MATH685Z – Mathematic Models in Financial Economics

Homework Five

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1. Suppose $T = 2, K = 4, N = 1$, the interest rate r is a constant satisfying $0 \le r < 0.125$, and the price process for the risky security and the probability measure are as follows:

ω	$S_0(\omega)$	$S_1(\omega)$	$S_2(\omega)$	$P(\omega)$
ω_1	5	8	9	1/4
ω_2	5	8	6	1/4
ω_3	5	4	6	1/4
ω_4	5	4	3	1/4

In addition, suppose the investor has an exponential utility function:

$$u(w) = 1 - \exp\{-w\}.$$

In view of the predictability requirement, the strategy H_1 for trading the risky security entails the specification of three scalar values: the position, denoted by H^5, carried forward from time 0 when the price $S_0 = 5$, the position, denoted by H^8, carried forward from time 1 when the price $S_1 = 8$, and the position, denoted by H^4, carried forward from time 1 when the price $S_1 = 4$. Show that

$$\begin{aligned} H^5 &= \frac{3\ln(3-5r) + (2-4r)\ln(2-4r) + (1+4r)\ln(1+4r)}{12(1+r)} \\ &\quad - \frac{3\ln(1+5r) + (2+8r)\ln(2+8r) + (1-8r)\ln(1-8r)}{12(1+r)}; \\ H^8 &= -\frac{1}{3}\ln\left(\frac{2+8r}{1-8r}\right); \quad H^4 = \frac{1}{3}\ln\left(\frac{2-4r}{1+4r}\right). \end{aligned}$$

It remains to compute H_0, the strategy for trading the bank account. Clearly $H_0(1) = v - 5H^5$, where v is the initial wealth. Show that

$$H_0(2) = \begin{cases} v - 5H^5 + \frac{8}{1+r}(H^5 - H^8) & \text{for } \omega_1 \text{ and } \omega_2 \\ v - 5H^5 + \frac{8}{1+r}(H^5 - H^4) & \text{for } \omega_3 \text{ and } \omega_4 \end{cases}.$$

2. For the securities model in Question 1, we now set $r = 0$. Compute the optimal trading strategy for the following utility functions. Verify that the strategies are the same and explain why this is so.

   (a) $u(w) = -\frac{1}{w}$;

   (b) $u(w) = \beta w - \frac{w^2}{2}$.

3. For the securities model in Question 1, with constant interest rate r where $0 \le r < 0.125$. Take $u(w) = \ln w$, compute the optimal attainable wealth, the optimal objective value, and the optimal trading strategy using the risk neutral computation approach.

4. Suppose $u(w) = \ln w$. Show that the inverse function $I(i) = i^{-1}$, the Lagrange multiplier $\lambda = v^{-1}$, the optimal attainable wealth is $W = vL^{-1}B_1$, and the optimal objective value is $\ln(v) - E[\ln(L/B_1)]$. Compute these expressions and solve for the optimal trading strategy in the case where $N = 1, K = 2, r = 1/9, S_0 = 5, S_1(\omega_1) = 20/3, S_1(\omega_2) = 40/9$, and $P(\omega_1) = 3/5$.

5. Suppose $u(w) = \gamma^{-1}w^\gamma$, where $-\infty < \gamma < 1$ and $\gamma \neq 0$. Show that the inverse function $I(i) = i^{-1/(1-\gamma)}$, the Lagrange multiplier

$$\lambda = v^{-(1-\gamma)}\{E[(L/B_1)^{-\gamma/(1-\gamma)}]\}^{(1-\gamma)}$$

the optimal attainable wealth

$$W = \frac{v(L/B_1)^{-1/(1-\gamma)}}{E[(L/B_1)^{-\gamma/(1-\gamma)}]}$$

and the optimal objective value $E[u(W)] = \lambda v/\gamma$. Compute these expressions and solve for the optimal trading strategy in the case where the underlying model is as in Problem 2.

6. Derive formulas for λ, C_0, and C_1 for the consumption investment problem in the case where the utility function is:

   (a) $u(c) = -\exp(-c)$.

   (b) $u(c) = \gamma^{-1}c^\gamma$, where $-\infty < \gamma < 1$ and $\gamma \neq 0$.

7. Suppose we allow the customer to have income or endowment $\widetilde{E}$ at time $t = 1$, where $\widetilde{E}$ is a specified random variable. Consider the optimization problem:

$$\begin{array}{rl} \text{maximize} & u(C_0) - E[u(C_1)] \\ \text{subject to} & C_0 + H_0B_0 + \sum_{n=1}^N H_nS_n(0) = v \\ & C_1 - H_0B_1 - \sum_{n=1}^N H_nS_n(1) = \widetilde{E} \\ & C_0 \geq 0, \quad C_1 \geq 0, \quad H \in \mathbb{R}^{N+1}. \end{array}$$

The pair $(v, \widetilde{E})$ is sometimes called the *endowment process* for the consumer. Show that the consumption-investment plan (C, H) is admissible if and only if

$$C_0 + E_Q[C_1 - \widetilde{E}/B_1] = \nu$$

for every risk neutral probability measure Q.

8. Assume a one-period model. The aggregate consumption at time 0 is 8 units. There are three states at time 1, $\{\omega_1, \omega_2, \omega_3\}$. All agents have homogeneous beliefs, and the probability of each state is 1/3. (This is the P measure.) The aggregate consumption in these states is

$$C(\omega_1) = 64, \quad C(\omega_2) = 27, \quad C(\omega_3) = 125.$$

The representative agent's utility function is of the form

$$v(c_0, C_1) = c_0^{1/3} + C_1^{1/3}.$$

Suppose the three Arrow-Debreu securities are traded in this model. Compute the prices of these three securities. A traded asset exists that pays 1% of aggregate consumption at time 1 in each state. Find the price of this asset at time 0.

9. There are $K = 3$ states and $N = 3$ securities with the payouts

$$d_1(\omega_1) = 24, \quad d_2(\omega_1) = 44, \quad d_3(\omega_1) = 12$$
$$d_1(\omega_2) = 20, \quad d_2(\omega_2) = 44, \quad d_3(\omega_2) = 12.$$

The prices of these securities are

$$p_1 = 35, \quad p_2 = 40, \quad \text{and} \quad p_3 = 12.$$

(a) Find the set of all the attainable consumption processes.

(b) Is the consumption process

$$c(0) = 10, \quad c(T, \omega_1) = 6, \quad c(T, \omega_2) = 5, \quad c(T, \omega_3) = 12$$

attainable? Find the initial endowment and the trading strategy that attain it.

(c) Is the consumption process

$$c(0) = 0, \quad c(T, \omega_1) = 9, \quad c(T, \omega_2) = 1, \quad c(T, \omega_3) = 17$$

attainable? Find the initial endowment and the trading strategy and attain it.

(d) Does the given price system permit arbitrage strategies?

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