MATH685Z – Mathematical Models in Financial Economics

Topic 7 — Capital asset pricing model and factor models

7.1 Capital asset pricing model and beta values

7.2 Interpretation and uses of the capital asset pricing model

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7.1 Capital asset pricing model and beta values

Capital market line (CML)

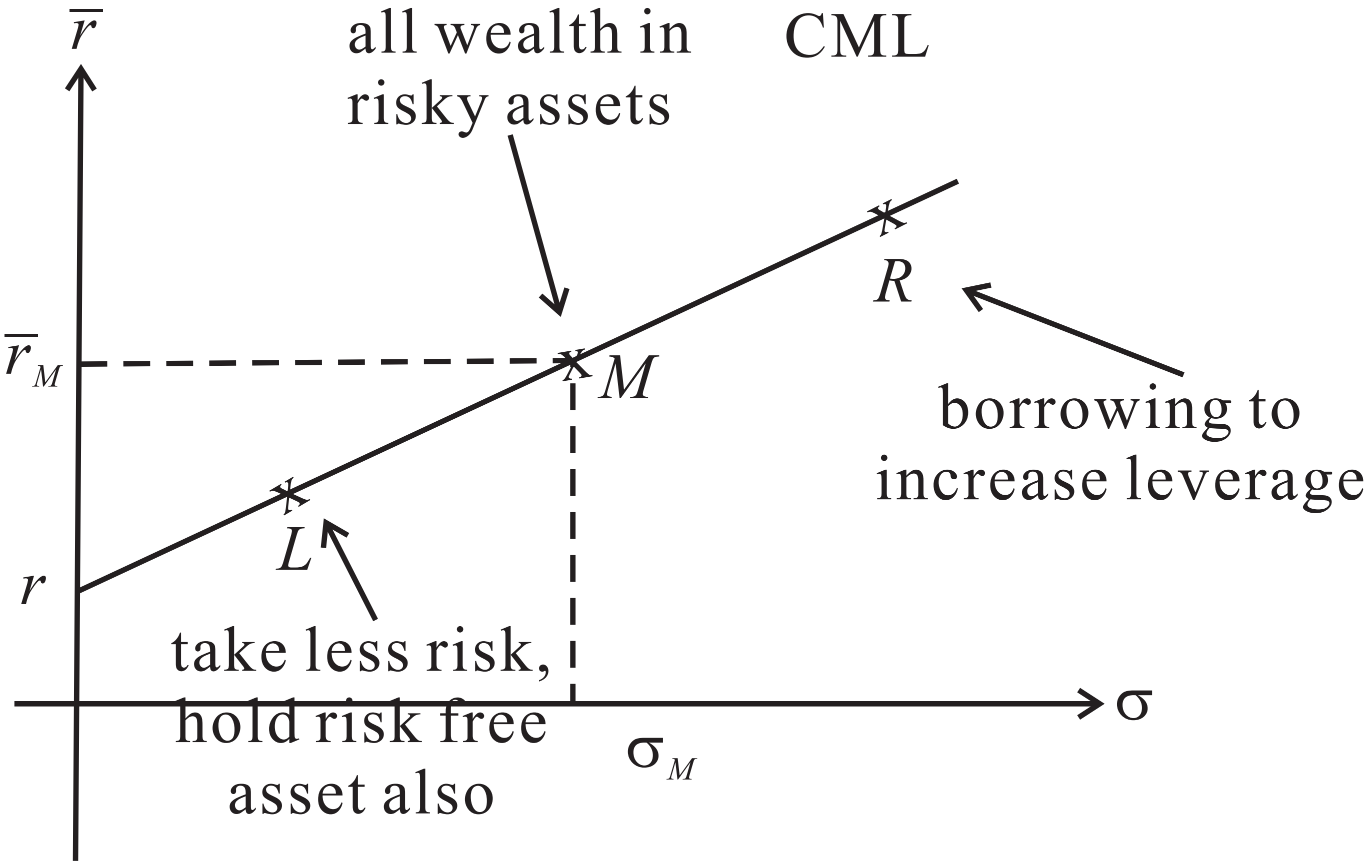
The CML is the tangent line drawn from the risk free point to the feasible region for risky assets. This line shows the relation between $\overline{r}_P$ and σ_P for efficient portfolios (risky assets plus the risk free asset).

The tangency point M represents the *market portfolio*, so named since all rational investors (minimum variance criterion) should hold their risky assets in the same proportions as their weights in the market portfolio.

- Every investor is a mean-variance investor and they all have homogeneous expectations on means and variances, then everyone buys the same portfolio. Prices adjust to drive the market to efficiency.

All portfolios on the CML are efficient, and they are composed of various mixes of the market portfolio and the risk free asset.

Based on the risk level that an investor can take, she combines the market portfolio of risky assets with the risk free asset.



Equation of the CML:

$$\overline{r} = r + \frac{\overline{r}_M - r}{\sigma_M}\sigma,$$

where $\overline{r}$ and σ are the mean and standard deviation of the rate of return of an efficient portfolio.

Slope of the CML $= \dfrac{\overline{r}_M - r}{\sigma_M} =$ price of risk of an efficient portfolio.

This indicates how much the expected rate of return must increase when the standard deviation increases by one unit.

The CML does not apply to an individual asset or portfolios that are inefficient.

Sharpe ratio

One index that is commonly used in performance measure is the Sharpe ratio, defined as

$$\frac{\overline{r}_i - r}{\sigma_i} = \frac{\text{excess expected rate of return above riskfree rate}}{\text{standard deviation}}.$$

We expect

$$\text{Sharpe ratio} \leq \text{slope of CML}.$$

Closer the Sharpe ratio to the slope of CML, the better the performance of the fund in terms of return against risk.

In the previous example,

$$\begin{aligned} \text{Slope of CML} &= \frac{17\% - 10\%}{12\%} = \frac{7}{12} = 0.583 \\ \text{Sharpe ratio} &= \frac{14\% - 10\%}{40\%} = 0.1 < \text{ Slope of CML.} \end{aligned}$$

Capital Asset Pricing Model

Let M be the market portfolio M, then the expected rate of return $\overline{r}_i$ of any asset i satisfies

$$\overline{r}_i - r = \beta_i(\overline{r}_M - r)$$

where

$$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2}.$$

Here, $\sigma_{iM} = \text{cov}(r_i, r_M)$ is the covariance between the rate of return of risky asset i and the rate of return of the market portfolio M.

Remark

Expected excess rate of return of a risky asset above r is related to the correlation of r_i with r_M.

Assumptions underlying the standard CAPM

1. No transaction costs.
2. Assets are infinitely divisible.
3. Absence of personal income tax.
4. An individual cannot affect the price of a stock by his buying or selling action. All investors are *price takers*.
5. Unlimited short sales are allowed.
6. Unlimited lending and borrowing at the riskless rate.

7. Investors are assumed to be concerned with the mean and variance of returns, and all investors are assumed to define the relevant period in exactly the same manner.

8. All investors are assumed to have identical expectations with respect to the necessary inputs to the portfolio decision.

Both (7) and (8) are called the "homogeneity of expectations".

The CAPM relies on the mean-variance approach, homogeneity of expectation of investors, and no market frictions. In equilibrium, every investor must invest in the same fund of risky assets and in the risk free asset.

Alternative proof of CAPM

Consider

$$\sigma_{iM} = \text{cov}(r_i, r_M) = \boldsymbol{e}_i^T \Omega \boldsymbol{w}_M^*,$$

where $\boldsymbol{e}_i = (0 \cdots 1 \cdots 0) = i^{\text{th}}$ co-ordinate vector which represents the weight of asset i. Recall $\boldsymbol{w}_M^* = \dfrac{\Omega^{-1}(\boldsymbol{\mu} - r\mathbf{1})}{b - ar}$ so that

$$\sigma_{iM} = \frac{(\boldsymbol{\mu} - r\mathbf{1})_i}{b - ar} = \frac{\overline{r}_i - r}{b - ar}, \text{ provided } b - ar \neq 0. \tag{1}$$

Recall

$$\sigma_M^2 = \boldsymbol{w}_M^* \Omega \boldsymbol{w}_M^* = \frac{\boldsymbol{w}_M^*(\boldsymbol{\mu} - r\mathbf{1})}{b - ar} = \frac{\mu_M - r}{b - ar}. \tag{2}$$

Alternatively, we may obtain (2) by setting $i \equiv M$ in (1). This gives $\sigma_M^2 = (\mu_M - r)/(b - ar)$. Eliminating $b - ar$ from eqs. (1) and (2), we obtain

$$\overline{r}_i - r = \frac{\sigma_{iM}}{\sigma_M^2}(\mu_M - r).$$

Beta of a portfolio

Consider a portfolio containing n risky assets with weights $w_1, \cdots, w_n$. Since $r_P = \sum_{i=1}^{n} w_i r_i$, we have $\text{cov}(r_P, r_M) = \sum_{i=1}^{n} w_i \text{cov}(r_i, r_M)$ so that

$$\beta_P = \frac{\text{cov}(r_P, r_M)}{\sigma_M^2} = \frac{\sum_{i=1}^{n} w_i \text{cov}(r_i, r_M)}{\sigma_M^2} = \sum_{i=1}^{n} w_i \beta_i.$$

The portfolio beta is given by the weighted average of the beta values of the risky assets in the portfolio.

Since $\overline{r}_P = \sum_{i=1}^{n} w_i \overline{r}_i$ and $\beta_P = \sum_{i=1}^{n} w_i \beta_i$, and for each asset i, the CAPM gives: $\overline{r}_i - r = \beta_i(\overline{r}_M - r)$. Noting $\sum_{i=1}^{n} w_i = 1$, we then have

$$\overline{r}_P - r = \beta_P(\overline{r}_M - r).$$

Various interpretations of the CAPM

- If we write $\sigma_{iM} = \rho_{iM}\sigma_i\sigma_M$, then the CAPM can be rewritten as
$$\frac{\overline{r}_i - r}{\sigma_i} = \rho_{iM}\frac{\overline{r}_M - r}{\sigma_M}.$$
The Sharpe ratio of asset i is given by the product of ρ_{iM} and the slope of CML. When ρ_{iM} is closer to one, the asset is closer to (but always below) the CML. For an efficient portfolio e that lies on the CML, we then have $\rho_{eM} = 1$.

