MATH685X – Mathematical Models in Financial Economics

Topic 3 – Portfolio choice under utility maximization

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Two-asset portfolio analysis – single risky asset and riskfree asset

Demand function for risky asset and absolute risk aversion

A risky asset and the riskfree asset are available for an investor to choose to invest with a given amount of initial wealth W_0.

Let a denote the number of units of risky asset

b denote the number of units of riskfree asset

$r_s =$ return from the risky asset in state s

$R =$ return from the riskfree asset.

Return from the portfolio (a, b) in state s

$$W_s(a, b) = r_s a + Rb.$$

Let the price of the risky asset be q and the price of the riskfree asset be the numeraire.

The investor's budget constraint is $W_0 = aq + b$, where W_0 is the initial wealth of the investor; $b = W_0 - qa$. We assume *no short selling*[*] so that $a > 0$.

Assume a finite set of states $S = \{1, \cdots, s\}$ with probability distribution $p = (p_1, \cdots, p_s)$.

[*]It will be shown later that positive risk premium of the risky asset $E[\widetilde{r}] - r_f$ would induce positive holding of the risky investment.

The optimization problem of an expected utility-maximizing investor:

Choose (a, b) to maximize

$$\sum_{s \in S} p_s u(W_s(a, b))$$

subject to $qa + b = W_0$.

Alternatively, choose a to maximize $\sum_{s \in S} p_s u(RW_0 + (r_s - Rq)a)$.

The first order condition is

$$\sum_{s \in S} p_s u'(RW_0 + (r_s - Rq)a)[r_s - Rq] = 0.$$

If the investor is risk-averse, $u''(\cdot)$ is strictly negative, then the (sufficient) second order condition is

$$\sum_{s \in S} p_s u''(RW_0 + (r_s - Rq)a)(r_s - Rq)^2 < 0.$$

A solution to the first order condition would yield a maximum if the investor is risk averse.

Define $a(W_0) =$ demand function for the risky asset, which is the *optimal solution* to the portfolio choice problem.

Question

Is the demand function $a(W_0)$ for the number of units of the risky asset increasing or decreasing in initial wealth W_0?

- Relation between absolute risk aversion coefficient R_a and demand function $a(W_0)$.

Lemma

$$\begin{aligned} a'(W_0) > 0 &\quad \text{if} \quad R_a'(W) < 0 \\ a'(W_0) = 0 &\quad \text{if} \quad R_a'(W) = 0 \\ a'(W_0) < 0 &\quad \text{if} \quad R_a'(W) > 0 \end{aligned}$$

Proof

Consider the derivative with respect to W_0 of the first order condition:

$$\sum_{s\in S} p_s u''(RW_0 + (r_s - Rq)a(W_0))(r_s - Rq)R$$
$$+ \sum_{s\in S} p_s u''(RW_0 + (r_s - Rq)a(W_0))(r_s - Rq)^2 a'(W_0) = 0.$$

Solving for $a'(W_0)$:

$$a'(W_0) = -\left[\sum_{s\in S} p_s u''(RW_0 + (r_s - Rq)a(W_0))(r_s - Rq)^2\right]^{-1} R$$
$$\left[\sum_{s\in S} p_s u''(RW_0 + (r_s - Rq)a(W_0))(r_s - Rq)\right].$$

If the investor is risk-averse, $u''(\cdot) < 0$. Hence, the sign of $a'(W_0)$ should be the same as the sign of

$$\sum_{s\in S} p_s u''(RW_0 + (r_s - Rq)a(W_0)) \underbrace{(r_s - Rq)}_{\text{can be positive or negative}}.$$

Recall the definition: $R_a(W) = -\dfrac{u''(W)}{u'(W)}$; the above term can be expressed as

$$-\sum_{s\in S} p_s u'(RW_0 + (r_s - Rq)a(W_0))(r_s - Rq)$$
$$R_a(RW_0 + (r_s - Rq)a(W_0)).$$

We would like to establish that for all $s \in S$

$$(r_s - Rq)R_a(RW_0) \lesseqqgtr (r_s - Rq)R_a(RW_0 + (r_s - Rq)a(W_0))$$

if and only if $R_a'(x) \gtreqqless 0$.

For convenience, we write

$$y = r_s - Rq, \quad x_0 = RW_0, \quad \lambda = a(W_0) > 0.$$

Taking the case $R_a'(x) < 0$, we have

$$yR_a(x_0) > yR_a(x_0 + \lambda y) \quad \Leftrightarrow \quad R_a'(x) < 0.$$

To continue on, again considering $R_a'(x) < 0$, the sign of $a'(W_0)$ depends on the sign of

$$
\begin{aligned}
& -\sum_{s\in S} p_s u'(RW_0 + (r_s - Rq)a(W_0)) \\
& \qquad R_a(RW_0 + (r_s - Rq)a(W_0))(r_s - Rq) \\
& > -R_a(RW_0) \sum_{s\in S} p_s u'(RW_0 + (r_s - Rq)a(W_0))(r_s - Rq) \\
& = \; 0 \text{ [due to the first order condition]}
\end{aligned}
$$

Hence, we obtain $a'(W_0) > 0$.

- When the absolute risk aversion is a decreasing function, investors would invest more on risky asset when the initial wealth level is higher. In this case, the risky asset is *a normal good*. Otherwise, $R_a' > 0$ would make the risky asset be an *inferior good*. As a consequence, investor's utility function satisfying $R_a' < 0$ seems to be a more plausible hypothesis.

Elasticity of demand of risky asset and relative risk aversion coefficient

Define the elasticity of demand of the risky asset with respect to the wealth by

$$\eta = \frac{\frac{da}{a}}{\frac{dW_0}{W_0}}.$$

For a risk averse investor whose utility function is an increasing, strictly concave utility function, and three times differentiable, we have

$$\eta \lesseqqgtr 1 \quad \text{if} \quad R_R(w) \gtreqqless 0.$$

Proof

Recall that the random terminal wealth $\widetilde{W} = W_0(1+r_f)+a(\widetilde{r}-r_f)$, where $\widetilde{r}$ is the random rate of return of the risky asset. We write

$$\eta = 1 + \frac{\left(\frac{da}{dW_0}\right) W_0 - a}{a}.$$

From previous result on $\dfrac{da}{dW_0}$, we have

$$\begin{aligned}
\eta &= 1 + \frac{W_0(1+r_f)E[u''(\widetilde{W})(\tilde{r}-r_f)] + aE[u''(\widetilde{W})(\tilde{r}-r_f)^2]}{aE[-u''(\widetilde{W})(\tilde{r}-r_f)^2]} \\
&= 1 + \frac{E[u''(\widetilde{W})\{W_0(1+r_f)+a(\tilde{r}-r_f)\}(\tilde{r}-r_f)]}{aE[-u''(\widetilde{W})(\tilde{r}-r_f)^2]} \\
&= 1 + \frac{E[u''(\widetilde{W})\widetilde{W}(\tilde{r}-r_f)]}{aE[-u''(\widetilde{W})(\tilde{r}-r_f)^2]} \\
&= 1 + \frac{E[R_R(\widetilde{W})u'(W)(\tilde{r}-r_f)]}{aE[u''(\widetilde{W})(\tilde{r}-r_f)^2]}.
\end{aligned}$$

