MATH685X – Mathematical Models in Financial Economics

Topic 2 – Risk aversion and stochastic dominance

2.1 Risk aversion

– Certainty equivalent, and insurance premium

– Risk aversion coefficients and characterizations of utility functions

– Two-asset portfolio analysis (risky and riskfree) – discrete model

2.2 Stochastic dominance

– First order stochastic dominance

– Second order stochastic dominance

– Higher order stochastic dominance

2.1 Risk aversion

1. More is being preferred to less: $u'(w) > 0$

2. Investors' taste for risk (certainty equivalent as price of the game)

   - averse to risk (certainty equivalent $<$ mean value of game)
   - neutral toward risk (indifferent to a fair gamble)
   - seek risk (certainty equivalent $>$ mean value of game)

3. In a portfolio choice between the riskfree asset and single risky asset, the percentage of wealth invested in risky asset changes as wealth changes.

Concave function

A function $u(\cdot)$ is said to be concave if

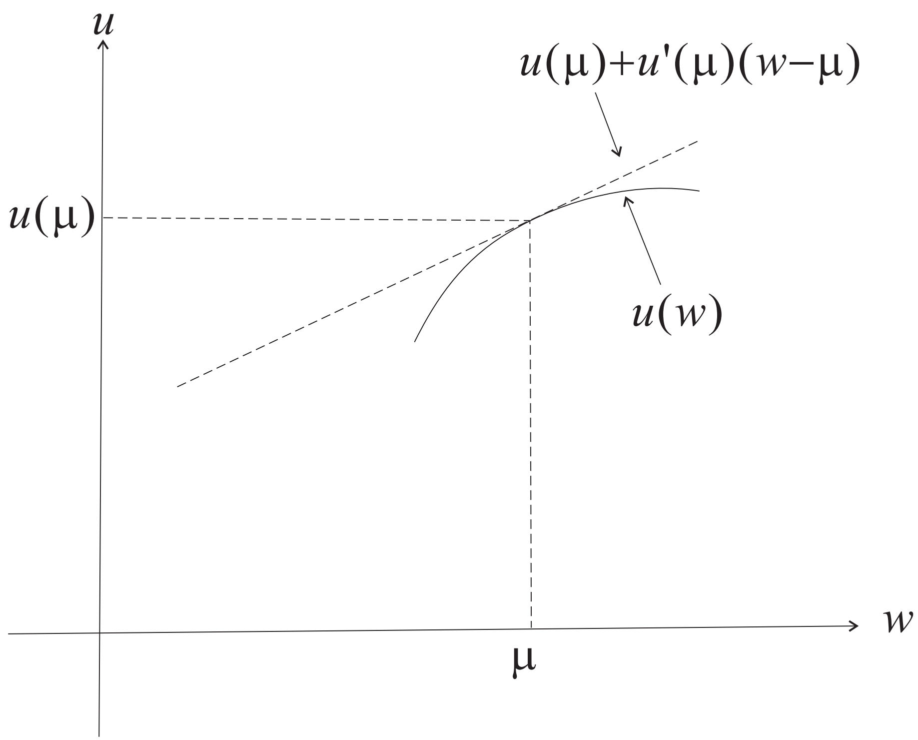
$$u(\alpha x + (1-\alpha)y) \geq \alpha u(x) + (1-\alpha)u(y)$$

for all x and y, and $\alpha \in [0,1]$. Furthermore, suppose $u(\cdot)$ is twice differentiable, $u'' \leq 0 \Leftrightarrow u$ is concave.

Jensen's inequality

Suppose $u''(w) \leq 0$ and X is a random variable, then

$$u(E[X]) \geq E[u(X)].$$



Write $E[X] = \mu$; since $u(w)$ is concave, we have

$$u(w) \leq u(\mu) + u'(\mu)(w - \mu) \quad \text{for all values of } w.$$

Replace w by X and take the expectation on each side

$$E[u(X)] \leq u(\mu) = u(E[X]).$$

Interpretation

$E[u(X)]$ represents the expected utility of the gamble associated with X. The investor prefers a sure wealth of $\mu = E[X]$ rather than playing the game, if $u''(w) \leq 0$. This indicates *risk aversion*.

Recall that the certainty equivalent c is given by

$$u(c) = E[u(X)] \leq u(\mu)$$

so that $c \leq \mu$ since u is an increasing function (more wealth is preferred to less). The certainty equivalent may be visualized as the price of the game. The investor visualizes the price to be less than its mean value.

Certainty equivalent, risk premium and risk aversion

1. The certainty equivalent of a probability distribution F is the real number $C(F)$ that satisfies

$$u(C(F)) = \int_{\mathbb{C}} u(x)\, dF(x) \stackrel{\triangle}{=} U(F).$$

2. The risk premium is the real number $q(F)$ that satisfies

$$q(F) = \mu(F) - C(F)$$

   where $\mu(F) = \int_{\mathbb{C}} x\, dF(x) =$ expected value of F.