- For any two risky assets i and j, we have
$$\frac{\overline{r}_i - r}{\beta_i} = \frac{\overline{r}_j - r}{\beta_j} = \overline{r}_M - r.$$
Under the CAPM, the expected excess return above r normalized by the beta value is constant for all assets. On the other hand, the Sharpe ratios are related by
$$\frac{(\text{Sharpe ratio})_i}{\rho_{iM}} = \frac{\overline{r}_i - r}{\rho_{iM}\sigma_i} = \frac{\overline{r}_j - r}{\rho_{jM}\sigma_j} = \frac{(\text{Sharpe ratio})_j}{\rho_{jM}} = \frac{\overline{r}_M - r}{\sigma_M}.$$

- Let P be an efficient portfolio on the CML, then

$$r_P = \alpha r_M + (1-\alpha)r$$

where α is the proportional weight of the market portfolio M. Consider

$$\begin{aligned}\text{cov}(r_P, r_M) &= \text{cov}(\alpha r_M + (1-\alpha)r, r_M) = \alpha\text{var}(r_M) = \alpha\sigma_M^2 \\ \text{var}(r_P) &= \alpha^2\sigma_M^2; \text{hence} \\ \rho_{PM} &= \frac{\text{cov}(r_P, r_M)}{\sigma_P\sigma_M} = \frac{\alpha\sigma_M^2}{\alpha\sigma_M\sigma_M} = 1,\end{aligned}$$

thus verifying the earlier claim. Furthermore, it is seen that

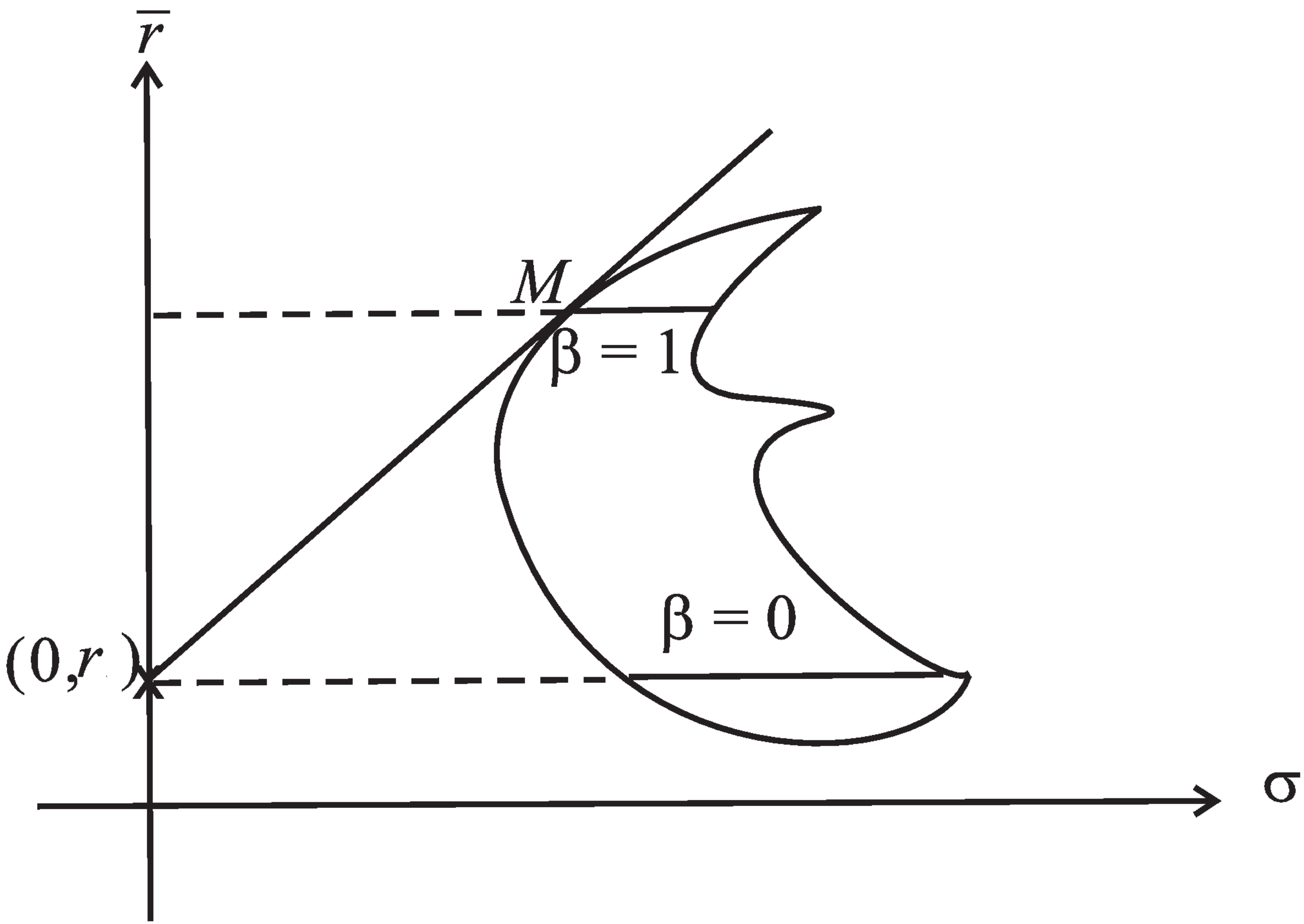
$$\beta_P = \frac{\text{cov}(r_P, r_M)}{\text{var}(r_M)} = \alpha\frac{\text{var}(r_M)}{\text{var}(r_M)} = \alpha.$$

- The beta value of an efficient portfolio is equal to the proportional weight α of the market portfolio in the efficient portfolio. This is obvious since the expected excess rate of return above the riskfree rate is contributed by the proportion of market portfolio in the efficient portfolio only while the proportion of riskfree asset does not contribute.

Some special cases of beta values

1. When $\beta_i = 0, \overline{r}_i = r$. A risky asset (with $\sigma_i > 0$) that is uncorrelated with the market portfolio will have an expected rate of return equal to the risk free rate. There is no expected excess return over r even the investor bears some risk in holding a risky asset with zero beta.

2. When $\beta_i = 1, \overline{r}_i = r_M$. The risky asset has the same expected rate of return as that of the market portfolio.



Representation of the risky assets or portfolios of risky assets with $\beta = 0$ and $\beta = 1$ in the $\sigma - \overline{r}$ diagram.

3. When $\beta_i > 1$, the expected excess rate of return is higher than that of market portfolio - *aggressive asset*. When $\beta_i < 1$, the asset is said to be *defensive*.

4. When $\beta_i < 0, \overline{r}_i < r$. Since $\dfrac{d\sigma_M}{\sigma_M} = \beta_i \, dw_i^M$, so a risky asset with *negative beta reduces the variance* of the portfolio. This risk reduction potential of an asset with negative β is something like paying a premium to reduce risk. When more units of the negative beta asset are added to the portfolio, the expected return is reduced while the risk of the market portfolio (as quantified by σ_M) is also reduced at the same time.

Example

Assume that the expected rate of return on the market portfolio is 12% per annum and the rate of return on the riskfree asset is 7% per annum. The standard deviation of the market portfolio is 32% per annum. Assume that the market portfolio is efficient.

(a) What is the equation of the capital market line?

CML is given by

$$\overline{r} = r + \left(\frac{\overline{r}_M - r}{\sigma_M}\right)\sigma = 0.07 + 0.1562\sigma.$$

(b) (i) If an expected return of 18% is desired for an efficient portfolio, what is the standard deviation of this portfolio?

Substituting $\overline{r} = 0.18$ into the CML equation, we obtain

$$\sigma = \frac{(0.18 - 0.07)}{0.1562} = 0.7042.$$

(ii) If you have $1,000 to invest, how should you allocate the wealth among the market portfolio and the riskfree asset to achieve the above portfolio?

Recall

$$\overline{r}_P = \alpha \overline{r}_M + (1 - \alpha) r$$

so that

$$\alpha = \frac{\overline{r}_P - r}{\overline{r}_M - r} = \frac{0.18 - 0.07}{0.12 - 0.07} = \frac{0.11}{0.05} = 2.2.$$

Note that $1 - \alpha = -1.2$. The investor should short sell $1,200 of the riskfree asset and long $2,200 of the market portfolio.

(iii) What is the beta value of this portfolio?

The beta value equals the weight of investment on the market portfolio in the efficient portfolio, so

$$\beta = \alpha = 2.2.$$

(c) If you invest \$300 in the riskfree asset and \$700 in the market portfolio, how much money should you expect to have at the end of the year?

The expected return per annum is given by

$$E[r_P] = 0.3r + 0.7\overline{r}_M = 0.105.$$

The expected amount of money at the end of the year is

$$(\$300 + \$700)(1 + E[r_P]) = \$1,105.$$

Extension of CAPM – no reference to the market portfolio

Let P be any efficient portfolio lying along the CML and Q be any portfolio. An extension of the CAPM gives

$$\overline{r}_Q - r = \beta_{QP}(\overline{r}_P - r), \quad \beta_{QP} = \frac{\sigma_{QP}}{\sigma_P^2}, \tag{A}$$

that is, we may replace the market portfolio M by an efficient portfolio P.

More generally, the random rates of return r_P and r_Q are related by

$$r_Q - r = \beta_{QP}(r_P - r) + \epsilon_{QP} \tag{B}$$

with $\text{cov}(r_P, \epsilon_{QP}) = E[\epsilon_{QP}] = 0$. The residual ϵ_{QP} has zero expected value and it is uncorrelated with r_P.

Proof

Since Portfolio P is efficient (lying on the CML), then

$$r_P = \alpha r_M + (1-\alpha)r, \quad \alpha > 0.$$

The first result (A) can be deduced from the CAPM by observing

$$\begin{aligned} \sigma_{QP} &= \text{cov}(r_Q, \alpha r_M + (1-\alpha)r) = \alpha \text{cov}(r_Q, r_M) = \alpha\sigma_{QM}, \quad \alpha > 0 \\ \sigma_P^2 &= \alpha^2\sigma_M^2 \quad \text{and} \quad \overline{r}_P - r = \alpha(\overline{r}_M - r). \end{aligned}$$

Consider

$$\begin{aligned} \overline{r}_Q - r &= \beta_{MQ}(\overline{r}_M - r) = \frac{\sigma_{QM}}{\sigma_M^2}(\overline{r}_M - r) \\ &= \frac{\sigma_{QP}/\alpha}{\sigma_P^2/\alpha^2}(\overline{r}_P - r)/\alpha = \beta_{QP}(\overline{r}_P - r). \end{aligned}$$

By performing the regression of r_Q on r_P, the relationship among r_Q and r_P can be formally expressed as

$$r_Q = \alpha_0 + \alpha_1 r_P + \epsilon_{QP},$$

where α_0 and α_1 are the resulting coefficients estimated from the regression. The residual ϵ_{QP} is taken to have zero expected value.

Observe that

$$\overline{r}_Q = \alpha_0 + \alpha_1 \overline{r}_P$$

and from result (A), we obtain

$$\overline{r}_Q = \beta_{QP}\overline{r}_P + r(1 - \beta_{QP})$$

so that

$$\alpha_0 = r(1 - \beta_{QP}) \quad \text{and} \quad \alpha_1 = \beta_{PQ}.$$

Hence, we obtain result (B).

Zero-beta CAPM: absence of the risk free asset

There exists a portfolio Z_M whose beta is zero. Since $\beta_{MZ_M} = 0$, we have $\bar{r}_{Z_M} = r$. Consider the following relation from CAPM

$$\bar{r}_Q = r + \beta_{QM}(\bar{r}_M - r),$$

it can be expressed in terms of the market portfolio M and its zero-beta counterpart Z_M as follows

$$\bar{r}_Q = \bar{r}_{Z_M} + \beta_{QM}(\bar{r}_M - \bar{r}_{Z_M}).$$

In this form, the role of the riskfree asset is replaced by the zero-beta portfolio Z_M. However, the formula is still referencing the market portfolio (implicitly implies the presence of the risk free asset).

⋆ The more general version of the CAPM allows the choice of *any* efficient (mean-variance) portfolio and its zero-beta counterpart. In this sense, we allow the absence of the risk free asset.

Zero-beta counterpart of a given efficient portfolio

Let P and Q be any two frontier portfolios of risky assets. Recall

$$\boldsymbol{w}_P^* = \Omega^{-1}(\lambda_1^P \mathbf{1} + \lambda_2^P \boldsymbol{\mu}) \quad \text{and} \quad \boldsymbol{w}_Q^* = \Omega^{-1}(\lambda_1^Q \mathbf{1} + \lambda_2^Q \boldsymbol{\mu})$$

where

$$\lambda_1^P = \frac{c - b\mu_P}{\Delta}, \quad \lambda_2^P = \frac{a\mu_P - b}{\Delta}, \quad \lambda_1^Q = \frac{c - b\mu_Q}{\Delta}, \quad \lambda_2^Q = \frac{a\mu_Q - b}{\Delta},$$

$$a = \mathbf{1}^T \Omega^{-1} \mathbf{1}, \quad b = \mathbf{1}^T \Omega^{-1} \boldsymbol{\mu}, \quad c = \boldsymbol{\mu}^T \Omega^{-1} \boldsymbol{\mu}, \quad \Delta = ac - b^2.$$

The covariance between R_P and R_Q is given by

$$\begin{aligned} \text{cov}(r_P, r_Q) &= \boldsymbol{w}_P^{*T} \Omega \boldsymbol{w}_Q^* = \left[\Omega^{-1}(\lambda_1^P \mathbf{1} + \lambda_2^P \boldsymbol{\mu})\right]^T (\lambda_1^Q \mathbf{1} + \lambda_2^Q \boldsymbol{\mu}) \\ &= \lambda_1^P \lambda_1^Q a + (\lambda_1^P \lambda_2^Q + \lambda_1^Q \lambda_2^P) b + \lambda_2^P \lambda_2^Q c \\ &= \frac{a}{\Delta}\left(\mu_P - \frac{b}{a}\right)\left(\mu_Q - \frac{b}{a}\right) + \frac{1}{a}. \qquad (A) \end{aligned}$$

Setting Q to be P, we obtain $\sigma_P^2 = \frac{a}{\Delta}\left(\mu_P - \frac{b}{a}\right)^2 + \frac{1}{a}$.

