MATH685Z – Mathematical Models in Financial Economics

Homework Seven

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1. Some risky assets are not easily marketable yet the standard CAPM, in principle, applies only to the choice between the complete set of all risky assets. For example, human capital, an individual's future lifetime income, cannot be sold. Some assets such as one's house may not be sold for psychological reasons or inertia, so it may be considered non-marketable. When some assets are not marketable the CAPM can be reworked. Show that the equilibrium rate of return on asset i is given by

$$\overline{R}_i = r + \beta_i^*(\overline{R}_m - r)$$

where

$$\beta_i^* = \frac{\text{cov}(R_i, R_m) + (V_N/V_m)\text{cov}(R_i, R_N)}{\sigma_m^2 + (V_N/V_m)\text{cov}(R_m, R_N)}$$

and V_N = value of all non-marketable assets, V_m = value of marketable assets and R_N = one-period rate of return on non-marketable assets.

Hint: Let w_j be the weight of the j^{th} risky asset within the universe of marketable assets and let $w_N = V_N/V_m$. The objective function to be minimized is

$$\text{var}\left(\sum_{j=1}^{J} w_j r_j + w_N r_N\right).$$

Here, $w_1, \cdots, w_J$ are the control variables while w_N is not a control variable. The model has close resemblance to the asset-liability model. The random rate of return r_N can be visualized as the liability's rate of growth.

2. Let r_j denote the equilibrium rate of return of risky asset j as deduced from CAPM and S be its equilibrium price. Let P_0 be the market price of asset j and $\widetilde{P}_e$ be the random return of the asset. Let r_j' be the rate of return as deduced from the market price of the asset. Let r_m denote the equilibrium rate of return of the market portfolio. Show that

$$E[r_j'] - r_f = (1 + r_f)\left(\frac{S}{P_0} - 1\right) + \frac{\text{cov}(\widetilde{P}_e/P_0, r_m)}{\sigma_m^2}(\mu_m - r_f)$$

where $\sigma_m^2 = \text{var}(r_m)$ and $\mu_m = E[r_m]$, r_f is the risk free interest rate.

Hint: First, find the relation between $E[r_j']$ and $E[r_j]$. Note that they are the same if and only if $S = P_0$.

3. Take a subset of N risky assets from the financial market and assume their beta values to be $\boldsymbol{\beta}_m = (\beta_{1m} \quad \beta_{2m} \quad \beta_{Nm})^T$. We would like to construct the market proxy $\widehat{m}$ from these N risky assets such that the beta of $\widehat{m}$ is one and $\boldsymbol{\beta}_{\widehat{m}} = \boldsymbol{\beta}_m$, that is, $\beta_{jm} = \beta_{j\widehat{m}}$ for all j. Consider the following minimization problem

$$\min_{\substack{\boldsymbol{w} \\ \boldsymbol{\beta}_m^T \boldsymbol{w} = 1}} \boldsymbol{w}^T \Omega \boldsymbol{w}$$

where Ω is the covariance matrix of the random returns of the N assets. The constraint indicates that the beta of the market proxy $\widehat{m}$ is one. The first order conditions give

$$\Omega \boldsymbol{w} - \lambda \boldsymbol{\beta}_m = 0 \quad \text{and} \quad \boldsymbol{\beta}_m^T \boldsymbol{w} = 1.$$

Show that

$$\lambda = \frac{1}{\boldsymbol{\beta}_m^T \Omega^{-1} \boldsymbol{\beta}_m} \quad \text{and} \quad \boldsymbol{w}^* = \frac{\Omega^{-1} \boldsymbol{\beta}_m}{\boldsymbol{\beta}_m^T \Omega^{-1} \boldsymbol{\beta}_m}.$$

We set the market proxy to be $\boldsymbol{w}^*$. Check whether $\boldsymbol{\beta}_{\widehat{m}} = \boldsymbol{\beta}_m$, where $\boldsymbol{\beta}_{\widehat{m}} = (\beta_{1\widehat{m}} \cdots \beta_{N\widehat{m}})^T$, and recall $\beta_{j\widehat{m}} = \dfrac{\mathrm{cov}(r_j, r_{\widehat{m}})}{\sigma_{\widehat{m}}^2}$.

4. Someone who believes that the collection of all stocks satisfies a single-factor model with the market portfolio serving as the factor gives you information on three stocks which make up a portfolio. (See Table.) In addition, you know that the market portfolio has an expected rate of return of 12% and a standard deviation of 18%. The riskfree rate is 5%.

   (a) What is the portfolio's expected rate of return?

   (b) Assuming the factor model is accurate, what is the standard deviation of this rate of return?

Simple Portfolio

Stock	Beta	Standard deviation of random error term	Weight in portfolio
A	1.10	7.0%	20%
B	0.80	2.3%	50%
C	1.00	1.0%	30%

5. Two stocks are believed to satisfy the two-factor model

$$\begin{aligned} r_1 &= a_1 + 2f_1 + f_2 \\ r_2 &= a_2 + 3f_1 + 4f_2. \end{aligned}$$

   In addition, there is a riskfree asset with a rate of return of 10%. It is known that $\overline{r}_1 = 15\%, \overline{r}_2 = 20\%$. What are the values of λ_0, λ_1 and λ_2 for this model?

6. David and Sue are portfolio managers for a defined benefit pension fund that has \$20 billion invested in U.S. stocks. A large portion of these funds are passively invested to match the returns of the S & P 500 index. Recently, top administrators of the pension plan have asked David and Sue to develop an active investment strategy that will initially be used on a small portion of the \$20 billion. If the active strategy they develop is successful, it will be used on a larger percentage of the \$20 billion. David and Sue have decided to use the concept of arbitrage pricing theory (APT) in developing their active investment strategy.

   (a) They decide their initial APT model will include the following three common factors:
   N = percentage change in consumer non-durable good purchases
   D = percentage change in consumer durable good purchases
   I = percentage change in consumer price inflation
   Following the standard way of symbolically expressing the APT, they write the following equation for stock i:

$$\widetilde{R}_{it} = a_{0t} + b_{iN}(\widetilde{N}_t) + b_{iD}(\widetilde{D}_t) + b_{iI}(\widetilde{I}_t) + e_{it}.$$

   Define what the following terms mean: $a_{0t}, b_{iN}, \widetilde{N}_t$.

(b) After considerable statistical analysis, they develop the following factor estimates for stock 1:

$$B_{1N} = 1.0; b_{1D} = 1.5; b_{1I} = -0.5.$$

They believe a_{01} should be the one-year risk-free rate at 4%. In addition, they believe the security markets expect N, D, and I to be the following during the next year.

Factor	Expected value
N	2.0%
D	3.0%
I	1.5%

If these assessments are correct, what expected return does the market expect on stock 1 during the next year?

(c) David and Sue agree with the expected value of N and I. But they believe the percentage growth of durable consumer purchases (common factor D) during the next year will be 5.0%. How should they use this opinion in developing their active management strategy?

(d) What is the role of the e_{it} term in the equation?

(e) What types of difficulties do you see in using the APT to develop an active management strategy?