Since $u''(W) < 0$ for any strictly concave utility function, we have

$$\mathrm{sign}\,(\eta - 1) = -\mathrm{sign}\,(E[R_R(\widetilde{W})u'(\widetilde{W})(\tilde{r}-r_f)]).$$

We consider the case where $R_R(W)$ is an increasing function. For $a > 0$, note that

$$\begin{array}{rcl} R_R(\widetilde{W}) & = & R_R(W_0(1+r_f)+a(\tilde{r}-r_f)) \\ & & \begin{cases} \geq R_R(W_0(1+r_f)) & \text{when} \quad \tilde{r} \geq r_f \\ < R_R(W_0(1+r_f)) & \text{when} \quad \tilde{r} < r_f. \end{cases} \end{array}$$

By the rule of conditional probability, we have

$$\begin{array}{rl} & E[R_R(\widetilde{W})u'(\widetilde{W})(\tilde{r}-r_f)] \\ = & E[R_R(\widetilde{W})u'(\widetilde{W})(\tilde{r}-r_f)|\tilde{r}-r_f \geq 0]\text{Prob}\,(\tilde{r}-r_f \geq 0) \\ & +\ E[R_R(\widetilde{W})u'(\widetilde{W})(\tilde{r}-r_f)|\tilde{r}-r_f < 0]\text{Prob}\,(\tilde{r}-r_f < 0). \end{array}$$

The first term is always positive. How about the sign of the second term?

Consider

$$E[R_R(\widetilde{W})u'(\widetilde{W})(\tilde{r}-r_f)|(\tilde{r}-r_f)<0],$$

since $u'(\widetilde{W})>0$ and $R_R(\widetilde{W})>0$, we have

$$R_R(\widetilde{W})(\tilde{r}-r_f)>R_R(W_0(1+r_f))(\tilde{r}-r_f) \text{ for } \tilde{r}-r_f<0.$$

We observe

$$\begin{aligned} & E[R_R(\widetilde{W})u'(\widetilde{W})(\tilde{r}-r_f)] \\ > \ & R_R(W_0(1+r_f))E[u'(\widetilde{W})(\tilde{r}-r_f)]=0 \end{aligned}$$

due to the first order condition. Combining all these results, we obtain $\eta<1$.

- For most investors, it may be reasonable to assume decreasing absolute risk aversion but neutral on relative risk aversion.

In deriving the above results, we have assumed $a > 0$. Under what condition that $a > 0$ in the optimal portfolio.

Lemma: If the investor is risk averse (with strictly concave utility), then

(i) $a > 0 \Leftrightarrow E[\tilde{r}] > r_f$;

(ii) $a = 0 \Leftrightarrow E[\tilde{r}] = r_f$;

(iii) $a < 0 \Leftrightarrow E[\tilde{r}] < r_f$.

Proof: We write

$$\overline{u}(a) = E[u(\widetilde{W})] = E[u(W_0(1 + r_f) + a(\tilde{r} - r_f))].$$

Since u is strictly concave, we have

$$\overline{u}''(a) = E[u''(\widetilde{W})(\tilde{r} - r_f)^2] < 0.$$

Hence, $\overline{u}$ is strictly concave and has a maximum value a^* given by the first order condition: $\overline{u}'(a^*) = 0$. On the other hand, we have

$$\overline{u}'(0) = u'(W_0(1 + r_f))(E[\widetilde{r}] - r_f).$$

Since $u' > 0$, so $\overline{u}'(0)$ and $E[\widetilde{r}] - r_f$ have the same sign. Therefore,

$$E[\widetilde{r}] - r_f > 0 \Leftrightarrow \overline{u}'(0) > 0.$$

Lastly, since $\overline{u}'(a)$ is decreasing in a and $\overline{u}'(a^*) = 0$, so

$$\overline{u}(0) > 0 \Leftrightarrow a^* > 0.$$

What is the minimum level of risk premium required to induce the investor to invest α portion of his wealth in the risky asset?

- In subsequent discussions, unless otherwise specified, we assume positive risk premium for any risky asset.

If αW_0 is invested in the risky asset, then

$$\widetilde{W}(\alpha) = W_0(1 + r_f) + +\alpha W_0(\tilde{r} - r_f).$$

Treating $E[u(\widetilde{W}(\alpha))]$ as a function of α, the necessary and sufficient condition for investing at least α portion of wealth in the risky asset is

$$\begin{aligned} & \frac{d}{d\alpha} E[u(\widetilde{W}(\alpha))] \geq 0 \\ \Leftrightarrow \quad & E[u'((W_0(1 + r_f) + \alpha W_0(\tilde{r} - r_f))(\tilde{r} - r_f))] \geq 0. \qquad (1) \end{aligned}$$

- We perform the Taylor series expansion of $u'(W(\alpha))$ at $W_0(1+r_f)$ and assume a small risk of the risky asset so that terms involving $E[(\tilde{r}-r_f)^2]$ and higher order terms are neglected.

- As an approximation to inequality (1), we obtain

$$\begin{aligned} E[u'(W(\alpha))(\tilde{r}-r_f)] &\approx u'(W_0(1+r_f))E[\tilde{r}-r_f] \\ &\quad + u''(W_0(1+r_f))\alpha W_0 E[(\tilde{r}-r_f)^2] \geq 0 \end{aligned}$$

  so that the risky premium has to satisfy

$$\begin{aligned} E[\tilde{r}]-r_f &\geq -\frac{\alpha W_0 u''(W_0(1+r_f))}{u'(W_0(1+r_f))}E[(\tilde{r}-r_f)^2] \\ &= \alpha R_A(W_0(1+r_f))E[(\tilde{r}-r_f)^2]. \end{aligned}$$

Demand function a^* and riskfree return R_f

Let $\widetilde{R}$ denote the random return of the risky asset. Recall the first order condition: $E[u'(W_0R_f + a^*(\tilde{R} - R_f))(\tilde{R} - R_f)] = 0$, which solves for the demand function a^* in the optimal portfolio. Recall $R_f = 1 + r_f$ and $\widetilde{R} = 1 + \tilde{r}$.

Differentiating the first order condition with respect to R_f, we obtain

$$-E[u'(\widetilde{W})] + E[u''(\widetilde{W})](W_0 - a^*) + \frac{da^*}{dR_f}E[u''(\widetilde{W})(\tilde{R} - r_f)] = 0$$

so that

$$\begin{aligned}\frac{da^*}{dR_f} &= \frac{E[u'(\widetilde{W})] - E[u''(\widetilde{W})](W_0 - a^*)}{E[u''(\widetilde{W})(\tilde{R} - R_f)]} \\ &= \frac{E[u'(\widetilde{W})]}{E[u''(\widetilde{W})(\tilde{R} - R_f)]} + \frac{W_0 - a^*}{R_f}\frac{da^*}{dW_0}.\end{aligned}$$

- The first term is strictly negative due to absolute risk aversion.

- For increasing absolute risk aversion, $\frac{da^*}{dW_0} < 0$; so the second term is also negative.

- Overall speaking, $\frac{da^*}{dR_f} < 0$ if the risk premium of the risky asset is positive and absolute risk aversion is increasing.

- The first term expresses the substitution effect of the risky asset. When R_f increases, the riskfree asset becomes more attractive leading to a decrease in a^*.

Comparison of two investors with different levels of absolute risk aversion

Let $u_i(z)$ and $u_k(z)$ denote the twice-differentiable, increasing and concave utility function of investor i and investor k, respectively.