Find the frontier portfolio Z such that $\text{cov}(R_P, R_Z) = 0$. We find μ_Z such that [see Eq. (A)]

$$\frac{a}{\Delta}\left(\mu_P - \frac{b}{a}\right)\left(\mu_Z - \frac{b}{a}\right) + \frac{1}{a} = 0.$$

This gives

$$\mu_Z = \frac{b}{a} - \frac{\frac{\Delta}{a^2}}{\mu_P - \frac{b}{a}}.$$

Since $(\mu_P - \mu_g)(\mu_Z - \mu_g) = -\dfrac{\Delta}{a^2} < 0$, where $\mu_g = \dfrac{b}{a}$, so when one portfolio is efficient, then its zero-covariance counterpart is non-efficient.

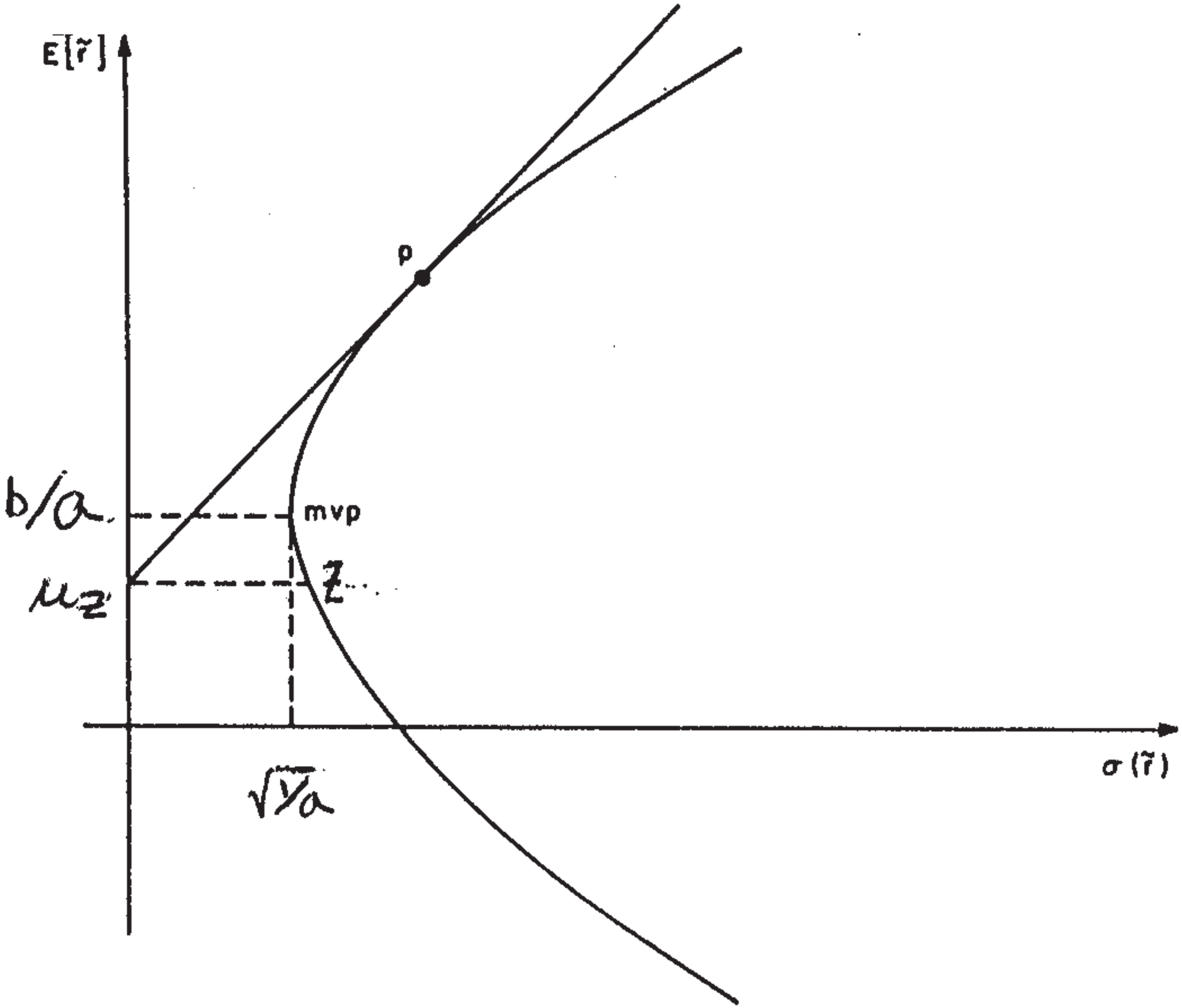
Slope of the tangent at P to the frontier curve:

$$\frac{d\mu_P}{d\sigma_P} = \frac{\Delta\sigma_P}{a\mu_P - b}.$$

The intercept of the tangent line at the vertical axis is

$$\begin{aligned}\mu_P - \frac{d\mu_P}{d\sigma_P}\sigma_P &= \mu_P - \frac{\Delta\sigma_P^2}{a\mu_P - b} \\ &= \mu_P - \frac{a\mu_P^2 - 2b\mu_P + c}{a\mu_P - b} = \frac{b}{a} - \frac{\Delta/a^2}{\mu_P - b/a} = \mu_Z.\end{aligned}$$

These calculations verify that the uncorrelated counterpart Z can be obtained by drawing a tangent to the frontier curve at P and finding the intercept of the tangent line at the vertical axis. Draw a horizontal line from the intercept to hit the frontier curve at Z.



The Location of a Zero Covariance Portfolio in the $\sigma(\tilde{r})$–$E[\tilde{r}]$ Space

Intuition behind the geometric construction of the uncorrelated counterpart

- Fix the riskfree point, we determine the market portfolio by the tangency method. Subsequently, all zero-beta funds (uncorrelated with the market portfolio) lie on the same horizonal line through the riskfree point in the σ_P-μ_P diagram.

- Conversely, we consider the scenario where the riskfree point is NOT specified. Actually, the riskfree asset is absent in the present context. Apparently, given an efficient fund, we determine the corresponding "riskfree point" such that the efficient fund is the market portfolio with reference to the riskfree point. In this case, the frontier fund with the same return as this pseudo "riskfree point" will have its random rate of return uncorrelated with that of the efficient fund. The pseudo "riskfree point" and this uncorrelated counterpart (itself is a minimum variance portfolio) lie on the same horizontal line in the σ-$\overline{r}$ diagram.

Let P be a frontier portfolio other than the global minimum variance portfolio and Q be any portfolio, then

$$\begin{aligned}\text{cov}(R_P, R_Q) &= \left[\Omega^{-1}\left(\lambda_1^P \mathbf{1} + \lambda_2^P \boldsymbol{\mu}\right)\right]^T \Omega \boldsymbol{w}_Q \\ &= \lambda_1^P \mathbf{1}^T \boldsymbol{w}_Q + \lambda_2^P \boldsymbol{\mu}^T \boldsymbol{w}_Q = \lambda_1^P + \lambda_2^P \mu_Q.\end{aligned}$$

Solving for μ_Q and substituting $\lambda_1^P = \dfrac{c - b\mu_P}{\Delta}$ and $\lambda_2^P = \dfrac{a\mu_P - b}{\Delta}$:

$$\begin{aligned}\mu_Q &= \frac{b\mu_P - c}{a\mu_P - b} + \text{cov}(r_P, r_Q)\frac{\Delta}{a\mu_P - b} \\ &= \frac{b}{a} - \frac{\Delta/a^2}{\mu_P - b/a} + \frac{\text{cov}(r_P, r_Q)}{\sigma_P^2}\left[\frac{\left(\mu_P - \frac{b}{a}\right)^2}{\Delta/a} + \frac{1}{a}\right]\frac{\Delta}{a\mu_P - b} \\ &= \mu_{Z_P} + \beta_{QP}\left(\mu_P - \frac{b}{a} + \frac{\Delta/a^2}{\mu_P - b/a}\right) \\ &= \mu_{Z_P} + \beta_{QP}(\mu_P - \mu_{Z_P})\end{aligned}$$

so that we obtain the following generalized CAPM (in terms of the given efficient portfolio and its uncorrelated counterpart)

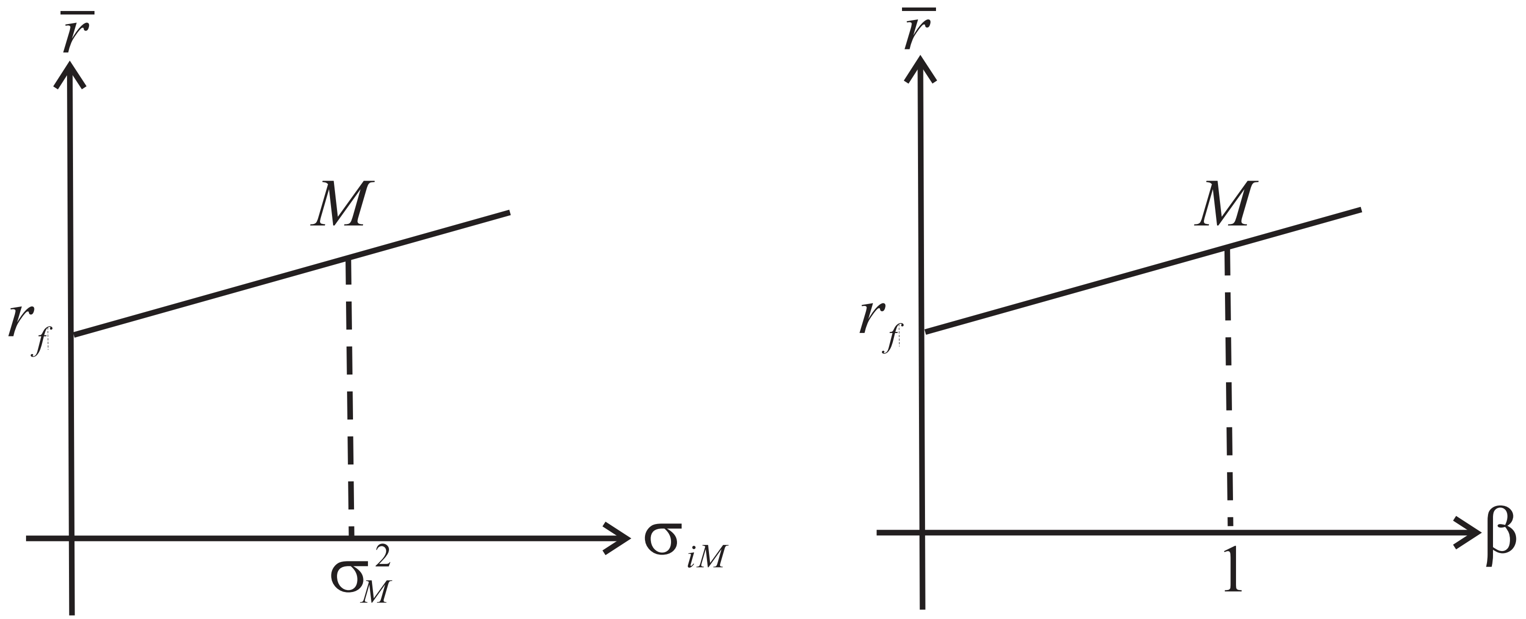
$$\mu_Q - \mu_{Z_P} = \beta_{QP}(\mu_P - \mu_{Z_P}).$$