Consider

$$u(\mu(F) - q(F)) = u(C(F)) = \int u(x)\, dF(x) \overset{\triangle}{=} U(F),$$

since $u(x)$ is strictly increasing, we have

$$q(F) \begin{array}{c} > \\ = \\ < \end{array} 0 \quad \Longleftrightarrow \quad u(\mu(F)) \begin{array}{c} > \\ = \\ < \end{array} U(F),$$

where $\mu(F)$ denotes the expected value of the distribution F and $U(F)$ is the expected utility of the distribution F.

$$\begin{cases} \text{risk-averse} \\ \text{risk-neutral} \\ \text{risk-loving} \end{cases} \text{if} \quad \begin{cases} q(F) > 0 \\ q(F) = 0 \\ q(F) < 0 \end{cases} \text{for all probability distribution } F.$$

Consider an arbitrary distribution F that is concentrated on the two outcomes x_1 and x_2

$$u(\mu(F)) = u(p_1 x_1 + p_2 x_2) \begin{array}{c} > \\ = \\ < \end{array} p_1 u(x_1) + p_2 u(x_2) = U(F)$$

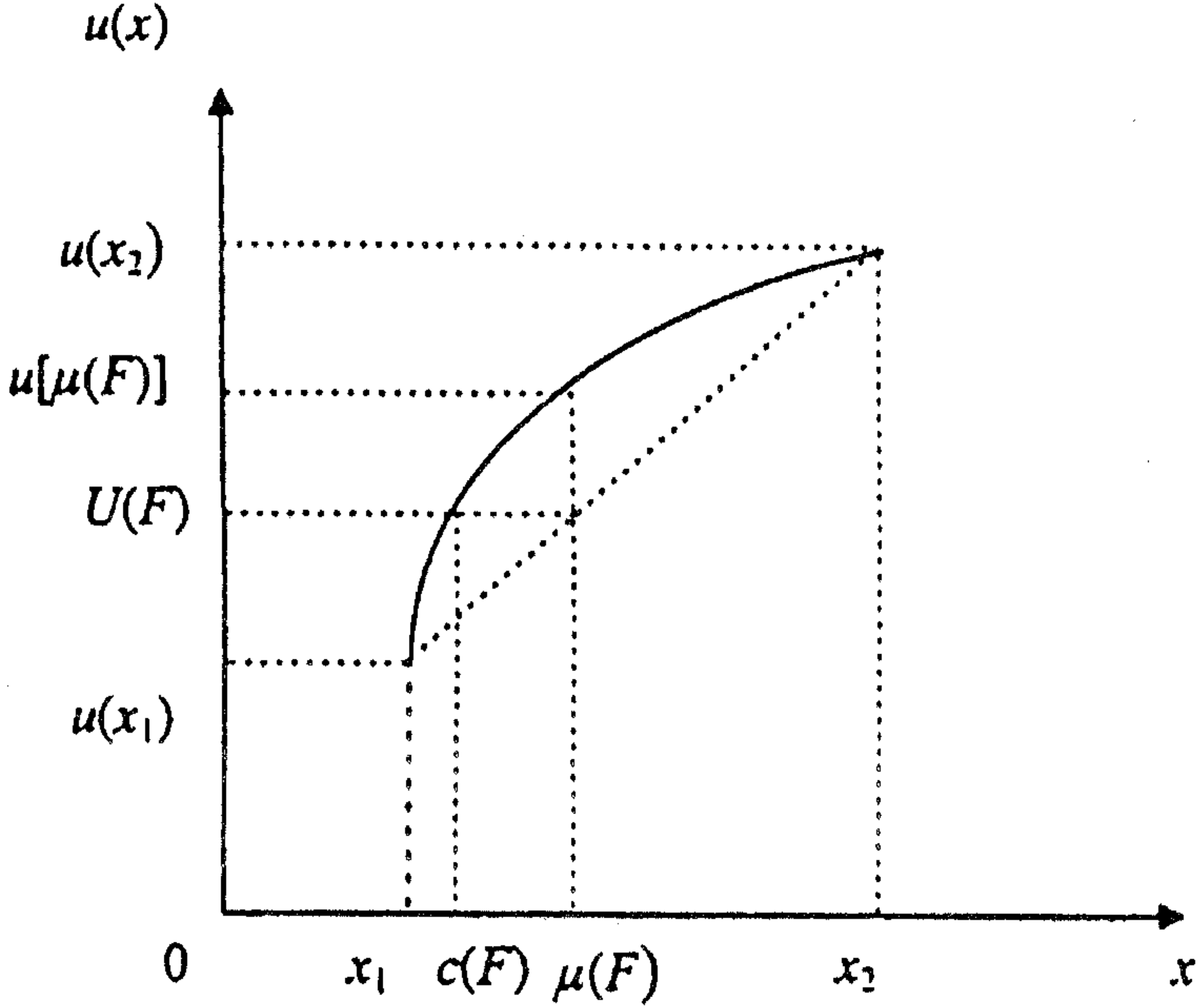
depending on whether the agent is $\left\{ \begin{array}{l} \text{risk-averse} \\ \text{risk-neutral} \\ \text{risk-loving} \end{array} \right.$.

A function $u : \mathbb{R} \to \mathbb{R}$ is $\left\{ \begin{array}{l} \text{concave} \\ \text{linear} \\ \text{convex} \end{array} \right.$

$$u(\lambda x_1 + (1-\lambda) x_2) \begin{array}{c} > \\ = \\ < \end{array} \lambda u(x_1) + (1-\lambda) u(x_2), \quad 0 \leq \lambda \leq 1.$$

Conclusion: An expected-utility maximizing agent is

$$\left\{ \begin{array}{l} \text{risk-averse} \\ \text{risk-neutral} \\ \text{risk-loving} \end{array} \right. \quad \text{if} \quad u(x) \text{ is } \left\{ \begin{array}{l} \text{concave} \\ \text{linear} \\ \text{convex} \end{array} \right. .$$



Risk-aversion ($q(F) > 0$)

Theorem

A decision maker is (globally) risk averse if and only if his von Neumann-Morgenstein utility function of wealth is strictly concave at the relevant (all) wealth levels.

