1. Suppose $T = 2, K = 4, N = 1$, the interest rate $r$ is a constant satisfying $0 \leq r < 0.125$, and the price process for the risky security and the probability measure are as follows:

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$S_0(\omega)$</th>
<th>$S_1(\omega)$</th>
<th>$S_2(\omega)$</th>
<th>$P(\omega)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>5</td>
<td>8</td>
<td>9</td>
<td>1/4</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>5</td>
<td>8</td>
<td>6</td>
<td>1/4</td>
</tr>
<tr>
<td>$\omega_3$</td>
<td>5</td>
<td>4</td>
<td>6</td>
<td>1/4</td>
</tr>
<tr>
<td>$\omega_4$</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>1/4</td>
</tr>
</tbody>
</table>

In addition, suppose the investor has an exponential utility function:

$$u(w) = 1 - \exp\{-w\}.$$

In view of the predictability requirement, the strategy $H_1$ for trading the risky security entails the specification of three scalar values: the position, denoted by $H^5$, carried forward from time 0 when the price $S_0 = 5$, the position, denoted by $H^8$, carried forward from time 1 when the price $S_1 = 8$, and the position, denoted by $H^4$, carried forward from time 1 when the price $S_1 = 4$. Show that

$$H^5 = \frac{3\ln(3 - 5r) + (2 - 4r)\ln(2 - 4r) + (1 + 4r)\ln(1 + 4r)}{12(1 + r)} - \frac{3\ln(1 + 5r) + (2 + 8r)\ln(2 + 8r) + (1 - 8r)\ln(1 - 8r)}{12(1 + r)};$$

$$H^8 = -\frac{1}{3}\ln\left(\frac{2 + 8r}{1 - 8r}\right); \quad H^4 = \frac{1}{3}\ln\left(\frac{2 - 4r}{1 + 4r}\right).$$

It remains to compute $H_0$, the strategy for trading the bank account. Clearly $H_0(1) = v - 5H^5$, where $v$ is the initial wealth. Show that

$$H_0(2) = \begin{cases} v - 5H^5 + \frac{8}{1 + r}(H^5 - H^8) & \text{for } \omega_1 \text{ and } \omega_2 \\ v - 5H^5 + \frac{8}{1 + r}(H^5 - H^4) & \text{for } \omega_3 \text{ and } \omega_4 \end{cases}.$$

2. For the securities model in Question 1, we now set $r = 0$. Compute the optimal trading strategy for the following utility functions. Verify that the strategies are the same and explain why this is so.

(a) $u(w) = -\frac{1}{w}$;

(b) $u(w) = \beta w - \frac{w^2}{2}$.

3. For the securities model in Question 1, with constant interest rate $r$ where $0 \leq r < 0.125$. Take $u(w) = \ln w$, compute the optimal attainable wealth, the optimal objective value, and the optimal trading strategy using the risk neutral computation approach.
4. Suppose \( u(w) = \ln w \). Show that the inverse function \( I(i) = i^{-1} \), the Lagrange multiplier \( \lambda = v^{-1} \), the optimal attainable wealth is \( W = vL^{-1}B_1 \), and the optimal objective value is \( \ln(v) - E[\ln(L/B_1)] \). Compute these expressions and solve for the optimal trading strategy in the case where \( N = 1, K = 2, r = 1/9, S_0 = 5, S_1(\omega_1) = 20/3, S_1(\omega_2) = 40/9, \) and \( P(\omega_1) = 3/5 \).

5. Suppose \( u(w) = \gamma^{-1}w^\gamma \), where \( -\infty < \gamma < 1 \) and \( \gamma \neq 0 \). Show that the inverse function \( I(i) = i^{-1/(1-\gamma)} \), the Lagrange multiplier

\[
\lambda = v^{-(1-\gamma)} \left\{ E[(L/B_1)^{-\gamma/(1-\gamma)}] \right\}^{(1-\gamma)}
\]

the optimal attainable wealth

\[
W = \frac{v(L/B_1)^{-1/(1-\gamma)}}{E[(L/B_1)^{-\gamma/(1-\gamma)}]}
\]

and the optimal objective value \( E[u(W)] = \lambda v/\gamma \). Compute these expressions and solve for the optimal trading strategy in the case where the underlying model is as in Problem 2.

6. Derive formulas for \( \lambda, C_0, \) and \( C_1 \) for the consumption investment problem in the case where the utility function is:

(a) \( u(c) = -\exp(-c) \).

(b) \( u(c) = \gamma^{-1}c^\gamma \), where \( -\infty < \gamma < 1 \) and \( \gamma \neq 0 \).

7. Suppose we allow the customer to have income or endowment \( \tilde{E} \) at time \( t = 1 \), where \( \tilde{E} \) is a specified random variable. Consider the optimization problem:

\[
\begin{align*}
\text{maximize} \quad & u(C_0) - E[u(C_1)] \\
\text{subject to} \quad & C_0 + H_0B_0 + \sum_{n=1}^{N} H_nS_n(0) = v \\
& C_1 - H_0B_1 - \sum_{n=1}^{N} H_nS_n(1) = \tilde{E} \\
& C_0 \geq 0, \quad C_1 \geq 0, \quad H \in \mathbb{R}^{N+1}.
\end{align*}
\]

The pair \((v, \tilde{E})\) is sometimes called the endowment process for the consumer. Show that the consumption-investment plan \((C, H)\) is admissible if and only if

\[
C_0 + E_Q[C_1 - \tilde{E}/B_1] = v
\]

for every risk neutral probability measure \( Q \).

8. Assume a one-period model. The aggregate consumption at time 0 is 8 units. There are three states at time 1, \( \{\omega_1, \omega_2, \omega_3\} \). All agents have homogeneous beliefs, and the probability of each state is 1/3. (This is the \( P \) measure.) The aggregate consumption in these states is

\[
C(\omega_1) = 64, \quad C(\omega_2) = 27, \quad C(\omega_3) = 125.
\]

The representative agent’s utility function is of the form

\[
v(c_0, C_1) = c_0^{1/3} + C_1^{1/3}.
\]

Suppose the three Arrow-Debreu securities are traded in this model. Compute the prices of these three securities. A traded asset exists that pays 1% of aggregate consumption at time 1 in each state. Find the price of this asset at time 0.
9. There are $K = 3$ states and $N = 3$ securities with the payouts

$$
d_1(\omega_1) = 24, \quad d_2(\omega_1) = 44, \quad d_3(\omega_1) = 12 \\
d_1(\omega_2) = 20, \quad d_2(\omega_2) = 44, \quad d_3(\omega_2) = 12. $$

The prices of these securities are

$$
p_1 = 35, \quad p_2 = 40, \quad \text{and} \quad p_3 = 12. $$

(a) Find the set of all the attainable consumption processes.

(b) Is the consumption process

$$
c(0) = 10, \quad c(T, \omega_1) = 6, \quad c(T, \omega_2) = 5, \quad c(T, \omega_3) = 12 $$

attainable? Find the initial endowment and the trading strategy that attain it.

(c) Is the consumption process

$$
c(0) = 0, \quad c(T, \omega_1) = 9, \quad c(T, \omega_2) = 1, \quad c(T, \omega_3) = 17$$

attainable? Find the initial endowment and the trading strategy and attain it.

(d) Does the given price system permit arbitrage strategies?

---End---