7.2 Interpretation and uses of the capital asset pricing model

Security market line (SML)

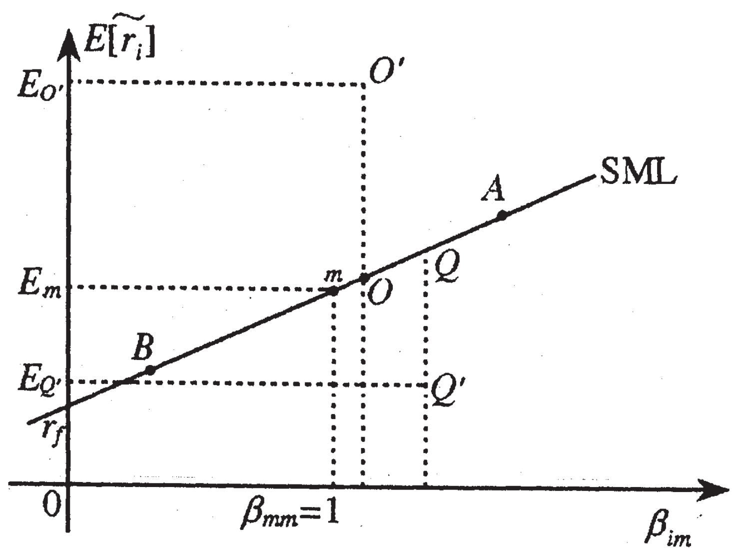
From the two relations: $\begin{cases} \overline{r} = r_f + \frac{\overline{r}_M - r_f}{\sigma_M^2}\sigma_{iM} \\ \overline{r} = r_f + (\overline{r}_M - r_f)\beta_i \end{cases}$,

we can plot either $\overline{r}$ against σ_{iM} or $\overline{r}$ against β_i.



Under the equilibrium conditions assumed by the CAPM, every asset should fall on the SML. The SML expresses the risk reward structure of assets according to the CAPM.

- Point O' represents under-priced security. This is because the expected return is higher than the return with reference to the risk. In this case, the demand for such security will increase and this results in price increase and lowering of the expected return.



Example

Consider the following set of data for 3 risky assets, market portfolio and risk free asset:

portfolio/security	σ_i	ρ_{iM}	β_i	actual expected rate of return $= \frac{E[P_1 + D_1]}{P_0} - 1.0$
1	10%	1.0	0.5	13%
2	20%	0.9	0.9	15.4%
3	20%	0.5	0.5	13%
market portfolio	20%	1.0	1.0	16%
risk free asset	0	0.0	0.0	10%

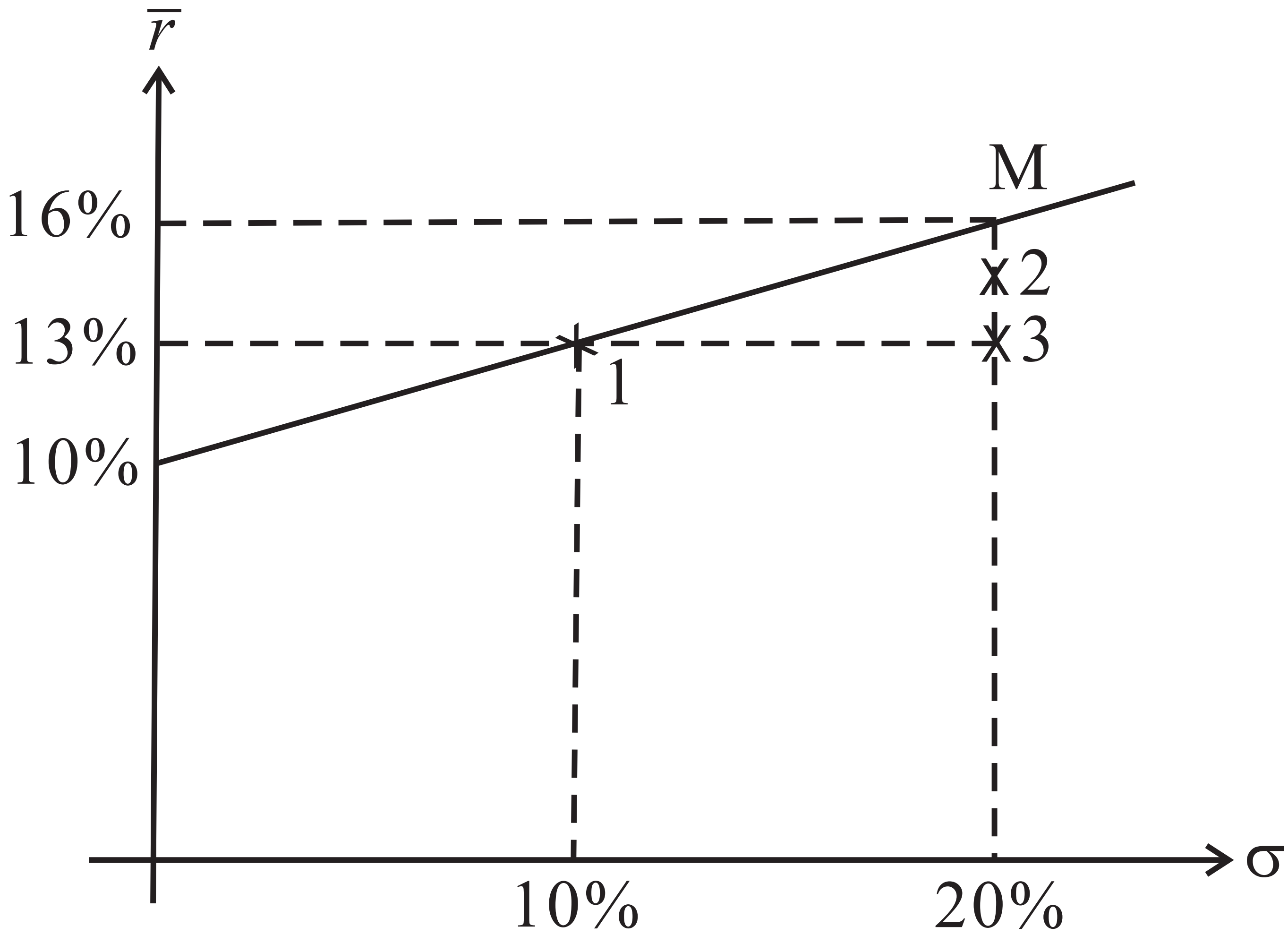
- Note that β can be computed using the data given for ρ_{iM}, σ_i and σ_M. For example, $\beta_1 = \rho_{1M}\sigma_1/\sigma_M = 0.5$.

Use of the CML

The CML identifies expected rates of return which are available on *efficient portfolios* of all possible risk levels. Portfolios 2 and 3 lie below the CML. The market portfolio, the risk free asset and Portfolio 1 all lie on the CML. Hence, Portfolio 1 is efficient while Portfolios 2 and 3 are non-efficient.

$$\text{At } \sigma = 10\%, \overline{r} = \underbrace{10\%}_{r_f} + \underbrace{10\%}_{\sigma} \times \underbrace{\frac{(16-10)\%}{20\%}}_{(\overline{r}_M - r_f)/\sigma_M} = 13\%.$$

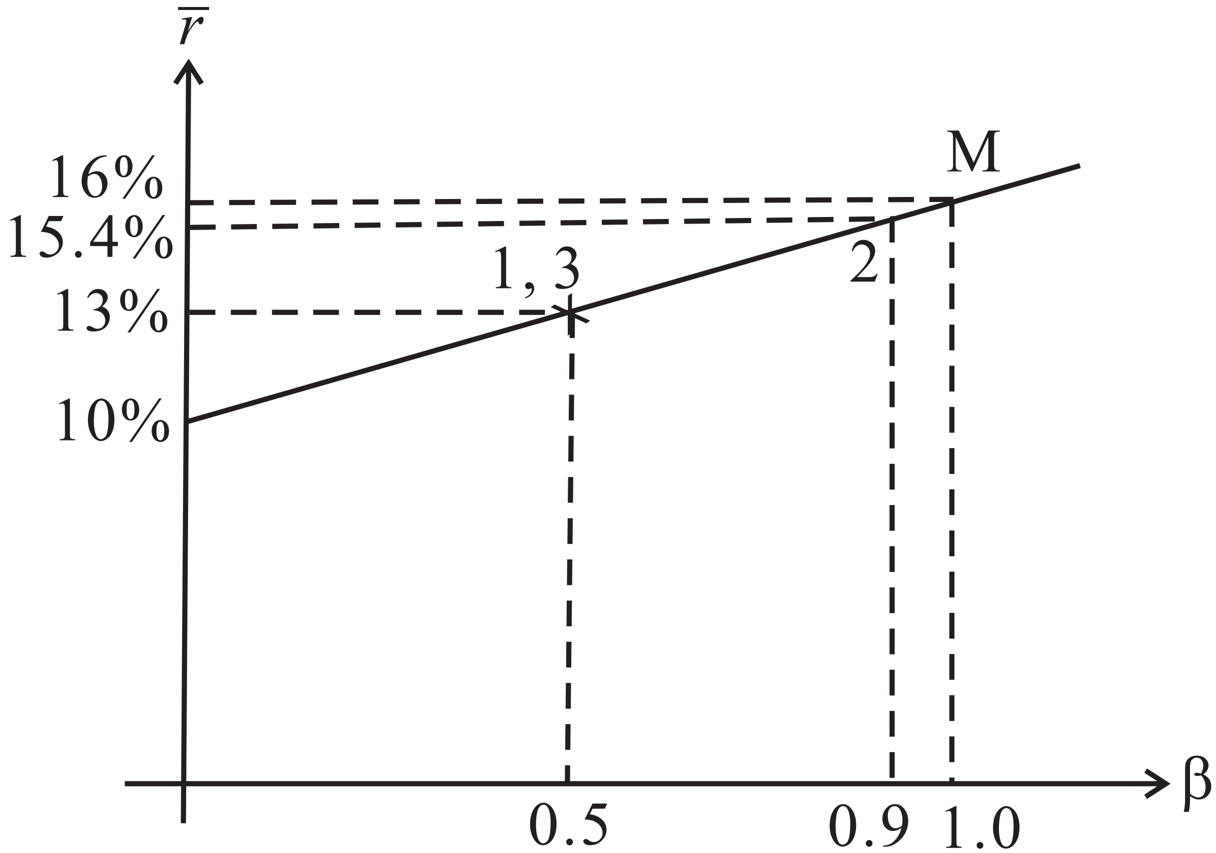
$$\text{At } \sigma = 20\%, \overline{r} = 10\% + 20\% \times \frac{(16-10)\%}{20\%} = 16\%.$$



Note that Asset 2 is closer to the CML since ρ_{2M} is 0.9, which is sufficiently close to 1. Asset 3 has high non-systematic risk (risk that does not contribute to expected return) as ρ_{3M} is seen to have a low value.

Use of the SML

The SML asks whether the portfolio provides a return equal to what equilibrium conditions suggest should be earned.