Proof

$\Rightarrow$ part

Consider the simple fair gamble

$$\epsilon = \begin{cases} \lambda a & \text{with probability } 1-\lambda \\ -(1-\lambda)a & \text{with probabilty } \lambda \end{cases},$$

where $0 \leq \lambda \leq 1$. By assumption, the fair gamble is unliked so that

$$u(w) > \lambda u(w-(1-\lambda)a) + (1-\lambda)u(w+\lambda a),$$

provided that both outcomes are in the domain of u.

Since $w = \lambda(w-(1-\lambda)a) + (1-\lambda)(w+\lambda a)$ for any a and λ, so u is concave.

$\Leftarrow$ part

Let $\tilde{\epsilon}$ denote the outcome of a generic game. If it is actuarially fair, then $E[\tilde{\epsilon}] = 0$. By Jensen's inequality, if $u(\cdot)$ is strictly concave at w, then

$$E[u(w + \tilde{\epsilon})] < u(E[w + \tilde{\epsilon}]) = u(w).$$

Therefore, higher expected utility results from avoiding every gamble.

The proof for global risk aversion is the same, point by point.

Insurance premium

Individual's total initial wealth is w, and the wealth is subject to random loss Y during the period, $0 \leq Y < w$.

Let π be the insurable premium payable at time 0 that fully reimburses the loss (neglecting the time value of money).

1. If the individual decides not to buy insurance, then the expected utility is $E[u(w - Y)]$. The expectation is based on investor's *own subjective assessment* of the loss.

2. If he buys the insurance, the utility at the end of the period is $u(w - \pi)$. Note that $w - \pi$ is the sure wealth.

If the individual is risk averse [$u''(w) \leq 0$], then from Jensen's inequality (change X to $w - Y$), we obtain

$$u(w - E[Y]) \geq E[u(w - Y)].$$

The fair value of insurance premium π is determined by

$$u(w - \pi) = E[u(w - Y)]$$

so that we can deduce that $\pi \geq E[Y]$.

Suppose the higher moments of Y are negligible, it can be deduced that the maximum premium that a risk-averse individual with wealth w is willing to pay to avoid a possible loss of Y is approximately

$$\pi \approx \mu_Y + \frac{\sigma_Y^2}{2} R_A(w - \mu),$$

where $R_A(w) = -u''(w)/u'(w)$, $0 \leq Y < w$ and $\mu = E[Y] < w$. With higher $R_A(w)$, the individual is willing to pay a higher premium to avoid risk.

Proof

We start from the governing equation for π

$$u(w-\pi) = E[u(w-Y)].$$

Write $Y = \mu + zV$, where V is a random variable with zero mean. Here, z is a small perturbation parameter. We then have

$$u(w-\pi) = E[u(w-\mu-zV)]. \tag{1}$$

We are seeking the perturbation expansion of π in powers of z in the form

$$\pi = a + bz + cz^2 + \cdots$$

(i) Setting $z = 0, u(w-a) = E[u(w-\mu)] = u(w-\mu)$ so that

$$a = \mu.$$

(ii) Differentiating (1) with respect to z and setting $z = 0$,

$$-\pi'(0)u'(w - \pi) = E[-Vu'(w - \mu)] \tag{2}$$

since $E[V] = 0$ and $\pi'(0) = b$, so $b = 0$.

(iii) Differentiating (1) twice with respect to z and setting $z = 0$

$$-\pi''(0)u'(w - \pi) \;=\; E[V^2 u''(w - \mu)]$$

and observing $\text{var}(V) = E[V^2]$ since $E[V] = 0$, we obtain

$$c = -\frac{\text{var}(V)}{2}\frac{u''}{u'}\bigg|_{w-\mu}.$$

Absolute risk aversion coefficient

Define the Arrow-Pratt absolute risk aversion coefficient: $R_A(w) = -\frac{u''(w)}{u'(w)}$, we have

$$\begin{aligned}\pi &\approx \mu + \frac{R_A(w-\mu)}{2} z^2 \text{var}(V) \\ &= \mu + \frac{\sigma_Y^2}{2} R_A(w-\mu).\end{aligned}$$

$\pi - \mu \approx \frac{\sigma_Y^2}{2} R_A(w-\mu)$ is called the risk premium. For low level of risks, $\pi - \mu$ is approximately proportional to the product of variance of the loss distribution and individual's absolute risk aversion.

Ross' measure of risk aversion

Individual i is said to be strongly more risk averse then individual k if

$$\inf_w \frac{u_i''(w)}{u_k''(w)} \geq \sup_w \frac{u_i'(w)}{u_k'(w)}. \tag{i}$$

The above relation implies that for arbitrary w, we have

$$\frac{u_i''(w)}{u_k''(w)} \leq \frac{u_i'(w)}{u_k'(w)}.$$

Rearranging, we obtain

$$-\frac{u_i''(w)}{u_i'(w)} \geq -\frac{u_k''(z)}{u_k'(z)}, \tag{ii}$$

which indicates that individual i is more risk averse than individual k in the sense of Arrow-Pratt.

The following example shows that (i) is strictly strongly than (ii).

Take $u_i(w) = -e^{-aw}$ and $u_k(w) = -e^{-bw}$, where $a > b$. Obviously, i is more risk averse in the sense of Arrow-Pratt than k. However

$$\frac{u_i'(w_1)}{u_k'(w_1)} = \frac{a}{b}\,e^{-(a-b)w_1} \quad \text{and} \quad \frac{u_i''(w_2)}{u_k''(w_2)} = \frac{a^2}{b^2}\,e^{-(a-b)w_2}.$$

When $w_2 - w_1$ is sufficiently large, we may have

$$\frac{u_i''(w_2)}{u_k''(w_2)} < \frac{u_i'(w_1)}{u_i'(w_1)}$$

contracting (i). Therefore, the Ross notion of risk aversion is strictly stronger than the Arrow-Pratt measure.

