i^* = \{[1-U(A_i)], A_1; U(A_i), A_n\}$, such a function $U(A_i)$ always exists, though not all investors would agree on the specific value of $U(A_i)$.

- By the monotonicity axiom, utility is non-decreasing.

- A utility function is determined up to a positive linear transformation, so its value is not limited to the range $[0, 1]$. "Determined" means that the ranking of the projects by the MEUC does not change.

- The absolute difference or ratio of the utilities of two investment choices gives no indication of the degree of preference of one over the other since utility values can be expanded or suppressed by a linear transformation.

Ellsberg paradox

- It is a paradox in decision theory and experimental economic in which people's choices violate the expected utility criterion. This is taken to be evidence for *ambiguity aversion*.

Games

- A box contains 30 red balls and 60 other balls (either black or yellow).

Gamble A	Gamble B
Receive \$100 if a red ball is drawn	Receive \$100 if a black ball is drawn

Gamble C	Gamble D
Receive \$100 if a red or yellow ball is drawn	Receive \$100 if a black or yellow ball is drawn

According to *expected utility criterion*,

Gamble A is preferred to Gamble B if and only if

drawing a red ball is more likely than drawing a black ball

Mathematically,

$$p_R U(\$100) + (1 - p_R) U(\$0) > p_B U(\$100) + (1 - p_B) U(\$0)$$
$$(p_R - p_B)[U(\$100) - U(\$0)] > 0$$

Given $U(\$100) > U(\$0)$, this is equivalent to $p_R > p_B$.

Gamble D is preferred to Gamble C

$$\begin{aligned} \Leftrightarrow \quad & p_B U(\$100) + p_Y U(\$100) + p_R U(\$0) \\ & > p_R U(\$100) + p_Y U(\$100) + p_B U(\$0) \\ \Leftrightarrow \quad & p_B > p_R. \end{aligned}$$

When surveyed, most people strictly prefer Gamble A to Gamble B and Gamble D to Gamble C. These preferences are seen to be inconsistent with the expected utility theory.

- Taken as evidence for some sort of ambiguity aversion. In Gamble A, the probability of a red ball is 30/90, which is precise number. No probability information is provided regarding other outcomes, so the player has very unclear subjective impressions of these probabilities.

- Deceit aversion mechanism
  If a person is not sold the probability of a certain event, it is to deceive them. When faced with the choice between a red ball and a black ball, the probability of 30/90 is compared to the lower part of the 0/90 – 60/90 range.

1.4 Choice of probability distribution over outcomes

Economic agents choose actions on the basis of consequences that the chosen actions produce. Other factors may interact with an action (state of the world) to produce a particular consequence.

$A =$ set of feasible actions

$S =$ set of possible states of the world

$C =$ set of consequences

A combination of an action $a \in A$ and a state $s \in S$ will produce a particular consequence $c \in C$.

$$(s, a) \to c = f(s, a).$$

Uncertainty about the state of the world is often modelled by a probability measure on S.

- Choosing an action "a" determines a consequence for each state of the world, $f(s, a)$. The decision over actions in A can therefore be viewed as a decision over state-dependent consequences.

  Write $(c_{11}, c_{21}, \cdots, c_{s1})$ as the state-contingent consequences associated with action a_1. Choosing a_1 over a_2 is the same as choosing $(c_{11}, \cdots, c_{s1})$ over $(c_{12}, \cdots, c_{s2})$.

- If f is constant with respect to the state of the world, then the decision is taken *under certainty*.

Alternative viewpoint – choice of probability distribution over outcome

The relationship among actions, states of the world and consequences is described by $f : S \times A \to C$.

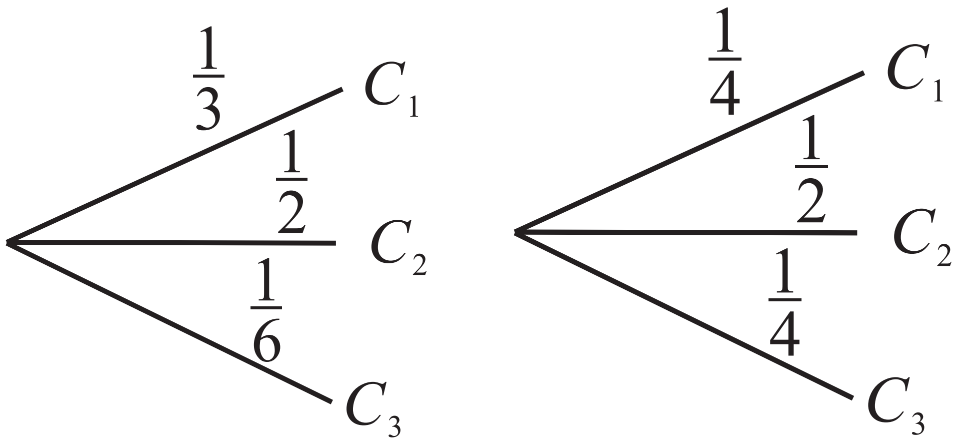
Since a probability distribution measure is defined on S, there is an induced probability distribution on the set of consequences for each action. Consider action $a \in A$, and any (measurable) subset of consequences $K \subset C$,