The expected rates of return of the portfolios for the given values of beta are given by

$$\begin{aligned}\overline{r}_1 &= \overline{r}_3 = \underbrace{10\%}_{r} + \underbrace{0.5}_{\beta} \times (\underbrace{16\% - 10\%}_{\overline{r}_M - r}) = 13\% \\ \overline{r}_2 &= 10\% + 0.9 \times (16\% - 10\%) = 15.4\%.\end{aligned}$$

These expected rates of return suggested by the SML agree with the actual expected rates of return. Hence, each investment is fairly priced.

Portfolio 1 has unit value of ρ_{iM}, that is, it is perfectly correlated with the market portfolio. Hence, Portfolio 1 has zero non-systematic risk.

Portfolios 2 and 3 both have ρ_{iM} less than one.

Portfolio 2 has ρ_{iM} closer to one and so it lies closer to the CML.

Decomposition of risks

Suppose we write the random rate of return r_i of asset i formally as

$$r_i = r + \beta_i(r_M - r) + \epsilon_i.$$

The CAPM tells us something about the residual term ϵ_i.

(i) Taking expectation on both sides

$$E[r_i] = r + \beta_i(\overline{r}_M - r) + E[\epsilon_i]$$

while $\overline{r}_i = r + \beta_i(\overline{r}_M - r)$ so that $E[\epsilon_i] = 0$.

(ii) Taking the covariance of r_i with r_M

$$\text{cov}(r_i, r_M) = \overbrace{\text{cov}(r_f, r_M)}^{\text{zero}} + \beta_i \left[\text{cov}(r_M, r_M) - \underbrace{\text{cov}(r_f, r_M)}_{\text{zero}}\right] + \text{cov}(\epsilon_i, r_M)$$

so that $\text{cov}(\epsilon_i, r_M) = 0$.

(iii) Consider the variance of r_i

$$\text{var}(r_i) = \beta_i^2 \underbrace{\text{cov}(r_M - r_f, r_M - r_f)}_{\text{var}(r_M)} + \text{var}(\epsilon_i)$$

so that $\sigma_i^2 = \beta_i^2 \sigma_M^2 + \text{var}(\epsilon_i)$.

The total risk consists of systematic risk $\beta_i^2 \sigma_M^2$ and firm-specific (idiosyncratic) risk $\text{var}(\epsilon_i)$.

Systematic risk $= \beta_i^2 \sigma_M^2$, this risk cannot be reduced by diversification because every asset with nonzero beta contains this risk.

It is the systematic risk where the investor is rewarded for excess return above the riskfree rate.

Efficient portfolios: zero non-systematic risk

Consider a portfolio P formed by the combination of the market portfolio and the risk free asset. This portfolio is an efficient portfolio (one fund theorem) and it lies on the CML with a beta value equal to β_P (say). Its rate of return can be expressed as

$$r_P = (1 - \beta_P)r + \beta_P r_M = r + \beta_P(r_M - r)$$

so that $\epsilon_P = 0$. The portfolio variance is $\beta_P^2 \sigma_M^2$. This portfolio has only systematic risk (zero non-systematic risk).

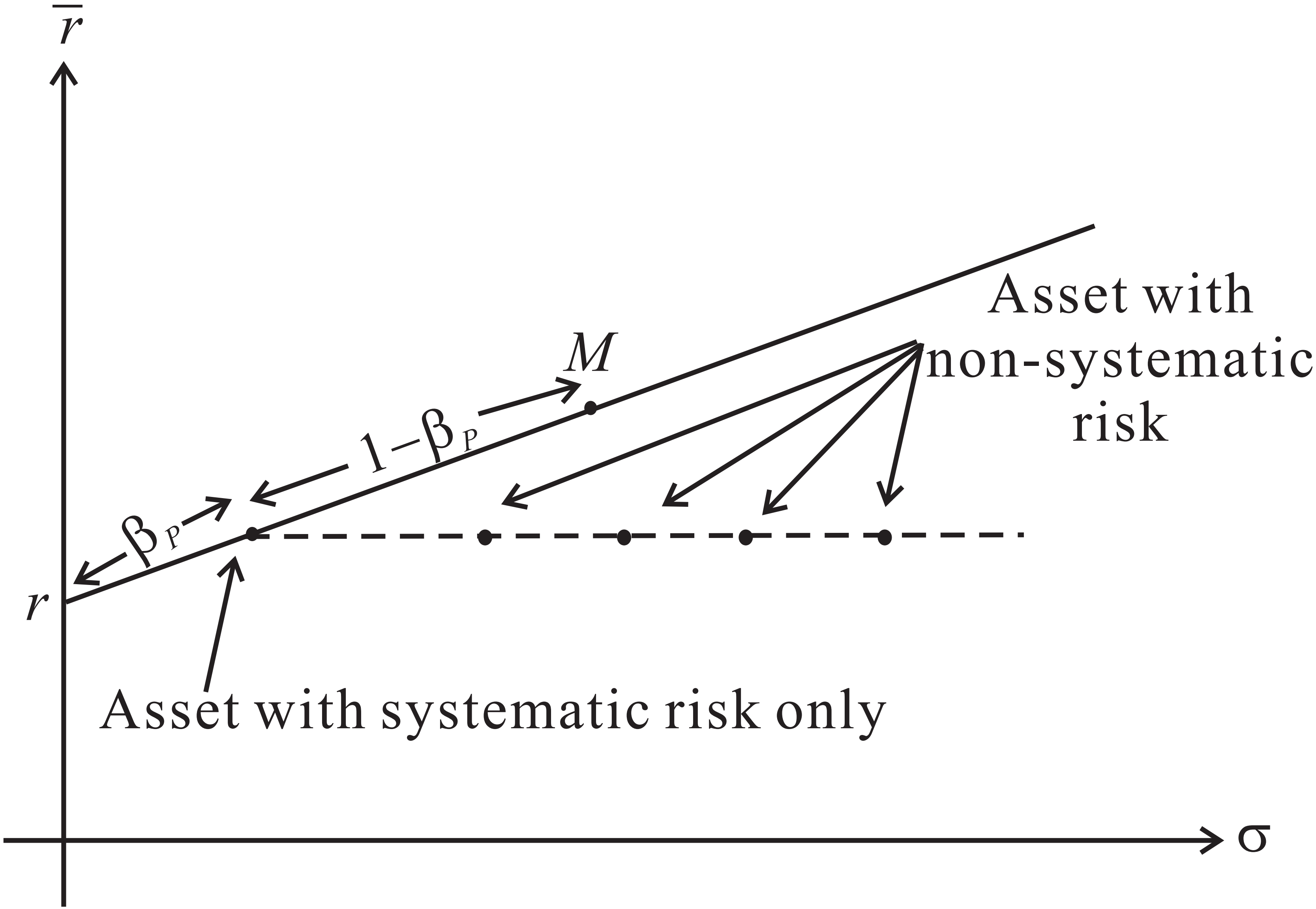
For an efficient portfolio P, we have $\rho_{PM} = 1$ so that $\beta_P = \dfrac{\sigma_P}{\sigma_M}$.

Portfolios not on the CML – non-efficient portfolios

For other portfolios with the same value of β_P but not lying on the CML, they lie below the CML since they are non-efficient portfolios. With the same value of β_P, they all have the same expected rate of return given by

$$\overline{r} = r + \beta_P(\overline{r}_M - r)$$

but the portfolio variance is greater than $\beta_P^2 \sigma_M^2$. The extra part of the portfolio variance is var(ϵ_i).



equation of CML: $\overline{r} = r + \frac{\overline{r}_M - r}{\sigma_M}\sigma$

Diversification effect

Note that ϵ_i is uncorrelated with r_M as revealed by $\text{cov}(\epsilon_i, r_M) = 0$. The term $\text{var}(\epsilon_i)$ is called the *non-systematic* or *specific* risk. This risk can be reduced by diversification.

Consider $r_i = (1 - \beta_{iM})r + \beta_{iM} r_M + \epsilon_i$ and observe $\text{cov}(\epsilon_i, \epsilon_j) \approx 0$ for $i \neq j$ and $\text{cov}(r_M, \epsilon_i) = 0$ for all i, then

$$
\begin{aligned}
\mu_P &= \sum_{i=1}^n w_i \overline{r}_i = \sum_{i=1}^n (1 - \beta_{iM}) w_i r + \sum_{i=1}^n \beta_{iM} w_i \mu_M \\
\sigma_P^2 &= \left(\sum_{i=1}^n w_i \beta_{iM} \right) \left(\sum_{j=1}^n w_j \beta_{jM} \right) \sigma_M^2 + \sum_{i=1}^n w_i^2 \sigma_{\epsilon_i}^2.
\end{aligned}
$$

Let $\beta_{PM} = \sum_{i=1}^n w_i \beta_{iM}$ and $\alpha_P = \sum_{i=1}^n w_i (1 - \beta_{iM}) r$, then

$$
\begin{aligned}
\mu_P &= \alpha_P + \beta_{PM} \mu_M \\
\sigma_P^2 &= \beta_{PM}^2 \sigma_M^2 + \sum_{i=1}^n w_i^2 \sigma_{\epsilon_i}^2.
\end{aligned}
$$

Suppose we take $w_i = 1/n$ so that

$$\sigma_P^2 = \beta_{PM}^2 \sigma_M^2 + \frac{1}{n^2} \sum_{i=1}^{n} \sigma_{\epsilon_i}^2 = \beta_{PM}^2 \sigma_M^2 + \overline{\sigma}^2/n,$$

where $\overline{\sigma}^2$ is the average of $\sigma_{\epsilon_1}^2, \cdots, \sigma_{\epsilon_n}^2$. When n is sufficiently large

$$\sigma_P \to \left(\sum_{i=1}^{n} w_i \beta_{iM} \right) \sigma_M = \beta_{PM} \sigma_M.$$

- We may view β_{iM} as the contribution of asset i to the portfolio variance σ_P^2.
- From $\sigma_i^2 = \beta_{iM}^2 \sigma_M^2 + \sigma_{\epsilon_i}^2$, the contribution from $\sigma_{\epsilon_i}^2$ to the portfolio variance σ_P^2 goes to zero as $n \to \infty$.

Example

Suppose that the relevant equilibrium model is the CAPM with unlimited borrowing and lending at the riskless rate of interest. Complete the blanks in the following table.

Stock	Expected Return	Standard Deviation	Beta	Residual Variance
1	0.15	—	2.00	0.10
2	—	0.25	0.75	0.04
3	0.09	—	0.50	0.17

Solution

Given our assumptions, the relationship between the expected rate of return and beta is linear.

From the information we have for stock 1 and 3, we know the risk premium accorded the market portfolio must be

$$E[r_M] - r = \frac{E[r_1] - E[r_3]}{\beta_1 - \beta_3} = \frac{0.15 - 0.09}{2.0 - 0.5} = 0.04.$$

Knowing this, we can use the information we have for stock 1 to find the risk-free rate

$$\begin{aligned} E[r_1] &= r + (E[r_M] - r)\beta_1 \\ r &= E[r_1] - (E[r_M] - r)\beta_1 = 0.15 - (0.04)(2.00) = 0.07. \end{aligned}$$

We can now find the expected return for stock 2

$$E[r_2] = r + (E[r_M] - r)\beta_2 = 0.07 + (0.04)(0.75) = 0.10.$$

The information given for stock 2 allows us to estimate the variance of returns of the market:

$$\begin{aligned} \sigma^2(r_2) &= \beta_2^2\sigma^2(r_M) + \sigma^2(\epsilon_2) \\ \sigma^2(r_M) &= \frac{\sigma^2(r_2) - \sigma^2(\epsilon_2)}{\beta_2^2} = \frac{(0.25)^2 - (0.04)}{(0.75)^2} = 0.04 \end{aligned}$$

The standard deviations of stock 1 and 3 can now be found:

$$\sigma^2(r_1) = (2.0)^2(0.04) + 0.10 = 0.26; \sigma(r_1) = 0.5099.$$
$$\sigma^2(r_3) = (0.5)^2(0.04) + 0.17 = 0.18; \sigma(r_3) = 0.4243.$$

Stock	Expected Return	Standard Deviation	Beta	Residual Variance
1	0.15	**0.51**	2.00	0.10
2	**0.10**	0.25	0.75	0.04
3	0.09	**0.42**	0.5	0.17
risk free asset	0.07	0	0	0
market portfolio	0.11	0.2	1.00	0

Stock 3 has very high firm specific risk; $\sigma(r_3) = 0.4243$ is much higher than $\sigma(r_M) = 0.2$ but the expected return is only 9% as compared to $E[r_M] = 11\%$. This represents an inferior stock.