Relative risk aversion coefficient

Let X be a fair game with $E[X] = 0$ and $\text{var}(X) = \sigma_X^2$. The whole wealth w is invested into the game.

Choice A	Choice B
$w + X$	w_C (with certainty)

The investor is indifferent to these two positions iff

$$E[u(w + X)] = u(w_C).$$

Note that $w_C = w-(w-w_C)$, indicating the payment of $w-w_C$ for Choice B. The difference $w - w_C$ represents the maximum amount the investor would be willing to pay in order to avoid the risk of the game.

Let q be the fraction of wealth an investor is giving up in order to avoid the gamble; then $q = \dfrac{w - w_C}{w}$ or $w_C = w(1 - q)$. Let Z be the return per dollar invested so that for a fair gamble, $E[Z] = 1$. Write $\text{var}(Z) = \sigma_Z^2$.

Suppose we invest w dollars, the return would be wZ. Expand $u(wZ)$ around w:

$$\begin{aligned} u(wZ) &= u(w) + u'(w)(wZ - w) + \frac{u''(w)}{2}(wZ - w)^2 + \cdots \\ E[u(wZ)] &= u(w) + 0 + \frac{u''(w)}{2} w^2 \sigma_Z^2 + \cdots \end{aligned}$$

since $\sigma_Z^2 = E[(Z - 1)^2]$.

On the other hand,

$$u(w_C) = u(w(1-q)) = u(w) - qwu'(w) + \cdots .$$

Equating $u(w_C)$ with $E[u(wZ)]$, we obtain

$$\frac{u''(w)}{2} w^2 \sigma_Z^2 = -u'(w)qw$$

so that

$$q = -\frac{\sigma_Z^2}{2} w \frac{u''(w)}{u'(w)}.$$

Define $R_R(w)$ = coefficient of relative risk aversion $= -w\dfrac{u''(w)}{u'(w)}$, then $q = \dfrac{w - w_C}{w}$ = percentage of risk premium $= \dfrac{\sigma_Z^2}{2} R_R(w)$.

Types of utility functions

1. *Exponential utility*

$$\begin{aligned} u(x) &= 1 - e^{-ax}, x > 0 \\ u'(x) &= ae^{-ax} \\ u''(x) &= -a^2 e^{-ax} < 0 \quad \text{(risk aversion)} \end{aligned}$$

so that $R_A(x) = a$ for all wealth level x.

2. *Power utility*

$$\begin{aligned} u(x) &= \frac{x^\alpha - 1}{\alpha}, \quad \alpha \leq 1 \\ u'(x) &= x^{\alpha-1} \\ u''(x) &= (\alpha - 1)x^{\alpha-2} \end{aligned}$$

$R_A(x) = \dfrac{1-\alpha}{x}$ and $R_R(x) = 1 - \alpha$.

3. *Logarithmic utility* (corresponds to $\alpha \to 0$ in power utility)

$$\begin{aligned} u(x) &= a \ln x + b, \quad a > 0 \\ u'(x) &= a/x \\ u''(x) &= -a/x^2 \end{aligned}$$

$R_A(x) = \frac{1}{x}$ and $R_R(x) = 1$.

Observe that

$$\lim_{\alpha \to 0} \frac{x^\alpha - 1}{\alpha} = \lim_{\alpha \to 0} \frac{(\ln x)x^\alpha}{1} = \ln x.$$

Properties of the power utility functions: $U(x) = x^{\alpha}/\alpha, \alpha \leq 1$

(i) $\alpha > 0$, aggressive utility

Consider $\alpha = 1$, corresponding to $U(x) = x$. This is the *expected value criterion*.

Recall that the strategy that maximizes the expected value bets all capital on the most favorable sector – prone to early bankruptcy.

For $\alpha = 1/2$; consider two opportunities:

(a) capital will double with a probability of 0.9 or it will go to zero with probability 0.10,

(b) capital will increase by 25% with certainty.

Since $0.9 \times \sqrt{2} > \sqrt{1.25}$, so opportunity (a) is preferred to (b). However, opportunity (a) is certain to go bankrupt.

(ii) $\alpha < 0$, conservative utility

For $\alpha = -1/2$, consider two opportunities

(a) capital quadruples in value with certainty

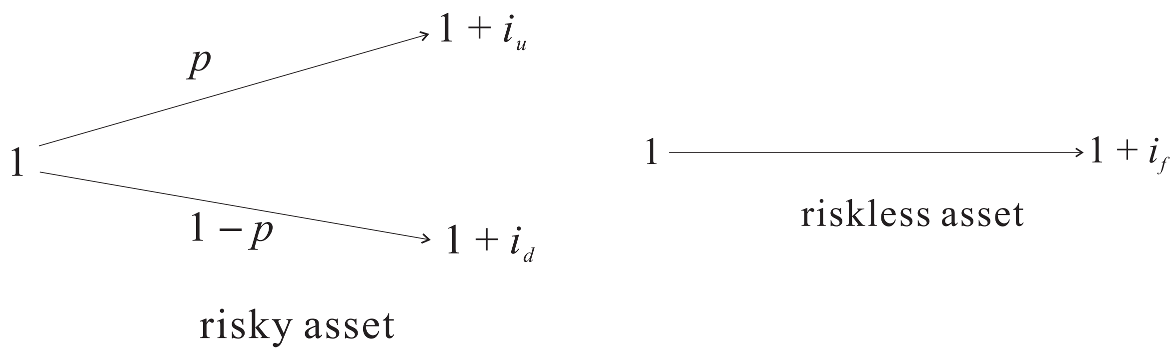
(b) with probability 0.5 capital remains constant and with probability 0.5 capital is multiplied by 10 million.