$$\text{Prob } \{K\} := \text{prob}\{s \in S | f(s,a) \in K\}.$$

The probability of a particular consequence is equal to the probability of the states of the world which lead to this consequence given a particular action.

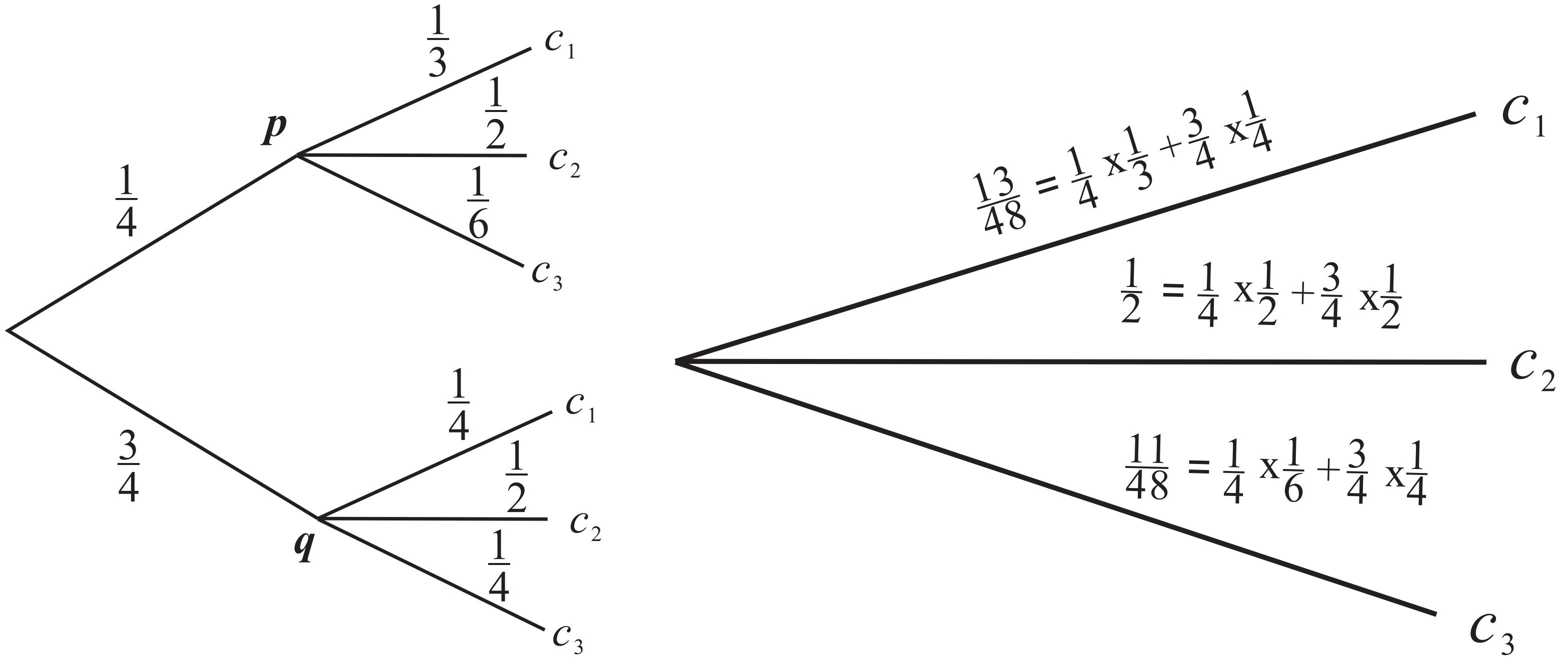
Hence, the choice of an action amounts to the choice of a probability distribution on consequences. The choice of different gambles or investment choices is the choice among alternative probability distributions.

$$\boldsymbol{p} = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{pmatrix} \qquad \boldsymbol{q} = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$$



Tree representation of lotteries

$$\boldsymbol{r} = \tfrac{1}{4}\boldsymbol{p} + \tfrac{3}{4}\boldsymbol{q} \qquad\qquad \boldsymbol{r} = \begin{pmatrix} \frac{13}{48} & \frac{1}{2} & \frac{11}{48} \end{pmatrix}$$



Compound lottery

Objects of choice can be viewed either as

- state-contingent outcomes
- probability distributions.