Remark

The CAPM predicts that the excess return on any stock (portfolio) adjusted for the risk on that stock (portfolio) should be the same

$$\frac{E[r_i] - r}{\beta_i} = \frac{E[r_j] - r}{\beta_j}. \tag{A}$$

This is in contrast to the Sharpe ratio, where

$$\frac{E[r_i] - r}{\sigma_i} \gtreqqless \frac{E[r_j] - r}{\sigma_j}. \tag{B}$$

The asset with a lower value of Sharpe ratio is considered inferior. Here, $\sigma_i^2 = \beta_i^2 \sigma_M^2 + \text{var}(\varepsilon_i)$ and $\sigma_j^2 = \beta_j^2 \sigma_M^2 + \text{var}(\varepsilon_j)$. The inferior asset has a high residual risk.

CAPM as a pricing formula

Suppose an asset is purchased at P and later sold at Q. The rate of return is $\dfrac{Q-P}{P}$, P is known and Q is random. Using the CAPM,

$$\frac{\overline{Q}-P}{P} = r + \beta(\overline{r}_M - r) \text{ so that } P = \frac{\overline{Q}}{1 + r + \beta(\overline{r}_M - r)}.$$

Here, P gives the fair price of the asset with expected value $\overline{Q}$ and beta β.

The factor $\dfrac{1}{1 + r + \beta(\overline{r}_M - r)}$ can be regarded as the risk adjusted discount rate.

Example (Investment in a mutual fund)

A mutual fund invests 10% of its funds at the risk free rate of 7% and the remaining 90% at a widely diversified portfolio with asymptotically low level of idiosyncratic risk that closely approximates the market portfolio, and $\overline{r}_M = 15\%$. The beta of the fund is then equal to 0.9.

Suppose the expected value of one share of the fund one year later is \$110, what should be the fair price of one share of the fund now?

According to the pricing form of the CAPM, the current fair price of one share $= \dfrac{\$110}{1 + 7\% + 0.9 \times (15 - 8)\%} = \dfrac{\$110}{1.142} = \$96.3$.

Implicitly, β also involves P since $\beta = \text{cov}\left(\frac{Q}{P} - 1, r_M\right) \bigg/ \sigma_M^2$ so that $\beta = \frac{\text{cov}(Q, r_M)}{P\sigma_M^2}$. We rearrange the terms in the CAPM pricing formula to solve for P explicitly

$$1 = \frac{\overline{Q}}{P(1+r) + \text{cov}(Q, r_M)(\overline{r}_M - r)/\sigma_M^2}$$

so that the fair price based on the CAPM is

$$P = \frac{1}{1+r}\left[\overline{Q} - \frac{\text{cov}(Q, r_M)(\overline{r}_M - r)}{\sigma_M^2}\right].$$

In this new form, the riskfree discount factor $\frac{1}{1+r}$ is applied on the certainty equivalent. Net present value of the asset is the difference between the fair price and the observed price, which is then given by

$$-P + \frac{1}{1+r}\left[\overline{Q} - \frac{\text{cov}(Q, r_M)(\overline{r}_M - r)}{\sigma_M^2}\right].$$

Difficulties with the mean-variance approach

1. Application of the mean-variance theory requires the determination of the parameter values: mean values of the asset returns and the covariances among them. Suppose there are n assets, then there are n mean values, n variances and $\frac{n(n-1)}{2}$ covariances. For example, when $n = 1,000$, the number of parameter values required $= 501,500$.

2. In the CAPM, there is really only one factor that influences the expected return, namely, β_{iM}.

The assumption of investors utilizing a mean variance framework is replaced by an assumption of the risk factors generating security returns.

7.3 Arbitrage pricing theory (APT) and factor models

- The APT rests on the law of one price in the financial market: portfolios with the same payoff have the same price. Arbitrage opportunities arise when two securities with the same payoff have different prices – buy the cheap one and sell the expensive one to secure a risk free profit. Absence of arbitrage $\Rightarrow$ Law of one price.

- The APT requires that the returns on any stock be linearly related to a number of risk factors.

- The return on a security can be broken down into an expected return and an unexpected (or surprise) component.

- Randomness displayed by the returns of n assets can be traced back to a smaller number of underlying basic sources of randomness (factors). Hopefully, this leads to a simpler covariance structure.

Single-factor model

The random rate of return r_i of asset i and the factor f are assumed to be linearly related by

$$r_i = a_i + b_i f + e_i \quad i = 1, 2, \cdots, n.$$

Here, f is the random quantity shared by all assets, a_i and b_i are fixed constants, e_i's are random errors (without loss of generality, take $E[e_i] = 0$). $b_i =$ factor loading; which measures the sensitivity of the return r_i to the factor. Further, we assume

$$\text{cov}(e_i, f) = 0 \quad \text{and} \quad E[e_i e_j] = 0, \quad i \neq j.$$

Interpret the CAPM model in terms of excess returns $r_i - r$ of any risky asset and $r_M - r$ of the market portfolio.

$$r_i - r = \beta_i (r_M - r) + e_i.$$

Here, $r_M - r$ is the single random factor that drives the return.

Specifying the factors (macroeconomic state variables) that affect the return-generating process

1. Inflation

   Inflation impacts both the level of the discount rate and the size of the future cash flows.

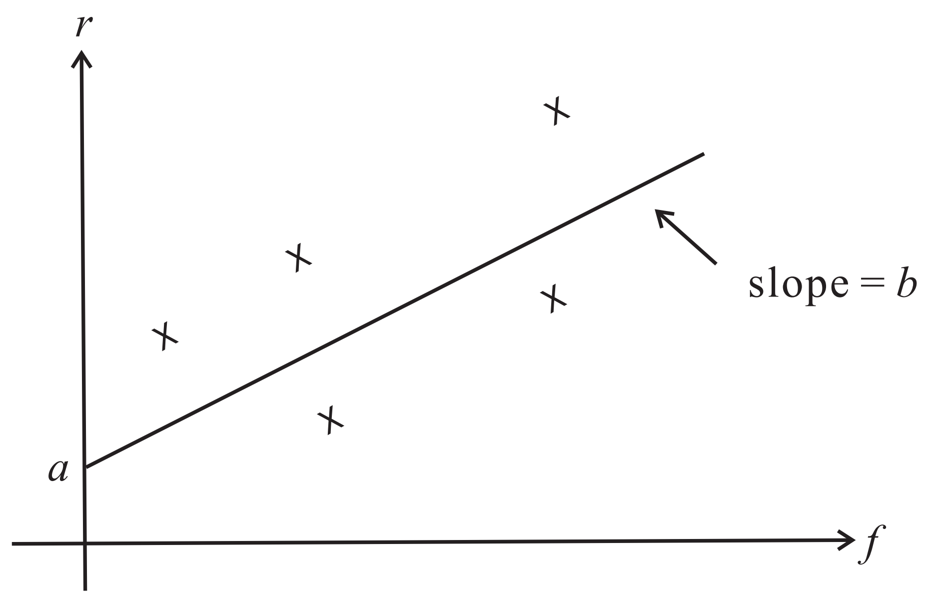
2. Risk premia

   Differences between the return on safe bonds and more risky bonds are used to measure the market's reaction to risk.

3. Industrial production

   Changes in industrial production affect the opportunities facing investors and the real value of cash flow.

Much of the empirical APT research has focused on the identification of these factors.



$\star$ Different data sets (past one month or two months data) may lead to different estimated values.

From $r_i = a_i + b_i f + e_i$, we deduce that

$$\begin{aligned}
\overline{r}_i &= a_i + b_i \overline{f} \\
\sigma_i^2 &= b_i^2 \sigma_f^2 + \sigma_{e_i}^2 && \text{[using cov}(f, e_i) = 0\text{]} \\
\sigma_{ij} &= b_i b_j \sigma_f^2, \quad i \neq j && \text{[using cov}(e_i, e_j) = 0 \text{ in addition]} \\
b_i &= \text{cov}(r_i, f)/\sigma_f^2.
\end{aligned}$$

Example (Four stocks and one index)

Historical rates of return for four stocks over 10 years, record of industrial price index over the same period.

Estimate of $\overline{r}_i$ is $\widehat{\overline{r}}_i = \dfrac{1}{10}\sum_{k=1}^{10} r_i^k$, where r_i^k is the observed rate of return of asset i in the k^{th} year. The estimated variances and co-variances are given by

$$\widehat{\text{var}}(r_i) = \frac{1}{9}\sum_{k=1}^{10}(r_i^k - \widehat{\overline{r}}_i)^2$$

$$\widehat{\text{cov}}(r_i, f) = \frac{1}{9}\sum_{k=1}^{10}(r_i^k - \widehat{\overline{r}}_i)(f^k - \widehat{\overline{f}}).$$

Once the covariances have been estimated, b_i and a_i are obtained:

$$b_i = \frac{\text{cov}(r_i, f)}{\text{var}(f)} \quad \text{and} \quad a_i = \widehat{\overline{r}}_i - b_i\widehat{\overline{f}}.$$

Also, e_i can be estimated once the estimated values of a_i and b_i are known.

We estimate the variance of the error under the assumption that these errors are uncorrelated with each other and with the index. The formula to be used is

$$\text{var}(e_i) = \text{var}(r_i) - b_i^2 \text{var}(f).$$

In addition, we estimate $\text{cov}(e_i, e_j)$ by following similar calculations as in $\widehat{\text{cov}}(r_i, f)$.

- Unfortunately, the error variances are almost as large as the variances of the stock returns.
- There is a high non-systematic risk, so the choice of this factor does not explain much of the variation in returns.
- Further, $\text{cov}(e_i, e_j)$ are not small so that the errors are highly correlated. We have

  $$\text{cov}(e_1, e_2) = 44 \quad \text{and} \quad \text{cov}(e_2, e_3) = 91.$$

  Recall that the factor model was constructed under the assumption of zero error covariances.

Year	Stock 1	Stock 2	Stock 3	Stock 4	Index
1	11.91	29.59	23.27	27.24	12.30
2	18.37	15.25	19.47	17.05	5.50
3	3.64	3.53	−6.58	10.20	4.30
4	24.37	17.67	15.08	20.26	6.70
5	30.42	12.74	16.24	19.84	9.70
6	−1.45	−2.56	−15.05	1.51	8.30
7	20.11	25.46	17.80	12.24	5.60
8	9.28	6.92	18.82	16.12	5.70
9	17.63	9.73	3.05	22.93	5.70
10	15.71	25.09	16.94	3.49	3.60
aver	15.00	14.34	10.90	15.09	6.74
var	90.28	107.24	162.19	68.27	6.99
cov	2.34	4.99	5.45	11.13	6.99
b	0.33	0.71	0.78	1.59	1.00
a	12.74	9.53	5.65	4.36	0.00
e-var	89.49	103.68	157.95	50.55	

The record of the rates of return for four stocks and an index of industrial prices are shown. The averages and variances are all computed, as well as the covariance of each with the index. From these quantities, the b_i*''s and the* a_i*'s are calculated. Finally, the computed error variances are also shown. The index does not explain the stock price variations very well.*

Portfolio risk under single-factor models – systematic and non-systematic risks

Let w_i denote the weight for asset $i, i = 1, 2, \cdots, n$.

$$r_P = \sum_{i=1}^{n} w_i a_i + \sum_{i=1}^{n} w_i b_i f + \sum_{i=1}^{n} w_i e_i$$

so that $r_P = a + bf + e$, where

$$a = \sum_{i=1}^{n} w_i a_i, \quad b = \sum_{i=1}^{n} w_i b_i \quad \text{and} \quad e = \sum_{i=1}^{n} w_i e_i.$$

Further, since $E[e_i] = 0, E[(f - \overline{f})e_i] = 0$ so that

$$E[e] = 0 \quad \text{and} \quad E[(f - \overline{f})e] = 0;$$

e and f are uncorrelated. Also, $\text{cov}(e_i, e_j) = 0, i \neq j$, so that $\sigma_e^2 = \sum_{i=1}^{n} w_i^2 \sigma_{e_i}^2$. Overall variance of portfolio $= \sigma^2 = b^2 \sigma_f^2 + \sigma_e^2$.