Lemma

There exists a strictly increasing and concave function G such that

$$u_i = G(u_k)$$

if and only if $R_A^i(z) \geq R_A^k(z)$, for all z.

Proof

(i) $R_A^i(z) \geq R_A^k(z) \Rightarrow$ existence of G with the given properties

Define the function $G(y) = u_i(u_k^{-1}(y))$, the inverse of u_k exists since u_k is increasing. Now,

$$u_i'(z) = G'(u_k(z))u_k'(z).$$

Since $u_i'(z) > 0$ and $u_k'(z) > 0$, so G is strictly increasing in its domain of definition. Furthermore,

$$u_i''(z) = G''(u_k(z))[u_k'(z)]^2 + G'(u_k(z))u_k''(z)$$

so that

$$R_A^i(z) = -\frac{G''(u_k(z))}{G'(u_k(z))}u_k'(z) + R_A^k(z). \tag{2}$$

Since $R'_A(z) \geq R_A^{(k)}(z)$ so G is concave.

(ii) Relation (2) dictates that if G is strictly increasing and concave, we have $R_A^i(z) \geq R_A^k(z)$.

Lemma

Suppose $R_A^i(z) \geq R_A^k(z)$ for all z. For the same amount of investment in the risky asset, investor i demands a higher risk premium than investor k.

Proof

The following first order condition

$$E[u_k'(W_0(1+r_f)+M(\widetilde{r}-r_f))(\widetilde{r}-r_f)] = 0$$

gives the minimum risk premium required for investing M units of the risky asset. In order to show that investor i requires a higher risk premium it suffices to show that

$$E[u_i'(\widetilde{W}_M)(\widetilde{r}-r_f)] < 0,$$

where $\widetilde{W}_M = W_0(1+r_f)+M(\widetilde{r}-r_f)$.

Given that $u_i(z) = G(u_k(z))$, then

$$\begin{aligned}
& E[u_i'(\widetilde{W}_M)(\tilde{r} - r_f)] \\
= \;& E[G'(u_k(\widetilde{W}_M)u_k'(\widetilde{W}_M)(\tilde{r} - r_f))] \\
= \;& E[G'(u_k(\widetilde{W}_M)u_k'(\widetilde{W}_M)(\tilde{r} - r_f)|\tilde{r} - r_f \geq 0]P_r[\tilde{r} - r_f \geq 0] \\
& + \; E[G'(u_k(\widetilde{W}_M)u_k'(\widetilde{W}_M)(\tilde{r} - r_f)|\tilde{r} - r_f < 0]P_r[\tilde{r} - r_f < 0].
\end{aligned}$$

For the first term, we observe $\widetilde{W}_M \geq W_0(1 + r_f)$ when $\tilde{r} - r_f \geq 0$ so that $G'(u_k(\widetilde{W}_M)) \leq G'(u_k(W_0(1 + r_f))$ since $G'' \leq 0$ and $u_k' > 0$. Therefore,

$$\begin{aligned}
& E[G'(u_k(\widetilde{W}_M)u_k'(\widetilde{W}_M)(\tilde{r} - r_f))|\tilde{r} - r_f \geq 0] \\
\leq \;& G'(u_k(W_0(1 + r_f))E[u_k'(\widetilde{W}_M)(\tilde{r} - r_f)|\tilde{r} - r_f].
\end{aligned}$$

In a similar manner, we obtain

$$
\begin{aligned}
& E[G'(u_k(\widetilde{W}_M)u_k'(\widetilde{W}_M)(\widetilde{r}-r_f))|\widetilde{r}-r_f<0] \\
\leq \; & G'(u_k(W_0(1+r_f)))E[u_k'(\widetilde{W}_M)(\widetilde{r}-r_f)|\widetilde{r}-r_f<0].
\end{aligned}
$$

Combining all the results, we obtain

$$
\begin{aligned}
& E[u_k'(\widetilde{W}_M)(\widetilde{r}-r_f)] \\
\leq \; & G'(u_k(W_0(1+r_f)))E[u_k'(\widetilde{W}_M)(\widetilde{r}-r_f)] = 0.
\end{aligned}
$$

Remark

The investor with stronger risk aversion invests less on the risky asset.

3.2 Portfolios with multiple risky assets and riskfree asset

Let α denote the proportion of initial wealth W_0 invested in the riskfree asset and let b_j denote the proportion of the remainder $W_0(1-\alpha)$ invested in the j^{th} risky asset, $j = 1, 2, \cdots, N$.

A risk averse investor solves

$$\max_{\{\alpha, b_j, \lambda\}} E[u(\widetilde{W})] + \lambda \left(1 - \sum_{j=1}^{N} b_j\right),$$

where the random terminal wealth is given by

$$\widetilde{W} = W_0 \left[1 + \alpha r_f + (1-\alpha) \sum_{j=1}^{N} b_j \widetilde{r}_j\right].$$

Alternatively, we may write a_j as the number of units of the j^{th} risky asset held in the portfolio, with $\boldsymbol{a} = (a_1 \quad a_2 \cdots a_N)^T$.

Write $\tilde{\boldsymbol{r}} - r_f \mathbf{1} = (\tilde{r}_1 - r_f \quad \tilde{r}_2 - r_f \ \cdots \tilde{r}_N - r_f)^T$ so that

$$\widetilde{W} = W_0(1 + r_f) + \boldsymbol{a}^T(\tilde{\boldsymbol{r}} - r_f \mathbf{1}).$$

Lemma

The optimal portfolio admits $\boldsymbol{a} = \mathbf{0}$ if $E[\tilde{r}_j] = r_f, j = 1, 2, \cdots, N$.

Proof

Suppose all risky premia are zero for the risky assets, then

$$E[\widetilde{W}] = W_0(1 + r_f).$$

This is the expected terminal portfolio value when $\boldsymbol{a} = \mathbf{0}$. On the other hand, we apply the Jensen inequality to obtain

$$E[u(\widetilde{W})] \leq u(E[\widetilde{W}]) = u(W_0(1 + r_f))$$

for any other portfolio choice. Therefore, $\boldsymbol{a} = \mathbf{0}$ is optimal.

3.2 Portfolios with multiple risky assets and riskfree asset

Consider a financial market with the riskfree asset and several risky assets, suppose the utility function satisfies

$$-\frac{u'(z)}{u''(z)} = a + bz, \quad \text{valid for all } z,$$

then the optimal portfolio at different wealth levels is given by the combination of the riskfree asset and market fund consisting of the risky assets. The relative proportions of risky assets in the market fund remain the same, irrespective of W_0.

Remark

The choices of utility functions include

(i) quadratic utility
(ii) log utility: $a = 0$
(iii) exponential utility: $b = 0$
(iv) power utility: $a = 0$.

Let W_0 be the initial wealth, then the wealth amount $a_j^*(W_0)$ of risky asset j in the optimal portfolio satisfies

$$a_j^*(W_0) = \alpha_j h(W_0) \quad j = 1, 2, \cdots, n, \text{ and } \alpha_j \text{ is independent of } W_0,$$

so that the relative proportion b_j is given by

$$b_j = \frac{a_j^*(W_0)}{\sum_{k=1}^{n} a_k^*(W_0)} = \frac{\alpha_j}{\sum_{k=1}^{n} \alpha_k}, \quad \text{independent of } W_0.$$

Lemma

1. Suppose the utility function satisfies

$$-\frac{u'(W_1)}{u''(W_1)} = a + bW_1, \quad \text{for all } W_1,$$

then the optimal portfolio is given by

$$a_j^*(W_0) = \alpha_j(a + bRW_0), \quad j = 1, 2, \cdots, n, \tag{A}$$

where $R = 1 + r_f$ and $\widetilde{R}_j = 1 + \widetilde{r}_j, j = 1, 2, \cdots, n$.