Formalism

Given a set of outcomes C and a probability distribution on the set of states, each action induces a probability distribution on the outcomes in C.

If the set of consequences is finite, $C = \{c_1, \cdots, c_n\}$, then each action determines a vector of probabilities from the set

$$\Delta^n = \left\{ (p_1, \cdots p_n) \in \mathbb{R}^n_+ \middle| \sum_{i=1}^n p_i = 1 \right\}$$

with $p_i = \text{prob}\ (\{s \in S | f(s,a) = c_i\})$.

The utility function over outcomes $u(c_i)$ is commonly called the **Von Neumann-Morgenstern utility index**.

Given a von Neumann-Morgenstern utility function u, one can treat the expected utility representation $\sum_{i=1}^{n} p_i u(c_i)$ as an utility function of the probability distribution $(p_1, \cdots, p_n)$:

$$U(p_1, \cdots, p_n) = \sum_{i=1}^{n} p_i u(c_i).$$

This expected utility representation evaluates a probability distribution $P(p_1, \cdots, p_n)$ over outcomes $(c_1, \cdots, c_n)$ by forming a weighted average of the utilities $u(c_i)$ derived from the different outcomes using the probabilities as weights.

Assumptions on a preference ordering over probability distributions:

1. Completeness requires the ordering to order any pair of probability distributions in Δ^n.

2. Transitivity: $p \succ q, q \succ r$ then $p \succ r$.

3. Continuity: For a continuous transformation of a probability distribution p into another probability distribution q, where $q \succ p$, the course of transformation leads to a probability distribution that is indifference to any probability distribution ranked between p and q. That is, preference for probability distributions do not change abruptly.