Since $-4^{-1/2} > -0.5 - 0.5(10,000,000)^{-1/2}$, opportunity (a) is preferred to (b).

Apparently, the best choice for α may be negative, but close to zero. This utility function is close to the logarithm function.

Two-asset portfolio analysis – discrete model

An investor has an initial wealth of w and can allocate funds between two assets: a risky asset and a riskless asset.



m = expected rate of return on the risky security $= pi_u + (1-p)i_d$ so that

$$p = \frac{m - i_d}{i_u - i_d}, \quad 1 - p = \frac{i_u - m}{i_u - i_d}.$$

- Impose the condition: $m > i_f$, otherwise a rational investor will never invest any positive amount in the risky asset. The risk averse investor should command an expected rate of return higher than r_f for bearing the risk.

- No arbitrage conditions: $i_u > i_f > i_d$. If otherwise, suppose $i_u > i_d > i_f$, then a riskfree profit can be secured by borrowing the riskfree asset as much as possible and use the proceeds to buy the risky asset. Also, m observes $i_u > m > i_d$ given that $0 < p < 1$.

- Let x be the fraction of initial wealth placed on the risky asset. Choose the power utility function: $\dfrac{w^\alpha}{\alpha}, 0 < \alpha < 1$.

Investor's expected utility:

$$p\frac{\{w[(1+i_f)+x(i_u-i_f)]\}^\alpha}{\alpha}+(1-p)\frac{\{w[1+i_f)+x(i_d-i_f)]\}^\alpha}{\alpha}.$$

To find the optimal strategy, we find x such that the expected utility is maximized. We differentiate the expected utility with respect to x and set it be zero.

Since the utility function is concave, the second order condition for a maximum is automatically satisfied.

Optimal proportion $\quad x^* = \dfrac{(1+i_f)(\theta-1)}{(i_u-i_f)+\theta(i_f-i_d)}$ where

$$\theta = \frac{[p(i_u - i_f)]^{1/(1-\alpha)}}{[(1-p)(i_f - i_d)]^{1/(1-\alpha)}} = \left[\left(\frac{m - i_d}{i_f - i_d}\right)\left(\frac{i_u - i_f}{i_u - m}\right)\right]^{1/(1-\alpha)}.$$

$$\theta > 1 \Leftrightarrow \frac{m - i_f}{i_f - i_d} > \frac{i_u - m}{i_u - i_f} \Leftrightarrow m > i_f$$

(i) $x^* > 0 \Longleftrightarrow \theta > 1$.

(ii) As $m \to i_f, \theta \to 1$ and $x^* \to 0$.

A risk-averse investor prefers the riskless asset if it has the same return as the expected return on the risky asset.

(iii) θ is an increasing function of m and $x^* < 1$ as long as $\theta < \dfrac{1 + i_u}{1 + i_d}$. When $x^* > 1$, the investor short sells the riskfree asset to increase his leverage.

2.2 Stochastic dominance

- Knowing the utility function, we have the full information on preference. Using the *maximum expected utility criterion*, we obtain a complete ordering of all the investments under consideration.

- What happens if we have only partial information on preferences (say, prefer more to less and/or risk aversion)?

- In *the First Order Stochastic Dominance Rule*, we only consider the class of utility functions, call $\mathbf{U}_1$, such that $u' \geq 0$. This is a very general assumption and it does not assume any specific utility function.

- Recall $E[u(x)] = \int u(x)\, dF(x) = U(F)$, where F is the probability distribution of the random variable x. We may consider expected utility value as a function of distribution on the underlying x.

Feasible set – set of all available investments under consideration.

Dominance in $\mathbf{U}_1$

Investment A dominates investment B in $\mathbf{U}_1$ if for all utility functions such that $u \in \mathbf{U}_1, E_A u(x) \geq E_B u(x)$; [equivalently, $U(F_A) \geq U(F_B)$, where F_A and F_B are the distribution function of choices A and B, respectively]; and for at least one utility function, there is a strict inequality.

- Choices among investments amount to choices on probability distributions.

Efficient set in $\mathbf{U}_1$ (collection of investments that are not being dominated)

- An investment is included in the efficient set if there is no other investment that dominates it.

- Suppose investments A and B are efficient, then neither A or B dominates the other. That is, there exists $u_1 \in \boldsymbol{U}_1$ such that $E_A u_1(x) > E_B u_1(x)$ while there exists another $u_2 \in \boldsymbol{U}_1$ such that $E_A u_2(x) < E_B u_2(x)$. Some prefer A and other prefer B (no dominance between A and B).

Inefficient set in $\mathbf{U}_1$ (being dominated)

The inefficient set includes all inefficient investments. An inefficient investment is that there is at least one investment in the efficient set that dominates it.

- There is no need for an inefficient investment to be dominated by all efficient investments. One dominance is enough to relegate an investment to the inefficient set.

- The efficient and inefficient sets form a partition of the feasible set. They are mutually exclusive and comprehensive.

- The partition into efficient and inefficient sets depends on the choice of the class of utility functions. In general, the smaller the efficient set relative to the feasible set, the better for the decision maker.

- When we have only one utility function, we have complete ordering of all investment choices. The efficient set may likely contain one element (possibly more than one if we have investments whose expected utility values tie with each other).

- Objective and subjective decisions
  The first stage provides the efficient set (objective decision) while the second state determines the optimal choice by maximizing the expected utility of an individual investor (subjective decision).