2. Define $V(W_0) = \max_{\{a_j\}_{j=1}^n} E[u(\widetilde{W}_1)]$,

where $\widetilde{W}_1 = \left(W_0 - \sum_{j=1}^m a_j\right) R + \sum_{j=1}^n a_j \widetilde{R}_j = RW_0 + \sum_{j=1}^n a_j(\widetilde{R}_j - R)$, then

$$-\frac{V'(W_0)}{V''(W_0)} = \frac{a}{R} + bW_0, \text{ for all initial wealth } W_0. \tag{B}$$

Proof

Assume that

$$a_j^*(W_0) = \alpha_j(W_0)(a + bRW_0)$$

where $\alpha_j(W_0), \quad j = 1, \cdots, n$, is a differentiable function.

For any value of W_0, from the optimality property of $a_j^*(W_0)$, we deduce that

$$\frac{\partial E[u(\widetilde{W}_1)]}{\partial a_k}$$

$$= E\left[u'\left(\underbrace{RW_0 + \sum_{j=1}^{N}(\tilde{R}_j - R)\alpha_j(W_0)(a + bRW_0)}_{\widetilde{W}_1}\right)(\tilde{R}_k - R)\right] = 0, \qquad (1)$$

$$k = 1, 2, \cdots, N.$$

Next, we differentiate eq (1) with respect to W_0. First, we observe that

$$\begin{aligned}\frac{d\widetilde{W}_1}{dW_0} &= R\left[1+\sum_{j=1}^{N}(\tilde{R}_j-R)\alpha_j(W_0)b\right]\\ &\quad+\sum_{j=1}^{N}\frac{d\alpha_j(W_0)}{dW_0}(\tilde{R}_j-R)(a+bRW_0).\end{aligned}$$

Hence, for the k^{th} component, we obtain

$$\begin{aligned}&\sum_{j=1}^{N}E[u''(\widetilde{W}_1)(\tilde{R}_j-R)(\tilde{R}_k-R)(a+bRW_0)]\frac{d\alpha_j(W_0)}{dW_0}\\ = &-E\left[u''(\widetilde{W}_1)(R_k-R)R\left[1+\sum_{j=1}^{N}(\tilde{R}_j-R)\alpha_j(W_0)b\right]\right],\\ &\qquad k=1,2,\cdots,N.\end{aligned}$$

In matrix form, we have

$$E\left[\begin{pmatrix} (\widetilde{R}_1-R)^2 & \cdots & (\widetilde{R}_1-R)(\widetilde{R}_N-R) \\ (\widetilde{R}_2-R)(\widetilde{R}_1-R) & \cdots & (\widetilde{R}_2-R)(\widetilde{R}_N-R) \\ \vdots & \ddots & \vdots \\ (\widetilde{R}_N-R)(\widetilde{R}_1-R) & \cdots & (\widetilde{R}_N-R)^2 \end{pmatrix} u''(\widetilde{W}_1)(a+bRW_0)\right]\begin{bmatrix} \frac{d\alpha_1(W_0)}{dW_0} \\ \frac{d\alpha_2(W_0)}{dW_0} \\ \vdots \\ \frac{d\alpha_N(W_0)}{dW_0} \end{bmatrix}$$

$$= -\begin{bmatrix} E\left\{u''(\widetilde{W}_1)(\widetilde{R}_1-R)R\left[1+\sum_{j=1}^N(\widetilde{R}_j-R)\alpha_j(W_0)b\right]\right\} \\ E\left\{u''(\widetilde{W}_1)(\widetilde{R}_2-R)R\left[1+\sum_{j=1}^N(\widetilde{R}_j-R)\alpha_j(W_0)b\right]\right\} \\ \vdots \\ E\left\{u''(\widetilde{W}_1)(\widetilde{R}_N-R)R\left[1+\sum_{j=1}^N(\widetilde{R}_j-R)\alpha_j(W_0)b\right]\right\} \end{bmatrix} \qquad (2)$$

From the assumption

$$-\frac{u'(W_1)}{u''(W_1)} = a + bW_1,$$

we obtain

$$\begin{aligned}
u''(\widetilde{W}_1) &= -\frac{u'(\widetilde{W}_1)}{a + b\left[RW_0 + \sum_{j=1}^{N}(\tilde{R}_j - R)\alpha_j(W_0)(a + bRW_0)\right]} \\
&= -\frac{u'(\widetilde{W}_1)}{(a + bRW_0)\left[1 + \sum_{j=1}^{N}(\tilde{R}_j - R)\alpha_j(W_0)b\right]}. \qquad (3)
\end{aligned}$$

observe that

$$\begin{aligned}
&-u''(\widetilde{W}_1)(\tilde{R}_k - R)R\left[1 + \sum_{j=1}^{N}(\tilde{R}_j - R)\alpha_j(W_0)b\right] \\
&= u'(\widetilde{W}_1)(\tilde{R}_k - R)\frac{R}{a + bRW_0}, \quad k = 1, 2, \cdots N. \qquad (4)
\end{aligned}$$

Recall the first order condition:

$$E[u'(\widetilde{W}_1)(\tilde{R}_k - R)] = 0 \quad k = 1, 2, \cdots, N.$$

Combining eqs (1) and (4), and knowing that the column vector on the right hand side of eq (2) is a zero vector, we deduce that

$$\frac{d\alpha_j}{dW_0}(W_0) = 0, \quad j = 1, 2, \cdots, n,$$

provided that the matrix in eq (2) is non-singular. We then have

$$\alpha_j(W_0) = \alpha_j, \quad \text{independent of } W_0.$$

Now, $a_j^* = \alpha_j(a + bW_0), a > 0$. When $b = 0, a_j^*$ is independent of the initial wealth W_0.

The portfolio $(a_1^*(W_0) \cdots a_N^*(W_0))$ is said to be *partially separated* if $a_j^*(W_0)/a_{j'}^*(W_0)$ is independent of W_0, and it is said to be *completely separated* if a_j^* is independent of W_0.