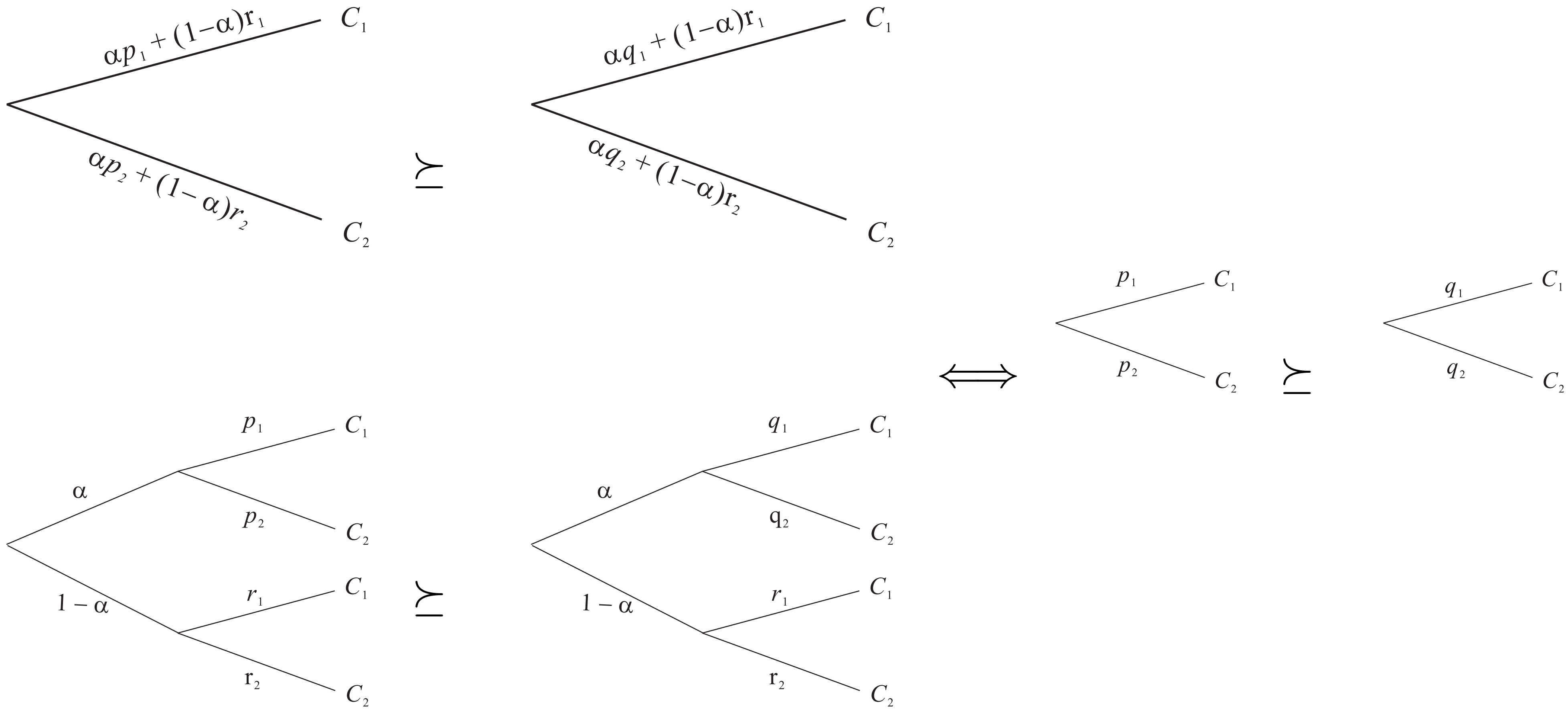
4. Independence axiom

The preference relation on Δ^n represented by the utility function $U(\cdot)$ satisfies for any $p, q, r \in \Delta^n$ and any $\alpha \in [0,1]$

$$U(\alpha p + (1-\alpha)r) \geq U(\alpha q + (1-\alpha)r) \text{ iff } U(p) \geq U(q).$$

For example, suppose an investor is indifferent between X and Y; Z is a third prospect. Investor should be indifferent to these 2 gambles:
X with prob p and Z with prob $1-p$
Y with prob p and Z with prob $1-p$

If a person were indifferent between having a Ford or a Datsun, she would be indifferent to buy a lottery ticket for $10 that gave a 1 in 500 chance of winning a Ford or a ticket for $10 that gave the same change of winning a Datsun.



One can decompose any two probability distributions into parts that are identical and parts that are different.

Existence of utility function on Δ^n

If a preference ordering over the probability distributions in Δ^n satisfies completeness, transitivity and continuity, there exists a utility function $U : \Delta^n \rightarrow \mathbb{R}$ that represents this preference ordering. The utility function $U(\cdot)$ is unique up to *monotone transformation*.

- One can take any strictly increasing function : $\mathbb{R} \rightarrow \mathbb{R}$, say, $f(x) =$ exp(x), to obtain another equivalent utility function $\widetilde{U}(p) = f(U(p)$.

Theorem (expected utility representation)

A utility function U on Δ^n satisfies the *independence axiom* iff there is a utility function over outcomes $u : C \rightarrow \mathbb{R}$ such that for all p and $q \in \Delta^n$

$$U(p) \geq U(q) \text{ if and only if } \sum_{i=1}^{n} p_i u(c_i) \geq \sum_{i=1}^{n} q_i u(c_i).$$

Allais Paradox (1952)

$C_1 = 5$ million, $C_2 = 1$ million, $C_3 = 0$

	prob $\{C_1\}$	prob $\{C_2\}$	prob $\{C_3\}$
p	0	1	0
q	0.1	0.89	0.01
r	0.1	0	0.9
s	0	0.11	0.89

Most people prefer p over q (did not consider the 10% chance of winning 5 million worth the risk of losing one million with 1% chance).

Most people prefer r over s since there is 10% chance of winning 5 million but only an additional 1% chance of getting nothing.

Mathematical proof of inconsistency

Based on the expected utility approach, we have

(i) p is preferred to q means

$$1.00U(1) > 0.89U(1) + 0.01U(0) + 0.1U(5) \qquad (i)$$

(ii) r is preferred to s means

$$0.9U(0) + 0.1U(5) > 0.89U(0) + 0.11U(1). \qquad (ii)$$

Adding $0.89[U(1) - U(0)]$ to both sides, we obtain

$$0.01U(0) + 0.89U(1) + 0.1U(5) > 1.00U(1),$$

a contradiction to (i).

The pair of gambles p and q (r and s) have 89% chance of giving the same outcome of 1 million (zero).

- If the 89% "common consequence" is disregarded, then both gambles offer the same choice: 10% chance of getting $5m$ and 1% chance of getting nothing against 11% chance of getting $1m$.

- The independence axiom overlooks the notion of complementaries. The 1% chance of getting nothing carries with it a great sense of disappointment if you were to pick that gamble and lose, knowing $1m$ would have won with 100% certainty. This feeling of disappointment is contingent on the outcome in the other portion of the gamble.

- Allais argues that it is not possible to evaluate portions of gambles or choices independently of the other choices presented, as the independence axiom requires.