First order stochastic dominance

Two Investment Alternatives: Outcomes and Associated Probabilities

Investment A		Investment B	
Outcome	Probability	Outcome	Probability
12	1/3	11	1/3
10	1/3	9	1/3
8	1/3	7	1/3

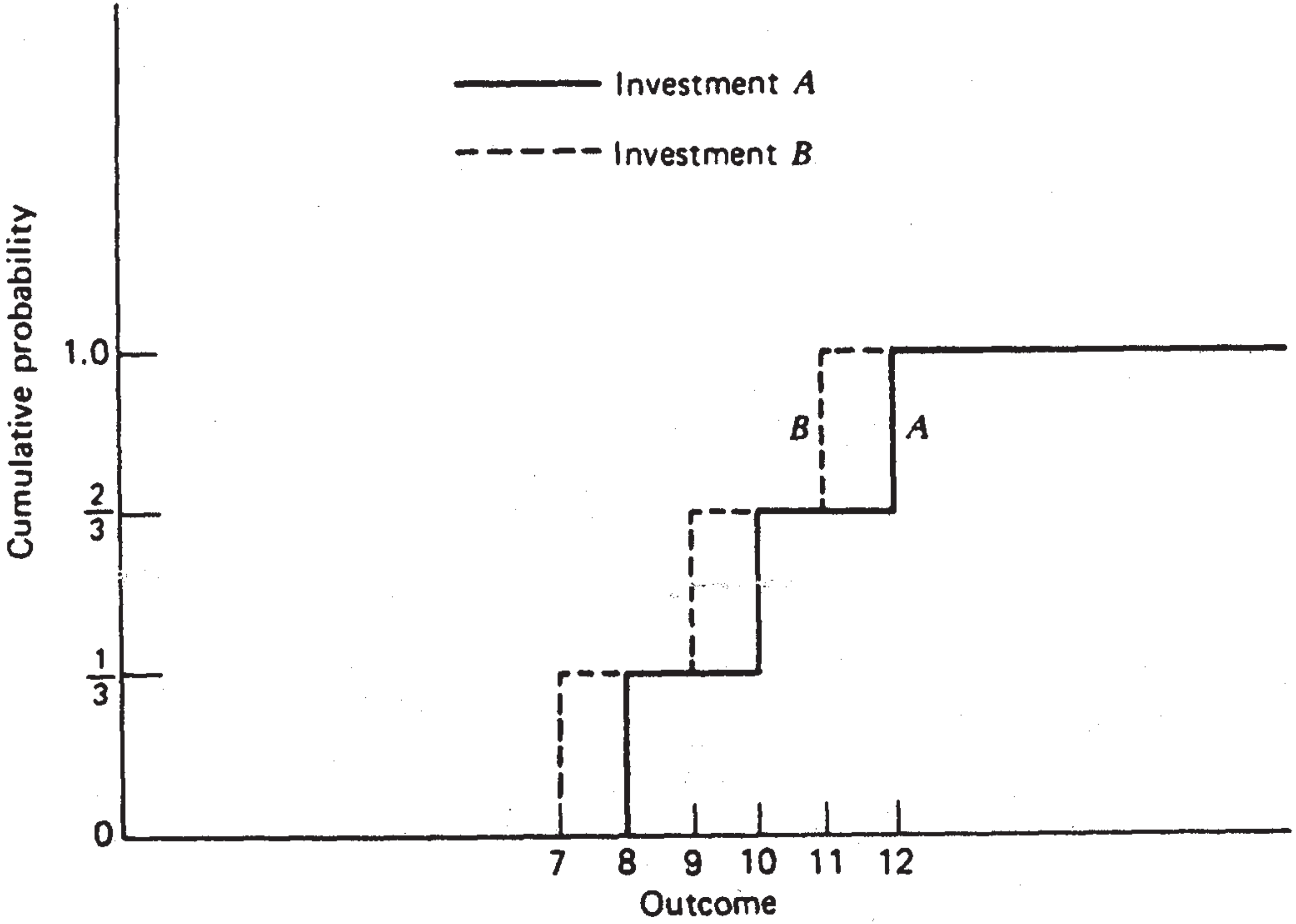
Can we argue that Investment A is better than Investment B? It is still possible that the return from investing in B is 11% but the return is only 8% from investing in A.

⋆ By looking at the cumulative probability distribution, we observe that for all returns and the odds of obtaining that return or less, B consistently has a higher or same value.

A Cumulative Probability Distribution

	Odds of Obtaining a Return Equal to or Less Than That shown in Column 1	
Return	*A*	*B*
7	0	1/3
8	1/3	1/3
9	1/3	2/3
10	2/3	2/3
11	2/3	1
12	1	1

Let F and G denote the distribution of the return of investments A and B, respectively.



Cumulative frequency function for gambles A and B.

To compare two investment choices, we examine their corresponding probability distribution, where $F_X(x) = P_r[X \leq x]$.

Definition

A probability distribution F dominates another probability distribution G according to the first-order stochastic dominance if and only if

$$F(x) \leq G(x) \quad \text{for all} \quad x \in C,$$

where C is the set of possible outcomes.

Lemma

F dominates G by FSD if and only if

$$\int_C u(x)\, dF(x) \geq \int_C u(x)\, dG(x)$$

for all strictly increasing utility functions $u(x)$.