To show eq (B), we start from the optimality condition on

$$a_j^*(W_0) = \alpha_j(a + bRW_0)$$

to obtain

$$\begin{aligned} V(W_0) &= E\left[u\left(RW_0 + \sum_{j=1}^{n}(\tilde{R}_j - R)\alpha_j(a + bRW_0)\right)\right] \\ &= E\left[u\left(\left(1 + \sum_{j=1}^{n}(\tilde{R}_j - R)\alpha_j b\right) RW_0 + \sum_{j=1}^{N}(\tilde{R}_j - R)\alpha_j a\right)\right]. \end{aligned}$$

Differentiate $V(W_0)$ twice with respect to W_0

$$\begin{aligned} V'(W_0) &= E\left[u'(\widetilde{W}_1)R\left(1+\sum_{j=1}^{n}(\tilde{R}_j-R)\alpha_j b\right)\right] \\ V''(W_0) &= E\left[u''(\widetilde{W}_1)R^2\left(1+\sum_{j=1}^{n}(\tilde{R}_j-R)\alpha_j b\right)^2\right]. \end{aligned}$$

Relating $u''(\widetilde{W}_1)$ with $u'(\widetilde{W}_1)$ using eq (3), we obtain

$$V''(W_0) = -\frac{RE\left[u'(\widetilde{W}_1)R\left(1+\sum_{j=1}^{n}(\tilde{R}_j-R)\alpha_j b\right)\right]}{a+bRW_0} = -\frac{R}{a+bRW_0}V'(W_0).$$

Combining the results

$$-\frac{V'(W_0)}{V''(W_0)} = \frac{a}{R}+bW_0.$$

Two-fund monetary separation

This refers to the phenomenon that an investor always chooses to hold the same portfolio of risky assets and only change the mix between that portfolio and the riskfree asset for differing levels of initial wealth. This particular portfolio of risky assets represents a risky mutual fund. For any wealth level, the portfolio problem consists only in determining the amount of money to be invested in the mutual fund.

Theorem

A necessary and sufficient condition for the two mutual funds monetary separation property to hold is

$$u'(x) = (a + bx)^c \quad \text{or} \quad u'(x) = ae^{bx},$$

where the parameters a, b and c are chosen such that u is increasing and concave.

Formulation for finding the optimal portfolio

Let α be the weight of the riskfree asset so that the wealth invested in risky assets is $W_0(1-\alpha)$. Let b_j be the weight of risky asset j within $W_0(1-\alpha)$ so that $\sum_{j=1}^{n} b_j = 1$. The random wealth $\widetilde{W}$ at the end of the investment period is

$$\max_{\{\alpha, b_j\}} E[u(\widetilde{W})]$$

where

$$\begin{aligned} \widetilde{W} &= W_0\alpha(1+r_f) + \sum_{j=1}^{n} W_0(1-\alpha)b_j(1+\widetilde{r}_j) \\ &= W_0\left[1+\alpha r_f + (1-\alpha)\sum_{j=1}^{n} b_j\widetilde{r}_j\right] \end{aligned}$$

subject to

$$\sum_{j=1}^{n} b_j = 1.$$

Lagrangian formulation

$$\max_{\{\alpha, b_j, \lambda\}} E[u(\widetilde{W})] + \lambda \left(1 - \sum_{j=1}^{n} b_j\right).$$

First order conditions give

$$E\left[u'(\widetilde{W})W_0\left(r_f - \sum_{j=1}^{n} b_j \widetilde{r}_j\right)\right] = 0 \tag{1}$$

$$E\left[u'(\widetilde{W})W_0(1-\alpha)\widetilde{r}_j\right] = \lambda, \quad j = 1, 2, \cdots, n, \tag{2}$$

$$\sum_{j=1}^{n} b_j = 1. \tag{3}$$

From eq. (1), $\quad E[u'(\widetilde{W})r_f] = E\left[u'(\widetilde{W}) \sum_{j=1}^{n} b_j \widetilde{r}_j\right],$

and from eqs (2) and (3), we have

$$\lambda = E\left[u'(\widetilde{W})W_0(1-\alpha) \sum_{j=1}^{n} b_j \widetilde{r}_j\right].$$

Substituting into eq (2)

$$E[u'(\widetilde{W})\tilde{r}_j] = E\left[u'(\widetilde{W})\sum_{j=1}^{n} b_j\tilde{r}_j\right], \quad j = 1, 2, \cdots, n,$$

and using eq (1), we obtain

$$E[u'(\widetilde{W})(\tilde{r}_j - r_f)] = 0$$

or equivalently,

$$E\left[u'\left(W_0\left[1 + r_f + (1-\alpha)\sum_{\ell=1}^{n} b_\ell(\tilde{r}_\ell - r_f)\right]\right)(\tilde{r}_j - r_f)\right] = 0,$$
$$j = 1, 2, \cdots, n. \tag{4}$$

Exponential utility

Consider $u'(z) = Ae^{-az}, \quad a > 0$, substituting into eq. (4)

$$E\left[A\exp\left(-a\left\{W_0\left[(1+r_f)+(1-\alpha)\sum_{\ell=1}^{n}b_\ell(\widetilde{r}_\ell - r_f)\right]\right\}\right)(\widetilde{r}_j - r_f)\right] = 0$$

and since $A\exp(-aW_0(1+r_f))$ is non-random, we have

$$E\left[e^{-a\sum_{\ell=1}^{n}W_0(1-\alpha)b_\ell(\widetilde{r}_\ell - r_f)}(\widetilde{r}_j - r_f)\right] = 0, \quad j = 1,2,\cdots,n. \tag{5a}$$

For another initial wealth W_0', we have similar result

$$E\left[e^{-a\sum_{\ell=1}^{n}W_0'(1-\alpha')b_\ell'(\widetilde{r}_\ell - r_f)}(\widetilde{r}_j - r_f)\right] = 0, \quad j = 1,2,\cdots,n. \tag{5b}$$

Suppose we **postulate** that the solution to the system of equations

$$E\left[e^{-a\sum_{\ell=1}^{n}\beta_\ell(\widetilde{r}_\ell - r_f)}(\widetilde{r}_j - r_f)\right] = 0, \quad j = 1, 2, \cdots, n,$$

is unique, then by comparing eqs (5a,b), we obtain

$$W_0(1-\alpha)b_\ell = W_0'(1-\alpha')b_\ell'.$$

Summing ℓ from 1 to n, we obtain

$$W_0(1-\alpha) = W_0'(1-\alpha'),$$

hence

$$b_\ell = b_\ell', \quad \ell = 1, 2, \cdots, n.$$

The total wealth amount $W_0(1-\alpha)$ invested in risky assets and the wealth amount in each asset are independent of W_0.

Remark

This reveals that fund F is a combination of two efficient portfolios.

$$\begin{aligned} \boldsymbol{v} &= (3.652 \quad 3.583 \quad 7.248 \quad 0.874 \quad 7.706)^T \\ &\quad - 10(0.141 \quad 0.401 \quad 0.452 \quad 0.166 \quad 0.440)^T \\ &= (2.242 \quad -0.427 \quad 2.728 \quad -0.786 \quad 3.306)^T. \end{aligned}$$

After normalization, $\boldsymbol{w} = (0.317 \quad -0.060 \quad 0.386 \quad -0.111 \quad 0.468)^T$. Investors seeking efficient portfolios need only to invest in this master fund of risky assets and in the risk free asset.

3.3 Logarithm utility and long term portfolio growth

Suppose there is an investment opportunity that the investor will either double her investment or return nothing. The probability of the favorable outcome is p. Suppose the investor has an initial capital of X_0, and she can repeat this investment many times. How much should she invest at each time in order to maximize the long-term growth of capital?

Statement of the problem

Let α be the proportion of capital invested during each play. The investor would like to find the optimal value of α which maximizes the long-term growth. The possible proportional changes are given by

$$\begin{cases} 1+\alpha & \text{if outcome is favorable} \\ 1-\alpha & \text{if outcome is unfavorable} \end{cases}, \quad 0 \leq \alpha \leq 1.$$

General formulation:-

Let X_k represent the capital after the k^{th} trial, then

$$X_k = R_k X_{k-1}$$

where R_k is the random return variable.

