MATH685Z – Mathematic Models in Financial Economics

Homework Four

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1. Show that a dominant trading strategy exists if and only if there exists a trading strategy satisfying $V_0 < 0$ and $V_1(\omega) \geq 0$ for all $\omega \in \Omega$.

   Hint: Consider the dominant trading strategy $\mathcal{H} = (h_0 \; h_1 \cdots h_M)^T$ satisfying $V_0 = 0$ and $V_1(\omega) > 0$ for all $\omega \in \Omega$. Take $G^*_{\min} = \min_\omega G^*(\omega) > 0$ and define a new trading strategy with $\widehat{h}_m = h_m, m = 1, \cdots, M$ and $\widehat{h}_0 = -G^*_{\min} - \sum_{m=1}^{M} h_m S^*_m(0)$.

2. Consider a portfolio with one risky security and the riskfree security. Suppose the price of the risky asset at time 0 is 4 and the possible values of the $t = 1$ price are $1.1, 2.2$ and 3.3 (3 possible states of the world at the end of a single trading period). Let the riskfree interest rate r be 0.1 and take the price of the riskfree security at $t = 0$ to be unity.

   (a) Show that the trading strategy: $h_0 = 4$ and $h_1 = -1$ is a dominant trading strategy that starts with zero wealth and ends with positive wealth with certainty.

   (b) Find the discounted gain G^* over the single trading period.

   (c) Find a trading strategy that starts with negative wealth and ends with non-negative wealth with certainty.

3. Given the discounted terminal payoff matrix

$$\widehat{S}^*(1;\Omega) = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 1 & 3 \\ 1 & 2 & 4 \end{pmatrix},$$

   and the current price vector $\widehat{\boldsymbol{S}}^*(0) = (1 \quad 2 \quad 3)$, find the state price of the Arrow security with discounted payoff $\boldsymbol{e}_k, k = 1, 2, 3$.

   Does the securities model admit dominant trading strategies? If so, find an example where one trading strategy dominates the other.

4. Consider the securities model with

$$\widehat{\boldsymbol{S}}^*(0) = (1 \quad 2 \quad 3 \quad k) \quad \text{and} \quad \widehat{S}^*(1;\Omega) = \begin{pmatrix} 1 & 2 & 6 & 9 \\ 1 & 3 & 3 & 7 \\ 1 & 6 & 12 & 19 \end{pmatrix},$$

   determine the value of k such that the law of one price holds. Taking this particular value of k in $\widehat{\boldsymbol{S}}^*(0)$, does the securities model admit dominant trading strategies. If yes, find one such dominant trading strategy.

5. Suppose a betting game has 3 possible outcomes. If a gambler bets on outcome i, then he receives a net gain of d_i dollars for one dollar betted, $i = 1, 2, 3$. The payoff matrix thus takes the form (consideration of discounting is not necessary in a betting game)

$$S(1;\Omega) = \begin{pmatrix} d_1 + 1 & 0 & 0 \\ 0 & d_2 + 1 & 0 \\ 0 & 0 & d_3 + 1 \end{pmatrix}.$$

   Find the condition on d_i such that a risk neutral probability measure exists for the above betting game (visualized as an investment model).

6. Define the pricing functional $F(\boldsymbol{x})$ on the asset span $\mathcal{S}$ by $F(\boldsymbol{x}) = \{y : y = \boldsymbol{S}^*(0)\boldsymbol{h}$ for some $\boldsymbol{h}$ such that $\boldsymbol{x} = S^*(1)\boldsymbol{h}$, where $\boldsymbol{x} \in \mathcal{S}\}$. Show that if the law of one price holds, then F is a *linear* functional.

7. Consider the following securities model

$$S^*(1;\Omega) = \begin{pmatrix} 3 & 4 \\ 2 & 5 \\ 2 & 4 \end{pmatrix}, \quad \boldsymbol{S}^*(0) = (2 \quad 4),$$

do risk neutral measures exist? If not, explain why. If yes, find the set of all risk neutral measures.

8. Suppose the set of risk neutral measures for a given securities model is non-empty, show that if the securities model is complete then the set of risk neutral measures must be singleton.

   Hint: Under market completeness, column rank of $\widehat{S}(1;\Omega)$ equals number of states. Since column rank = row rank, all rows of $\widehat{S}^*(1;\Omega)$ are independent.

9. Let P be the true probability measure, where $P(\omega)$ denotes the actual probability that the state ω occurs. Define the *state price density* by the random variable $L(\omega) = Q(\omega)/P(\omega)$, where Q is a risk neutral measure. Use R_m to denote the return of the risky security m, where $R_m = [S_m(1) - S_m(0)]/S_m(0), m = 1, \cdots, M$. Show that $E_Q[R_m] = r, m = 1, \cdots, M$, where r is the interest over one period. Let $E_P[R_m]$ denote the expectation of R_m under the actual probability measure P, show that

$$E_P[R_m] - r = -\text{cov}(R_m, L),$$

where cov denotes the covariance operator.