Proof

(i) $F(x) \leq G(x) \Rightarrow A \underset{\text{FSD}}{\geq} B$

Let a and b be the smallest and largest values that F and G can take on. Consider

$$\int_a^b u(x)\,d[F(x)-G(x)] = \underbrace{u(x)[F(x)-G(x)]_a^b}_{\substack{\text{zero since } F(a)=G(a)=0 \\ \text{and } F(b)=G(b)=1}} - \int_a^b u'(x)[F(x)-G(x)]\,dx;$$

given $F(x) \geq G(x)$ and $u^1(x) > 0$ so

$$-\int_a^b u'(x)[F(x)-G(x)]\,dx \geq 0.$$

Thus, $F(x) \leq G(x) \quad \Rightarrow \quad \int_C u(x)\,dF(x) \geq \int_C u(x)\,dG(x).$

(ii) $A \underset{\text{FSD}}{\geq} B \Rightarrow F(x) \leq G(x)$

We prove by contradiction. Assume the contrary, suppose there exists x_0 such that $F(x_0) > G(x_0)$. Since distribution functions are right continuous, there exists an interval $[x_0, c]$ such that $F(x) > G(x)$ for $x \in [x_0, c]$. Define the utility function

$$
\begin{aligned}
u(x) &= \int_a^x \mathbf{1}_{[x_0,c]}(t)\,dt, \text{ where} \\
\mathbf{1}_{[x_0,c]}(t) &= \begin{cases} 1 & t \in [x_0, c] \\ 0 & \text{otherwise} \end{cases}.
\end{aligned}
$$

Note that $u(x)$ is continuous and monotonically increasing, and

$$u'(x) = \mathbf{1}_{[x_0,c]}(x) \geq 0.$$

Now, consider

$$\begin{aligned} E_A u(x) - E_B u(x) &= -\int_a^b u'(x)[F(x) - G(x)]\,dx \\ &= -\int_{x_0}^c [F(x) - G(x)]\,dx < 0, \end{aligned}$$

a contradiction is encountered.

Properties of efficient and inefficient sets under FSD

- An inefficient investment would not be required to be dominated by all efficient investments. In order to be relegated into the inefficient set, it is sufficient to have one investment that dominates the inefficient investment.

- Dominance or non-dominance within the inefficient set is irrelevant since all investments included in this set are inferior.

- An investment within the inefficient set cannot dominate an investment within the efficient set since if such dominance were to exist then the latter would not be included in the efficient set.

- The distribution functions of all investments within FSD efficient set must intercept. If otherwise, one distribution would dominate the other, a contradiction.

Second order stochastic dominance

Two Investment Alternatives: Outcomes and Associated Probabilities

Investment A		Investment B	
Outcome	Probability	Outcome	Probability
6	1/4	5	1/4
8	1/4	9	1/4
10	1/4	10	1/4
12	1/4	12	1/4

If both investments turn out the worst, the investor obtains 6% from A and only 5% from B. If the second worst return occurs, the investor obtains 8% from A rather than 9% from B. If he is risk averse, then he should be willing to lose 1% in return at a higher level of return in order to obtain an extra 1% at a lower return level. If risk aversion is assumed, then A is preferred to B.

Definition

A probability distribution F dominates another probability distribution G according to the second order stochastic dominance if and only if for all $x \in C$

$$\int_{-\infty}^{x} F(y)\, dy \leq \int_{-\infty}^{x} G(y)\, dy.$$

Theorem

F dominates G by SSD if and only if

$$\int_C u(x)\, dF(x) \geq \int_C u(x)\, dG(x)$$

for all increasing and concave utility functions $u(x)$.

The Sum of the Cumulative Probability Distribution

Return	Cumulative Probability A	Cumulative Probability B	Sum of Cumulative Probability A	Sum of Cumulative Probability B
4	0	0	0	0
5	$\boxed{0}$	1/4	0	1/4
6	$\boxed{1/4}$	1/4	1/4	1/2
7	1/4	1/4	1/2	3/4
8	1/2	$\boxed{1/4}$	1	1
9	1/2	$\boxed{1/2}$	1 1/2	1 1/2
10	3/4	3/4	2 1/4	2 1/4
11	3/4	3/4	3	3
12	1	1	4	4

According to SSD, A is preferred to B since the sum of cumulative probability for A is always less than or equal to that for B.

Write $I_A(x) = \int_{-\infty}^{x} F_A(y)\, dy$

$$\begin{aligned} I_A(8.6) &= I_A(8) + F_A(8) \times 0.6 = 1 + \frac{1}{2} \times 0.6 = 1.3 \\ I_A(13.5) &= I_A(12) + F_A(12) \times 1.5 = 4 + 1 \times 1.5 = 5.5. \end{aligned}$$

Proof (only the "if" part is shown)

$$\begin{aligned}
\int_a^b u(x)\,d[F(x)-G(x)] &= -\int_a^b u'(x)[F(x)-G(x)]\,dx \\
&= -u'(x)\int_a^x [F(y)-G(y)]\,dy\Big|_a^b \\
&\quad + \int_a^b u''(x)\int_a^x [F(y)-G(y)]\,dydx \\
&= -u'(b)\int_a^b [F(y)-G(y)]\,dy \\
&\quad + \int_a^b u''(x)\int_a^x [F(y)-G(y)]\,dydx.
\end{aligned}$$

Given that $u'(b) > 0$ and $u''(x) < 0$,

$$\int_C u(x)\,dF(x) \geq \int_C u(x)\,dG(x)$$

if

$$\int_a^x [F(y)-G(y)]\,dy \leq 0,$$

for all x.