We assume that all R_k's have *identical probability distribution* and they are *mutually independent*. The capital at the end of n trials is

$$X_n = R_n R_{n-1} \cdots R_2 R_1 X_0.$$

Taking logarithm on both sides

$$\ln X_n = \ln X_0 + \sum_{k=1}^{n} \ln R_k$$

or

$$\ln \left(\frac{X_n}{X_0}\right)^{1/n} = \frac{1}{n} \sum_{k=1}^{n} \ln R_k.$$

Since the random variables $\ln R_k$ are independent and have identical probability distribution, by the law of large numbers, we have

$$\frac{1}{n} \sum_{k=1}^{n} \ln R_k \longrightarrow E[\ln R_1].$$

Remark

Since the expected value of $\ln R_k$ is independent of k, so we simply consider $E[\ln R_1]$. Suppose we write $m = E(\ln R_1)$, we have

$$\left(\frac{X_n}{X_0}\right)^{1/n} \longrightarrow e^m \quad \text{or} \quad X_n \longrightarrow X_0 e^{mn}.$$

For large n, the capital grows (roughly) exponentially with n at a rate m. Here, e^m is the growth factor for each investment period.

Log utility form

$$m + \ln X_0 = E[\ln R_1] + \ln X_0 = E[\ln R_1 X_0] = E[\ln X_1].$$

If we define the log utility form: $U(x) = \ln x$, then the problem of maximizing the growth rate m is equivalent to maximizing the expected utility $E[U(X_1)]$.

Essentially, we may treat the investment growth problem as a single-period model. The single-period maximization of log utility guarantees the maximum growth rate in the long run.

Back to the investment strategy problem, how to find the optimal value of α such that the growth factor is maximized:

$$m = E[\ln R_1] = p \ln(1 + \alpha) + (1 - p) \ln(1 - \alpha).$$

Setting $\dfrac{dm}{d\alpha} = 0$, we obtain

$$p(1 - \alpha) - (1 - p)(1 + \alpha) = 0$$

giving $\alpha = 2p - 1$.

Suppose we require $\alpha \geq 0$, then the existence of the above solution implicitly requires $p \geq 0.5$.

What happen when $p < 0.5$, the value for α for optimal growth is given by $\alpha = 0$?

Lesson learnt If the game is unfavorable to the player, then he should stay away from the game.

Example (volatility pumping)

Stock: In each period, its value either doubles or reduces by half.

riskless asset: just retain its value.

How to use these two instruments in combination to achieve growth?

$$\text{Return vector } \boldsymbol{R} = \begin{cases} \begin{pmatrix} \frac{1}{2} & 1 \end{pmatrix} & \text{if stock price goes down} \\ \begin{pmatrix} 2 & 1 \end{pmatrix} & \text{if stock price goes up} \end{cases}.$$

Strategy:

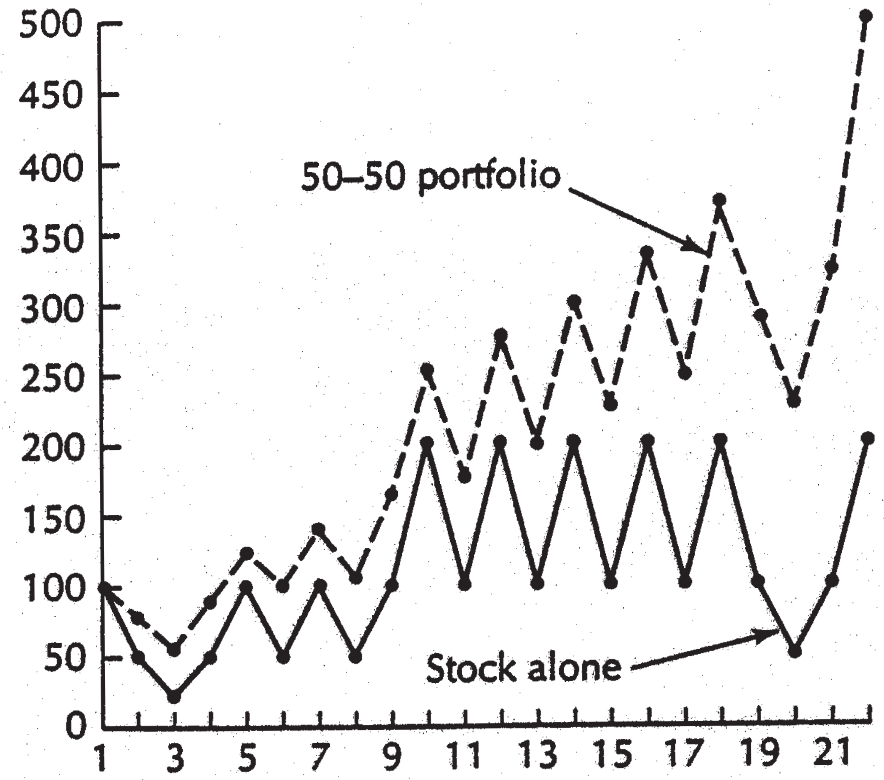
Invest *one half* of the capital in each asset for every period. Do the rebalancing at the beginning of each period so that one half of the capital is invested in each asset.

The expected growth rate

$$m \;=\; \underset{\substack{\uparrow \\ \text{prob of doubling}}}{\frac{1}{2}} \ln\left(\frac{1}{2}+1\right) + \underset{\substack{\uparrow \\ \text{prob of halving}}}{\frac{1}{2}} \ln\left(\frac{1}{2}+\frac{1}{4}\right) \approx 0.059.$$

We obtain $e^m \approx 1.0607$, so the gain on the portfolio is about 6% per period.

Remark This strategy follows the dictum of "buy low and sell high" by the process of rebalancing.



Combination of 50-50 **portfolio of risky stock and riskless asset gives an enhanced growth.**

$\boxed{\text{Example}}$ (pumping two stocks)

Both assets either double or halve in value over each period with probability 1/2; and the price moves are independent. Suppose we invest one half of the capital in each asset, and rebalance at the end of each period. The expected growth rate of the portfolio is found to be

$$m = \frac{1}{4}\ln 2 + \frac{1}{2}\ln\frac{5}{4} + \frac{1}{4}\ln\frac{1}{2} = \frac{1}{2}\ln\frac{5}{4} = 0.1116,$$

so that $e^m = \sqrt{\frac{5}{4}} = 1.118$. This gives an 11.8% growth rate for each period.

Remark

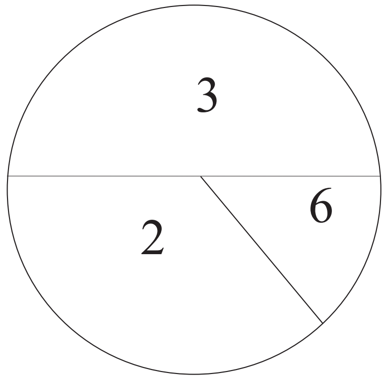
Advantage of the index tracking fund, say, Dow Jones Industrial Average. The index automatically

(i) exercises some form of volatility pumping due to stock splitting,

(ii) get rids of the weaker performers periodically.

Investment wheel

The numbers shown are the payoffs for one-dollar investment on that sector.