As an illustration suppose we take $\sigma_{e_i}^2 = S^2$ and $w_i = 1/n$ so that $\sigma_e^2 = \dfrac{S^2}{n}$. As $n \to \infty, \sigma_e^2 \to 0$. The overall variance of portfolio σ^2 tends to decrease as n increases since σ_e^2 goes to zero, but σ^2 does not go to zero since $b^2\sigma_f^2$ remains finite.

- The risk due to e_i is said to be *diversifiable* since its contribution to the overall risk is essentially zero in a well-diversified portfolio. This is because e_i's are uncorrelated and so each can be reduced by diversification. In simple sense, some of the ϵ_i will be positive and others negative, so their weighted sum is likely to be close to zero.

- The risk due to $b_i f$ is said to be systematic since it is present even in a diversified portfolio.

The return on the portfolio is made up of the expected returns on the individual securities and the random component arising from the single risk factor f.

Single-factor models with zero residual risk

Assume zero idiosyncratic (asset-specific) risk,

$$r_i = a_i + b_i f, \quad i = 1, 2, \cdots, n,$$

where the factor f is chosen to satisfy $E[f] = 0$ for convenience (with no loss of generality) so that $\overline{r}_i = a_i$.

Consider two assets which have two different factor loading b_i's, what should be the relation between their expected returns under the assumption of no arbitrage?

Consider a portfolio with weight w in asset i and weight $1 - w$ in asset j. The portfolio return is

$$r_P = w(a_i - a_j) + a_j + [w(b_i - b_j) + b_j]f.$$

By choosing $w^* = \dfrac{b_j}{b_j - b_i}$, the portfolio becomes risk free and

$$r_P^* = \frac{b_j(a_i - a_j)}{b_j - b_i} + a_j.$$

This must be equal to the return of the risk free asset, denoted by r. If otherwise, arbitrage opportunities arise. Suppose the risk free two-asset portfolio has a return higher than that of the riskfree asset, we then short sell the riskfree asset and long hold the risk free portfolio. We write the relation as

$$\frac{a_j - r}{b_j} = \frac{a_i - r}{b_i} = \lambda.$$

↑ set

Hence, $\overline{r}_i = r + b_i\lambda$, where λ is the factor risk premium. Note that when two assets have the same factor loading b, they have the same expected return.

1. The risk free return r is the expected return on a portfolio with zero factor loading.

2. In general, the term *risk premium* refers to the excess return above the riskfree rate of return demanded by an investor who bears the risk of the investment. The factor risk premium λ gives the extra return above r per unit loading of the risk factor,

$$\lambda = (\overline{r}_i - r)|_{b_i=1}.$$

3. Under the general single-factor model, where

$$r_i = a_i + b_i f + e_i,$$

$$\text{cov}(r_i, r_j) = \text{cov}(a_i + b_i f, a_j + b_j f) = b_i b_j \text{var}(f) = b_i b_j \sigma_f^2,$$

based on the usual assumption that

$$\text{cov}(e_i, f) = \text{cov}(e_j, f) = \text{cov}(e_i, e_j) = 0.$$

That is, the asset-specific risks are assumed to be uncorrelated with the factor risk and among themselves.

Numerical example

Given $a_1 = 0.10, b_1 = 2, a_2 = 0.08$ and $b_2 = 1$, and assuming $E[f] = e_1 = e_2 = 0$ for the two assets under the single-factor model, find the factor risk premium λ. How to construct the zero-beta portfolio from these two risky assets?

The two unknowns r and λ are determined from the no-arbitrage relation:

$$\frac{0.10 - r}{2} = \frac{0.08 - r}{1} = \lambda$$

so that $r = 0.06$ and $\lambda = 0.02$. The expected rate of return of the two assets are given by $\overline{r}_1 = 0.10 + 2\lambda$ and $\overline{r}_2 = 0.08 + \lambda$.

To construct a zero-beta portfolio, we long two units of asset 2 and short one unit of asset 1 so that

$$r_P = 2r_2 - r_1 = 2(0.08 + f) - (0.10 + 2f) = 0.06.$$

Two-factor extension

Consider the two-factor model

$$r_i = a_i + b_{i1} f_1 + b_{i2} f_2, \quad i = 1, 2, \cdots, n,$$

where the factors f_1 and f_2 are chosen such that

$$E[f_1 f_2] = 0, E[f_1^2] = E[f_2^2] = 1, E[f_1] = E[f_2] = 0.$$

Consider a 3-asset portfolio, with the assumption that $\mathbf{1}, \boldsymbol{b}_1 = \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \end{pmatrix}$ and $\boldsymbol{b}_2 = \begin{pmatrix} b_{12} \\ b_{22} \\ b_{32} \end{pmatrix}$ are linearly independent. Form the portfolio with weights w_1, w_2 and w_3 so that

$$r_P = \sum_{i=1}^{3} w_i a_i + f_1 \sum_{i=1}^{3} w_i b_{i1} + f_2 \sum_{i=1}^{3} w_i b_{i2}.$$

Since $\mathbf{1}, \boldsymbol{b}_1$ and $\boldsymbol{b}_2$ are independent, the following system of equations

$$\begin{pmatrix} 1 & 1 & 1 \\ b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad (A)$$

always has unique solution. By choosing this set of values for $w_i, i = 1, 2, 3$, the portfolio becomes riskfree. By applying the no-arbitrage argument again, the risk free portfolio should earn the return same as that of the riskfree asset, thus

$$r_P = \sum_{i=1}^{3} w_i a_i = r.$$

Rearranging, we obtain a new relation between w_1, w_2 and w_3:

$$\sum_{i=1}^{3} (a_i - r) w_i = 0. \qquad (B)$$

This implies that there exists a non-trivial solution to the following homogeneous system of linear equations:

$$\begin{pmatrix} a_1 - r & a_2 - r & a_3 - r \\ b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The above coefficient matrix must be singular. Since the second and third rows are independent, it must occur that the first row is formed by some linear combination of the second and third rows.

This gives

$$a_i - r = \overline{r}_i - r = \lambda_1 b_{i1} + \lambda_2 b_{i2}$$

for some constant parameters λ_1 and λ_2.

Remark

What happens if $\mathbf{1}, \boldsymbol{b}_1$ and $\boldsymbol{b}_2$ are not independent? In this case, we cannot form a riskfree portfolio using the 3 given assets as there is no solution to the linear system (A).

Factor risk premium: λ_1 *and* λ_2

– interpreted as the excess expected return per unit loading associated with the factors f_1 and f_2.

For example, $\lambda_1 = 3\%, \lambda_2 = 4\%$, factor loadings are $b_{i1} = 1.2, b_{i2} = 0.7, r = 7\%$, then

$$\overline{r}_i = 7\% + 1.2 \times 3\% + 0.7 \times 4\% = 13.6\%.$$

Absence of the riskfree asset

$$\overline{r}_i - r = \lambda_1 b_{i1} + \lambda_2 b_{i2}, \quad i = 1, 2, \cdots, n.$$

If no risk free asset exists naturally, then we replace r by λ_0, where λ_0 is the return of the zero-beta asset (whose factor loadings are all zero). Note that the zero-beta asset is riskfree. Once λ_0, λ_1 and λ_2 are known, the expected return of an asset is completely determined by the factor loadings b_{i1} and b_{i2}. Theoretically, a riskless asset can be constructed from any three risky assets so that λ_0 can be determined.

Indeed, we choose a solution $\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ that satisfies Eq. (A), we obtain a risk free portfolio. We then set

$$\lambda_0 = \sum_{i=1}^{3} w_i a_i.$$

The expected rate of return becomes $\overline{r}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2}$.

Numerical example

Consider 3 assets whose rates of return are governed by

$$\begin{aligned} r_1 &= 5 + 2f_1 + 3f_2 \\ r_2 &= 6 + f_1 + 2f_2 \\ r_3 &= 4 + 6f_1 + 10f_2, \end{aligned}$$

where f_1 and f_2 are the risk factors. We can form a riskfree portfolio by assigning weights w_1, w_2 and w_3, which can obtained by solving

$$\begin{aligned} w_1 + w_2 + w_3 &= 1 \\ 2w_1 + w_2 + 6w_3 &= 0 \\ 3w_1 + 2w_2 + 10w_3 &= 0. \end{aligned}$$

The solution of the above system of equations gives $w_1 = w_2 = \frac{2}{3}$ and $w_3 = -\frac{1}{3}$.

This riskfree portfolio has zero factor loading (or called zero-beta portfolio). It deterministic rate of return $= 5w_1 + 6w_2 + 4w_3 = 6$. This is the same as the riskfree rate, and it is called λ_0. To determine the factor risk premia λ_1 and λ_2, we observe

$$\begin{pmatrix} -1 & 0 & -2 \\ 2 & 1 & 6 \\ 3 & 2 & 10 \end{pmatrix} \begin{pmatrix} 2/3 \\ 2/3 \\ -1/3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The first row can be written as (-2) times the second row plus the third row, so $\lambda_1 = -2$ and $\lambda_2 = 1$. We then have

$$\begin{aligned} \overline{r}_1 &= \lambda_0 + 2\lambda_1 + 3\lambda_2 = 6 - 4 + 3 = 5; \\ \overline{r}_2 &= \lambda_0 + \lambda_1 + 2\lambda_2 = 6 - 2 + 2 = 6; \\ \overline{r}_3 &= \lambda_0 + 6\lambda_1 + 10\lambda_2 = 6 - 12 + 10 = 4. \end{aligned}$$

Remarks

1. In a well functioning market, the factor risk premia should be all positive. In this example, we obtain $\lambda_1 = -2$. This is because r_3 has a low value of expected value ($\overline{r}_3 = 4$), though r_3 has high factor loading. This leads to negative risk premium value.

   Suppose we modify the expected return values to assume some higher numerical values; for example

   $$r_1 = 13 + 2f_1 + 3f_2, \quad r_2 = 10 + f_1 + 2f_2, \quad r_3 = 28 + 6f_1 + 10f_2.$$

   The new $\lambda_0 = \frac{2}{3} \times 13 + \frac{2}{3} \times 10 - \frac{1}{3} \times 28 = 6$. The new first row $= (13 - \lambda_0 \quad 10 - \lambda_0 \quad 28 - \lambda_0) = (7 \quad 4 \quad 22)$, which can be expressed as 2 times the second row $= (2 \quad 1 \quad 6)$ plus the third row $= (3 \quad 2 \quad 10)$. We obtain $\lambda_1 = 2$ and $\lambda_2 = 1$.