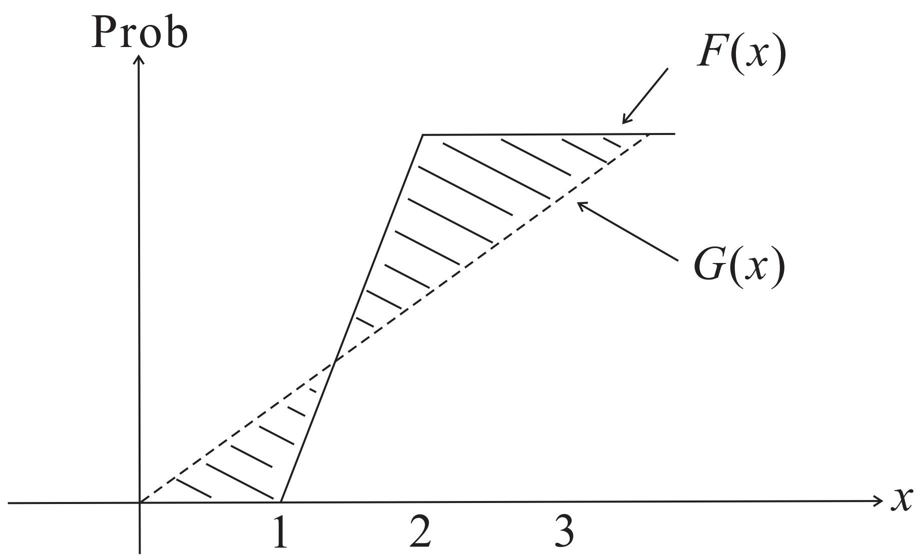
Example

$$F(x) = \begin{cases} 0 & \text{if } x < 1 \\ x - 1 & \text{if } 1 \leq x \leq 2 \\ 1 & \text{if } x \geq 2 \end{cases}, \qquad G(x) = \begin{cases} 0 & \text{if } x < 0 \\ x/3 & \text{if } 0 \leq x \leq 3 \\ 1 & \text{if } x \geq 3 \end{cases}.$$

F dominates G by SSD since

$$\int_{-\infty}^{x} F(y)\,dy \leq \int_{-\infty}^{x} G(y)\,dy.$$

$F(x)$ is seen to be more concentrated (less dispersed).



In this example, $F(x) \leq G(x)$ is not valid for all x.

Sufficient rules and necessary rules for second order stochastic dominance

Sufficient rule 1: FSD rule is sufficient for SSD

Proof: If F dominates G by FSD, then $F(x) \leq G(x), \forall x$.

This implies $\int_a^x [G(y) - F(y)]\, dy \geq 0$.

Remark

The inefficient set according to FSD is a subset of that of SSD.

Proof:

Suppose G lies in the inefficient set of FSD, say, it is dominated by F by FSD. Then F dominates G by SSD so that G must lie in the inefficient set of SSD.

Sufficient rule 2:

$\text{Min}_F(x) > \text{Max}_G(x)$ is a sufficient rule for SSD. Note that $\text{Min}_F(x) > \text{Max}_G(x)$ is a very strong requirement.

Example

F		G	
x	$p(x)$	x	$p(x)$
5	1/2	2	3/4
10	1/2	4	1/4

$\text{Min}_F(x) = 5 \geq \text{Max}_G(x) = 4$. Note that $F(x) = 0$ for $x \leq \text{min}_F(x)$ while $G(x) = 1$ for $x \geq \text{max}_G(x)$. Since $F(x)$ and $G(x)$ are non-decreasing functions in x, so $F(x) \leq G(x)$. Hence, F dominates G.

$$\text{Min}_F(x) \geq \text{Max}_G(x) \Rightarrow \text{FSD} \Rightarrow \text{SSD} \Rightarrow E_F u(x) \geq E_G u(x), \forall u \in \mathbf{U}_2.$$

Necessary rule 1 (Geometric means)

Given a risky project with the distribution $(x_i, p_i), i = 1, \cdots, n$, the geometric mean, $\overline{X}_{geo}$, is defined as

$$\overline{X}_{geo} = x_1^{p_1} \cdots x_n^{p_n} = \prod_{i=1}^{n} x_i^{p_i}, x_i \geq 0.$$

Taking logarithm on both sides

$$\ln \overline{X}_{geo} = \Sigma p_i \ln x_i = E[\ln X].$$

$\boxed{\overline{X}_{geo}(F) \geq \overline{X}_{geo}(G) \text{ is a necessary condition for dominance of } F \text{ over } G \text{ by SSD.}}$

Proof

Suppose F dominates G by SSD, we have

$$E_F u(x) \geq E_G u(x), \text{ for all } u \in \mathbf{U}_2.$$

Since $\ln x = u(x) \in \mathbf{U}_2$,

$$E_F \ln x = \ln \overline{X}_{geo}(F) \geq E_G \ln x = \ln \overline{X}_{geo}(G);$$

we obtain $\ln \overline{X}_{geo}(F) \geq \ln \overline{X}_{geo}(G)$.

Since the logarithm function is an increasing function, we deduce $\overline{X}_{geo}(F) \geq \overline{X}_{geo}(G)$.

Therefore, F dominates G by SSD $\Rightarrow \overline{X}_{geo}(F) \geq \overline{X}_{geo}(G)$.

Necessary rule 2 (left-tail rule)

Suppose F dominates G by SSD, then

$$\text{Min}_F(x) \geq \text{Min}_G(x),$$

that is, the left tail of G must be "thicker".

Proof by contradiction: Suppose $\text{Min}_F(x) < \text{Min}_G(x)$, and write $x_k = \text{Min}_F(x)$. At x_k, G will still be zero but F will be positive. Observe that

$$\int_{-\infty}^{x_k} [G(y) - F(y)]\, dy = \int_{-\infty}^{x_k} [0 - F(y)]\, dy < 0,$$

implying that F is not dominated by G by SSD. Hence, if F dominates G, then $\text{Min}_F(x) \geq \text{Min}_G(x)$.