1. Top sector: paying 3 to 1, though the area is 1/2 of the whole wheel (favorable odds).

2. Lower left sector: paying only 2 to 1 for an area of 1/3 of wheel (unfavorable odds).

3. Lower right sector: paying 6 to 1 for an area of 1/6 of the wheel (even odds).

Aggressive strategy

Invest all money in the top sector. This produces the highest single-period expected return. This is too risky for long-term investment! Why? The investor *goes broke* half of the time and cannot continue with later spins.

Fixed proportion strategy

Prescribe wealth proportions to each sector; apportion current wealth among the sectors as bets at each spin.

$$(\alpha_1, \alpha_2, \alpha_3) \text{ where } \alpha_i \geq 0 \quad \text{and} \quad \alpha_1 + \alpha_2 + \alpha_3 \leq 1.$$

α_1: top sector

α_2: lower left sector

α_3: lower right sector

If "top" occurs, $R(\omega_1) = 1 + 2\alpha_1 - \alpha_2 - \alpha_3$.

If "bottom left" occurs, $R(\omega_2) = 1 - \alpha_1 + \alpha_2 - \alpha_3$.

If "bottom right" occurs, $R(\omega_3) = 1 - \alpha_1 - \alpha_2 + 5\alpha_3$.

We have

$$m = \frac{1}{2}\ln(1+2\alpha_1-\alpha_2-\alpha_3)+\frac{1}{3}\ln(1-\alpha_1+\alpha_2-\alpha_3)+\frac{1}{6}\ln(1-\alpha_1-\alpha_2+5\alpha_3).$$

To maximize m, we compute $\frac{\partial m}{\partial \alpha_i}, i = 1, 2, 3$ and set them be zero:

$$\frac{2}{2(1+2\alpha_1-\alpha_2-\alpha_3)} - \frac{1}{3(1-\alpha_1+\alpha_2-\alpha_3)} - \frac{1}{6(1-\alpha_1-\alpha_2+5\alpha_3)} = 0$$
$$\frac{-1}{2(1+2\alpha_1-\alpha_2-\alpha_3)} + \frac{1}{3(1-\alpha_1+\alpha_2-\alpha_3)} - \frac{1}{6(1-\alpha_1-\alpha_2+5\alpha_3)} = 0$$
$$\frac{-1}{2(1+2\alpha_1-\alpha_2-\alpha_3)} - \frac{1}{3(1-\alpha_1+\alpha_2-\alpha_3)} + \frac{5}{6(1-\alpha_1-\alpha_2+5\alpha_3)} = 0.$$

There is a whole family of optimal solutions, and it can be shown that they all give the same value for m.

(i) $\alpha_1 = 1/2, \alpha_2 = 1/3, \alpha_3 = 1/6$

One should invest in every sector of the wheel, and the bet proportions are equal to the probabilities of occurrence.

$$m = \frac{1}{2} \ln \frac{3}{2} + \frac{1}{3} \ln \frac{2}{3} + \frac{1}{6} \ln 1 = \frac{1}{6} \ln \frac{3}{2}$$

so $e^m \approx 1.06991$ (a growth rate of about 7%).

Remark: Betting on the unfavorable sector is like buying insurance.

(ii) $\alpha_1 = 5/18, \alpha_2 = 0$ and $\alpha_3 = 1/18$.

Nothing is invested on the unfavorable sector.

Log utility and growth function

Let $\boldsymbol{w}_i = (w_{i1} \cdots w_{in})^T$ be the weight vector of holding n risky securities at the i^{th} period, where weight is defined in terms of wealth. Write the random return vector at the i^{th} period as $\boldsymbol{R}_i = (R_{i1} \cdots R_{in})^T$. Here, R_{ij} is the random return of holding the j^{th} security after the i^{th} play.

Write S_n as the total return of the portfolio after n periods:

$$S_n = \prod_{i=1}^{n} \boldsymbol{w}_i \cdot \boldsymbol{R}_i.$$

Define $B = \{\boldsymbol{w} \in \mathbb{R}^n : \mathbf{1} \cdot \boldsymbol{w} = 1 \text{ and } \boldsymbol{w} \geq \mathbf{0}\}$, where $\mathbf{1} = (1 \cdots 1)^T$. This represents a trading strategy that does not allow short selling. When the successive games are identical, we may drop the dependence on i.

Single-period growth function

Based on the *log-utility* criterion, we define the *growth function* by

$$W(\boldsymbol{w}; F) = E[\ln(\boldsymbol{w} \cdot \boldsymbol{R})],$$

where $F(\boldsymbol{R})$ is the distribution function of the stochastic return vector $\boldsymbol{R}$. The growth function is seen to be a function of the *trading strategy* $\boldsymbol{w}$ together with dependence on F. The optimal growth function is defined by

$$W^*(F) = \max_{\boldsymbol{w} \in B} W(\boldsymbol{w}; F).$$

Betting wheel revisited

Let the payoff upon the occurrence of the i^{th} event (denoted by ω_i, which corresponds to the pointer landing on the i^{th} sector) be $(0 \cdots a_i \cdot 0)^T$ with probability p_i. That is, $\boldsymbol{R}(\omega_i) = (0 \cdots a_i \cdot 0)^T$. Take the earlier example, the return vector is given by

$$\begin{aligned}
\boldsymbol{R}(\omega_1) &= (3 \quad 0 \quad 0)^T \\
\boldsymbol{R}(\omega_2) &= (0 \quad 2 \quad 0)^T \\
\boldsymbol{R}(\omega_3) &= (0 \quad 0 \quad 6)^T.
\end{aligned}$$

$\omega_1 =$ top sector, $\omega_2 =$ bottom left sector, $\omega_3 =$ bottom right sector.

For this betting wheel game, the gambler betting on the i^{th} sector (equivalent to investment on security i) is paid a_i if the pointer lands on the i^{th} sector and loses the whole bet if otherwise.

Suppose the gambler chooses the weights $\boldsymbol{w} = (w_1 \cdots w_n)$ as the betting strategy with $\displaystyle\sum_{i=1}^{n} w_i = 1$, then

$$\begin{aligned} W(\boldsymbol{w}; F) &= \sum_{i=1}^{n} p_i \ln(\boldsymbol{w} \cdot \boldsymbol{R}(\omega_i)) = \sum_{i=1}^{n} p_i \ln w_i a_i \\ &= \sum_{i=1}^{n} p_i \ln \frac{w_i}{p_i} + \sum_{i=1}^{n} p_i \ln p_i + \sum_{i=1}^{n} p_i \ln a_i, \end{aligned}$$

where the last two terms are known quantities.

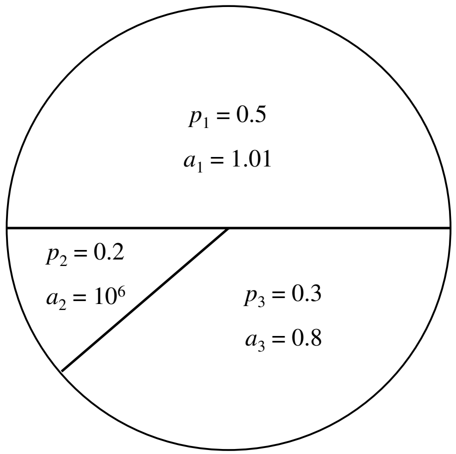
Using the inequality: $\ln x \leq x - 1$ for $x \geq 0$, with equality holds when $x = 1$, we have

$$\sum_{i=1}^{n} p_i \ln \frac{w_i}{p_i} \leq \sum_{i=1}^{n} p_i \left(\frac{w_i}{p_i} - 1 \right) = \sum_{i=1}^{n} w_i - \sum_{i=1}^{n} p_i = 0.$$

The upper bound of $\displaystyle\sum_{i=1}^{n} p_i \ln \frac{w_i}{p_i}$ is zero, and this maximum value is achieved when we choose $w_i = p_i$ for all i. Hence, an optimal portfolio is $w_i^* = p_i$, for all i.