2. Suppose we modify the risk factors by some scalar multiples, say, new factors $\widetilde{f}_1$ and $\widetilde{f}_2$ are chosen to be $\widetilde{f}_1 = 2f_1$ and $\widetilde{f}_2 = 3f_2$. The factor loading b_{i1} and b_{i2} are reduced by a factor of $\frac{1}{2}$ and $\frac{1}{3}$, respectively. We now have

$$\begin{aligned} r_1 &= 5 + \widetilde{f}_1 + \widetilde{f}_2 \\ r_2 &= 6 + \frac{\widetilde{f}_1}{2} + \frac{2}{3}\widetilde{f}_2 \\ r_3 &= 4 + 3\widetilde{f}_1 + \frac{10}{3}\widetilde{f}_2. \end{aligned}$$

The new factor risk premia become $\widetilde{\lambda}_1 = 2\lambda_1 = -4$ and $\widetilde{\lambda}_2 = 3\lambda_2 = 3$. Not surprisingly, we obtain the same results for the expected rates of return of the assets.

3. In the derivation of the factor risk premia, we have assumed zero idiosyncratic risk for all asset returns; that is, $e_j = 0$ for all assets. When idiosyncratic risks are present, we obtain the same result for the factor risk premia for a well diversified portfolio (under the assumption of so-called asymptotic arbitrage). That is, the expected excess return above the riskfree rate is given by the sum of the product of the factor loading and factor risk premium for each risk factor.

Expected excess return in terms of the expected excess return of two portfolios

Given any two portfolios P and M with $\dfrac{b_{P1}}{b_{P2}} \neq \dfrac{b_{M1}}{b_{M2}}$, we can solve for λ_1 and λ_2 in terms of the expected excess return on these two portfolios: $\overline{r}_M - r$ and $\overline{r}_P - r$. The governing equations for the determination of λ_1 and λ_2 are

$$\begin{aligned} \overline{r}_P - r &= \lambda_1 b_{P1} + \lambda_2 b_{P2} \\ \overline{r}_M - r &= \lambda_1 b_{M1} + \lambda_2 b_{M2}. \end{aligned}$$

Once λ_1 and λ_2 are obtained in terms of $\overline{r}_P - r, \overline{r}_M - r$ and factor loading coefficients, we then have the following CAPM-like formula:

$$\overline{r}_i = r + \lambda_1 b_{i1} + \lambda_2 b_{i2} = r + b'_{i1}(\overline{r}_M - r) + b'_{i2}(\overline{r}_P - r)$$

where

$$b'_{i1} = \frac{b_{i1}b_{P2} - b_{i2}b_{P1}}{b_{M1}b_{P2} - b_{M2}b_{P1}}, \qquad b'_{i2} = \frac{b_{i2}b_{M1} - b_{i1}b_{M2}}{b_{M1}b_{P2} - b_{M2}b_{P1}}.$$

Numerical example

Consider the previous example with the following 2 assets:

$$r_1 = 5 + 2f_1 + 3f_2, \quad r_2 = 6 + f_1 + 2f_2;$$

with riskfree rate $r = 6$ (here r is given). Now, λ_1 and λ_2 are governed by

$$\overline{r}_1 - 6 = 2\lambda_1 + 3\lambda_2, \quad \overline{r}_2 - 6 = \lambda_1 + 2\lambda_2;$$

so that

$$\lambda_1 = \frac{\begin{vmatrix} \overline{r}_1 - 6 & 3 \\ \overline{r}_2 - 6 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}} = 2(\overline{r}_1 - 6) - 3(\overline{r}_2 - 6)$$

$$\lambda_2 = \frac{\begin{vmatrix} 2 & \overline{r}_1 - 6 \\ 1 & \overline{r}_2 - 6 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}} = 2(\overline{r}_2 - 6) - (\overline{r}_1 - 6).$$

Lastly, we express the expected excess of the third asset in CAPM-like form:

$$\overline{r}_3 = r + 6\lambda_1 + 10\lambda_2 = 6 + 2(\overline{r}_1 - 6) + 2(\overline{r}_2 - 6).$$

Remark

With n risk factors, we may write the excess return above r of the asset j as sum of scalar multiples of $\overline{r}_1 - r, \overline{r}_2 - r, \cdots, \overline{r}_n - r$. That is,

$$\overline{r}_j - r = \sum_{k=1}^{n} b_{jk}^1 (\overline{r}_k - r).$$

Summary of assumptions on the capital market

- The capital market is characterized by perfect competition.

- For example, there are no investor who hold more market information than others. There are a large number of investors, each with wealth that is small relative to the total market value of all capital assets. Hence, the portfolio choice of individual investors has no noticeable effect on the prices of the securities; investors take the price as given.

- Capital market imperfections such as transaction costs and taxes do not occur.

- All investors have the same expectations regarding the future in terms of mean, variance and covariance terms (homogeneous expectations).

- The expectations are captured by a return distribution that is described by a factor risk model; rates of return depend on some common risk factors and some random asset-specific residual.

- The asset-specific residual has a zero mean, is uncorrelated across assets and is uncorrelated with the common factors.

Compare and contrast the CAPM and the APT

- The APT does not assume that investors make decisions according to the mean-variance rule.

- The primary assumption of the APT is that security returns are generated by a linear factor model. The APT is based on a no-arbitrage condition – riskfree portfolio should earn the same rate of return as that of the riskfree asset.

- The single-index model drastically reduces the inputs needed in solving for the optimum portfolios in the efficient frontier, since the covariances can be calculated easily: $\text{cov}(r_i, r_j) = \beta_i \beta_j \sigma_f^2$.

Sources of risk for the expected excess return on portfolios

Assume that we have identified four factors in the return-generating model

I_1 = unexpected change in inflation, denoted by I_I

I_2 = unexpected change in aggregate sales, denoted by I_S

I_3 = unexpected change in oil prices, denoted by I_O

I_4 = the return in the S&P index constructed to be orthogonal to the other factors, denoted by I_M.

Furthermore, assume that the oil risk is not priced, $\lambda_O = 0$; then

$$\overline{r}_i - r = \lambda_I b_{iI} + \lambda_S b_{iS} + \lambda_M b_{iM}.$$

Factor	b	λ	Contribution to mid-cap Expected Excess Return (%)
Inflation	-0.37	-4.32	1.59
Sales growth	1.71	1.49	2.54
Oil prices	0.00	0.00	0.00
Market	1.00	3.96	3.96
Expected excess return for mid-cap stock portfolio			8.09

The expected excess return for the mid-cap stock portfolio is 8.09%. Sales growth contributes 2.54% to the expected return for the mid-cap. In other words, sensitivity to sales growth accounts for $2.54 \div 8.09$ or 31.4% of the total expected excess return.

Factor	b	λ	Contribution to Growth Stock Portfolio Expected Excess Return (%)
Inflation	−0.50	−4.32	2.16
Sales growth	2.75	1.49	4.10
Oil prices	−1.00	0.00	0.00
Market	1.30	3.96	5.15
Expected excess return for growth stock portfolio			11.41

- The expected excess return for the growth stock portfolio (11.41%) is higher than it was for the mid-cap (8.09%). The growth stock portfolio has more risk, with respect to each index, than the mid-cap portfolio.

- Individual factors have a different absolute and relative contribution to the expected excess return on a growth stock portfolio than they have on the mid-cap index.

Portfolio management

Factor models are used to estimate short-run expected returns to the asset classes. The factors are usually macroeconomic variables, some of which are list below:

1. The rate of return on a treasury bill (T bill).
2. The difference between the rate of return on a short-term and long-term government bond (term).
3. Unexpected changes in the rate of inflation in consumer prices (inflation).
4. Expected percentage changes in industrial production (ind. prod.).
5. The ratio of dividend to market price for the S&P 500 in the month preceding the return (yield).
6. The difference between the rate of return on a low- and high-quality bond (confidence).
7. Unexpected percentage changes in the price of oil (oil).

Four distinctive phases of the market are identified which are based on the directional momentum in stock prices and earnings per share:

1. The initial phase of a bull market.

2. The intermediate phase of a bull market.

3. The final phase of a bull market.

4. The bear market.

Interestingly, for a given type of stock, the factor sensitivities can change dramatically as the market moves from one phase to the next.

Bear market $\longrightarrow$ *Initial phase of bull market*

The factor sensitivities for large versus small stocks in going from a bear market to the initial phase of a bull market are listed below:

	Phase IV		*Phase I*	
Factor	Small Stocks	Large Stocks	Small Stocks	Large Stocks
T bill	−6.45	−1.21	5.16	5.81
Term	0.34	0.45	0.86	0.92
Inflation	−3.82	−2.45	−3.23	−2.20
Ind. prod.	0.54	0.06	0.00	0.40
Yield	1.51	−0.16	−0.18	0.00
Confidence	−0.63	−0.43	2.46	1.45
Oil	−0.21	−0.07	0.26	0.20

Asset allocation decision procedure

- Identify the current market phase, calculate the factor values typically experienced in such a phase, and make modifications in these factor averages to reflect expectations for the forthcoming period (usually a year).

- Calculate expected returns for the asset classes (such as large and small stocks) on the basis of the factor sensitivities in the phase.

- These expected returns can then be imported to an optimizer to determine the mix of investments that maximizes expected return given risk exposure for the forthcoming year.

Summary of key concepts

Construction of the uncorrelated counterpart of a frontier fund

- Recall that zero-beta funds are those funds which lie on the same horizontal line with the riskfree point. Given the riskfree point, we determine the market portfolio by the tangency method. Conversely, given a frontier fund, we find the corresponding riskfree point such that the frontier fund becomes the market portfolio. This is done by drawing a tangent to the efficient frontier at the frontier fund and finding the intercept of this tangent line at the vertical $\overline{r}$-axis.

Capital market line and efficient portfolios

- All portfolios lying on the CML are efficient, and all are composed of various mixes of the market portfolio and the risk free asset.

- The beta value of an efficient portfolio is equal to the proportional weight of market portfolio in the efficient portfolio. This is obvious since the excess return above the riskfree rate is contributed by the portion of market portfolio.

- The CML does not apply to individual asset or portfolios that are inefficient, because investors do not require a compensation for non-systematic risk.

- Efficient portfolios have the same Sharpe ratio as that of the market portfolio.

- All portfolios are on or below the CML. When the correlation coefficient between portfolio's return and market return is closer to 100%, the portfolio is closer to being efficient and comes closer to the CML. The ratio of the Sharpe ratios is simply the correlation coefficient between the portfolio return and market returns.

- Efficient portfolios have zero specific (diversifiable or non-systematic) risk.

- An efficient portfolio has 100% correlation with the Market Portfolio. Security returns are driven by the market portfolio. The extended version of CAPM allow the replacement of the Market Portfolio by any efficient portfolio.

Security market line

- In equilibrium, all assets and portfolios lie on the security market line. All assets are priced correctly and one cannot find bargains. Any derivation from the SML implies that the market is not in the CAPM equilibrium.

- When equilibrium prevails, the expected excess return above the riskfree rate normalized by the beta is constant for all assets / portfolios.

Beta value

- According to CAPM, the higher the asset risk (beta), the higher the expected rate of return will be.

- All assets/portfolios with the same beta share the same amount of systematic risk, and they have the same excess return above the riskfree rate. The beta value (not portfolios standard deviation) is used as a measure of risk in CAPM since only the systematic risk is rewarded with extra returns. When the specific risk becomes zero, the portfolio standard deviation equals beta times market portfolio's standard deviation.

Systematic risk

- The variance of a security's returns stems from overall market movements and is measured by beta. It is only this risk that investors are rewarded for bearing.

- Systematic risk is given by β^2 times market return's variance.

Unsystematic (firm-specific or idiosyncratic) risk

Diversifiable risk that is unique to a particular stock / portfolio. The residual risks are uncorrelated to the market portfolio.

APT model

- Returns of assets are driven by a set of macroeconomic factors and noise / asset-specific component.

- The APT does not require identification of the market portfolio, but it does require the specification of the relevant macroeconomics factors. Much of the empirical APT research has focused on the identification of these factors.

- The APT and CAPM together complement each other. They both predict that positive returns will result from factor sensitivities that move with the market.