Remarks

1. Consider the following example



   Though the return of the second sector is highly favorable, we still apportion only $w_2 = 0.2$ to this sector, given that our goal is to achieve the long-term growth. However, if we would like to maximize the one-period return, we should place all bets in the second sector.

2. Normally, we should expect $a_i > 1$ for all i. However, the above result remains valid even if $0 < a_i \leq 1$.

Lemmas

1. For a given $\boldsymbol{w}$, $W(\boldsymbol{w}; F)$ is a linear function of the distribution function F. This follows directly from the linearity property of the expectation integral.
2. For a given function $F, W(\boldsymbol{w}; F)$ is a concave function on $\boldsymbol{w}$; and $W^*(F)$ is a convex function on F.

Proof

From the concave property of the logarithmic function, we have

$$\ln(\lambda \boldsymbol{w}_1 + (1-\lambda)\boldsymbol{w}_2) \cdot \boldsymbol{R} \geq \lambda \ln \boldsymbol{w}_1 \cdot \boldsymbol{R} + (1-\lambda) \ln \boldsymbol{w}_2 \cdot \boldsymbol{R}.$$

We then take the expectation on both side and obtain the concave property on $\boldsymbol{w}$.

To show the convexity property of $\boldsymbol{w}^*$, we consider two distribution functions F_1 and F_2. Let the corresponding optimal weights be denoted by $\boldsymbol{w}^*(F_1)$ and $\boldsymbol{w}^*(F_2)$.

Write $\boldsymbol{w}^*(\lambda F_1 + (1-\lambda)F_2)$ as the optimal weight vector corresponding to $\lambda F_1 + (1-\lambda)F_2$. Now, we consider

$$
\begin{aligned}
& W^*(\lambda F_1 + (1-\lambda)F_2) \\
= \;& W(\boldsymbol{w}^*(\lambda F_1 + (1-\lambda)F_2); \lambda F_1 + (1-\lambda)F_2) \\
= \;& \lambda W(\boldsymbol{w}^*(\lambda F_1 + (1-\lambda)F_2); F_1) + (1-\lambda)W(\boldsymbol{w}^*(\lambda F_1 + (1-\lambda)F_2; F_2) \\
\leq \;& \lambda W(\boldsymbol{w}^*(F_1); F_1) + (1-\lambda)W(\boldsymbol{w}^*(F_2); F_2) \\
= \;& \lambda W^*(F_1) + (1-\lambda)W^*(F_2).
\end{aligned}
$$

The inequality holds since $\boldsymbol{w}^*(F_1)$ and $\boldsymbol{w}^*(F_2)$ are the weights that lead to the maximization of $W(\boldsymbol{w}; F_1)$ and $W(\boldsymbol{w}; F_2)$, respectively.

Lemma

The log-utility optimal portfolio $\boldsymbol{w}^*$ that maximizes the growth function $W(\boldsymbol{w}; F)$ satisfies

$$E\left(\frac{R_j}{\boldsymbol{w}^* \cdot \boldsymbol{R}}\right) \leq 1.$$

Proof

Note that $W(\boldsymbol{w}; F)$ is a concave function on $\boldsymbol{w}$, and the domain of definition of $\boldsymbol{w}$ is a simplex. The necessary and sufficient condition for $\boldsymbol{w}^*$ to be an optimal solution is that the directional derivative of $W(\boldsymbol{w})$ at $\boldsymbol{w}^*$ along any path must be non-positive.

Let $\boldsymbol{w}_\lambda = (1 - \lambda)\boldsymbol{w}^* + \lambda \boldsymbol{w}, 0 \leq \lambda \leq 1$, where $\boldsymbol{w}_\lambda$ represents an element in B that moves from $\boldsymbol{w}^*$ to an arbitrary vector $\boldsymbol{w}$ in B.

The above necessary and sufficent condition can be represented by

$$\left.\frac{d}{d\lambda}W(\boldsymbol{w}_\lambda)\right|_{\lambda=0^+} \leq 0 \quad \text{for all } \boldsymbol{w} \in B.$$

Consider

$$\begin{aligned}
&\left.\frac{d}{d\lambda}E[\ln(\boldsymbol{w}\cdot\boldsymbol{R})]\right|_{\lambda=0^+}\\
=\;& \lim_{\Delta\lambda\to 0^+}\frac{1}{\Delta\lambda}E\left[\ln\left(\frac{(1-\Delta\lambda)\boldsymbol{w}^*\cdot\boldsymbol{R}+\Delta\lambda\boldsymbol{w}\cdot\boldsymbol{R}}{\boldsymbol{w}^*\cdot\boldsymbol{R}}\right)\right]\\
=\;& E\left[\lim_{\Delta\lambda\to 0^+}\frac{1}{\Delta\lambda}\ln\left(1+\Delta\lambda\left(\frac{\boldsymbol{w}\cdot\boldsymbol{R}}{\boldsymbol{w}^*\cdot\boldsymbol{R}}-1\right)\right)\right]\\
=\;& E\left[\frac{\boldsymbol{w}\cdot\boldsymbol{R}}{\boldsymbol{w}^*\cdot\boldsymbol{R}}\right]-1 \leq 0.
\end{aligned}$$

In particular, when $\boldsymbol{w}^*$ is an interior point of B, then

$$E\left[\frac{\boldsymbol{w}\cdot\boldsymbol{R}}{\boldsymbol{w}^*\cdot\boldsymbol{R}}\right] = 1 \quad \text{for all } \boldsymbol{w} \in B.$$

Suppose we take $\boldsymbol{w} = \boldsymbol{e}_j$, we then deduce that

$$E\left[\frac{R_j}{\boldsymbol{w}^* \cdot \boldsymbol{R}}\right] = 1, \quad j = 1, 2, \cdots, n.$$

Let P_j^0 be the price of security j at time 0 and P_j be the random payout of security j. The return of security j is

$$R_j = P_j / P_j^0$$

so that

$$P_j^0 = E\left[\frac{P_j}{\boldsymbol{w}^* \cdot \boldsymbol{R}}\right].$$

Note that $\boldsymbol{w}^* \cdot \boldsymbol{R}$ is the return on the log-optimal portfolio. Here, P_j^0 can be interpreted as the fair price of security j based on the knowledge of F.

Remark

Let w_j^* be the optimal weight invested on asset j, and R_j is its return per unit dollar betted. The random weight of asset j after one investment period is

$$\frac{w_j^* R_j}{w_1^* R_1 + \cdots + w_n^* R_n}.$$

Taking the expectation

$$E\left[\frac{w_j^* R_j}{\boldsymbol{w}^* \cdot \boldsymbol{R}}\right] = w_j^* E\left[\frac{R_j}{\boldsymbol{w}^* \cdot \boldsymbol{R}}\right] = w_j^*.$$

when $\boldsymbol{w}^*$ is an interior point of B. The expected weight of asset j after the game under the optimal trading strategy is simply the original optimal weight.