1. Given 3 events $R, B$ and $G$. Show that
$$P[R \cap B | G] = P[R | G] P[B | G]$$
is equivalent to
$$P[R | B \cap G] = P[R | G].$$

2. Let $G_Y$ denote the probability generating function (pgf) of a discrete non-negative integer-valued random variable $K$. Show that
(a) $\sigma_Y^2 = G_Y''(1) + G_Y'(1) - G_Y'(1)^2$
(b) $\sigma_Y^2 = (\ln G_Y)'(1) + (\ln G_Y)'(1)$.

3. Suppose we choose the mixture distribution in the Bernoulli mixture model to be the beta distribution whose density function is given by
$$f(\tilde{p}) = \frac{1}{\beta(a, b)} \tilde{p}^{a-1} (1 - \tilde{p})^{b-1}, \quad a, b > 0, \quad 0 < \tilde{p} < 1,$$
where the beta function $\beta(a, b)$ is defined by
$$\beta(a, b) = \int_0^1 x^{a-1} (1 - x)^{b-1} dx = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a + b)}, \quad \Gamma(a) = \int_0^\infty e^{-x} x^{a-1} dx.$$
Recall that the mean and variance of the beta distribution are given by
$$\text{mean} = \frac{a}{a + b} \quad \text{and} \quad \text{variance} = \frac{ab}{(a + b)^2 (a + b + 1)}.$$
(a) Based on the Bernoulli mixture model, show that the probability of $k$ defaults out of $m$ $(m \geq k)$ obligors is given by
$$\mathbb{P}[M = k] = C_m^k \int_0^1 \tilde{p}^k (1 - \tilde{p})^{m-k} f(\tilde{p}) \, d\tilde{p} = C_m^k \beta(a + k, b + m - k) \beta(a, b).$$
(b) Find the corresponding default-event correlation coefficient $\rho(X_i, X_j)$. Look at various combinations of the two parameters for which $\frac{a}{a + b} = \tilde{p}$ for a given level of expected default probability $\tilde{p}$. Assuming $\tilde{p}$ to be fixed, as $a$ increases, do we have higher or lower default-event correlation?

4. Consider a portfolio of $m$ risky bonds (of equal face value) with uniform default probability $p$. Let $L_i$ denote the default event indicator of bond $i$, where $L_i \sim B(1; p)$. Let $\rho$ be the uniform correlation coefficient between pairwise defaults of any two bonds. In this problem, we use the Binomial Expansion Technique where the defaults in a comparison portfolio are assumed to be independent. By matching the second order moment of the
original portfolio and the comparison portfolio consisting of \( n(\rho) \) independent bonds, show that the diversification score \( n(\rho) \) is given

\[
n(\rho) = \frac{m}{1 + \rho(m - 1)}.
\]

Show that the above diversification score is bounded from above by \( 1/\rho \).

5. Consider the Davis-Lo contagion model, show that

\[
E[D_n] = n[1 - (1 - p)(1 - pq)^{n-1}]
\]

\[
\text{cov}(Z_i, Z_j) = \beta^pq_n - (E[D_n/n])^2,
\]

where

\[
\beta^pq_n = p^2 + 2p(1 - p)[1 - (1 - q)(1 - pq)^{n-2}]
\]

\[
+ (1 - p)^2[1 - 2(1 - pq)^{n-2} + (1 - 2pq + pq^2)^{n-2}].
\]

**Hint:** The event \((Z_1 = 1, \ldots, Z_k = 1, Z_{k+1} = 0, \ldots, Z_n = 0)\) can be achieved in various disjoint combinations. Firstly, we may have \((X_1 = 1, X_k = 1, X_{k+1} = 0, X_n = 0, Y_{ij} = 0, i = 1, \ldots, k, j = k + 1, \ldots, n)\) i.e. bonds 1 to \( k \) default directly and do not infect bonds \( k + 1 \) to \( n \). On the other hand, bonds 1 to \( i \) (for some \( i < k \)) may default directly and infect the other bonds of the first \( k \) but none of the remaining.

6. This problem is an extension of the mixture approach to the Poisson model of default. Consider the Poisson mixture model where the loss statistics is a random vector \( L = (L_1, \ldots, L_m) \) of Poisson random variables \( L_i \sim Pois(\lambda_i) \), where \( \Lambda = (\lambda_1, \ldots, \lambda_m) \) is a random vector with some distribution function \( F \) with support in \([0, \infty)^m\). Note that the default probability of obligor \( i \) is given by \( p_i = P[L_i = 1] \). We assume that conditional on a realization \( \lambda = (\lambda_1, \ldots, \lambda_m) \) of \( \Lambda \), the variables \( L_1, L_2, \ldots, L_m \) are independent:

\[
L_i|\lambda_i = \lambda_i \sim Pois(\lambda_i), \quad (L_i|\lambda = \lambda_i)_{i=1,\ldots,m} \text{ are independent.}
\]

The (unconditional) joint distribution of the variables \( L_i \) is given by

\[
P[L_1 = \ell_1, \ldots, L_m = \ell_m] = \int_{(0,\infty)^m} e^{-(\lambda_1 + \cdots + \lambda_m)} \prod_{i=1}^m \frac{\lambda_i^{\ell_i}}{\ell_i!} dF(\lambda_1, \ldots, \lambda_m),
\]

where \( \ell_i \in \{0, 1\} \). Show that the correlation coefficient between pairwise default events is given by

\[
\rho(L_i, L_j) = \frac{\text{cov}(L_i, L_j)}{\sqrt{\text{var}(L_i)} \sqrt{\text{var}(L_j)}}.
\]

**Hint:**

\[
P[L_i = \ell_i] = \frac{e^{-\lambda_i} \lambda_i^{\ell_i}}{\ell_i!}, \quad \ell_i = 0 \text{ or } 1,
\]

\[
E[L_i] = E[\lambda_i],
\]

\[
\text{var}(L_i) = \text{var}(E[L_i|\lambda]) + E[\text{var}(L_i|\lambda)] = \text{var}(\lambda_i) + E[\lambda_i].
\]
7. The CreditRisk+ Model is designed to incorporate the effects of variability in the average rates of default. In this problem, we would like to show that the CreditRisk+ Model behaves as if the default rates were fixed when the standard deviation of the mean default rate for each sector tends to zero. Recall that the CreditRisk+ has the following probability generating function for the default losses:

\[ G(z) = \prod_{k=1}^{n} G_k(z) = \prod_{k=1}^{n} \left( \frac{1 - p_k}{1 - \frac{p_k}{\mu_k} \sum_{j=1}^{m(k)} \epsilon_j^{(k)} z^{v_j^{(k)}}} \right)^{\alpha_k} \]

where

\[ \alpha_k = \frac{\mu_k^2}{\sigma_k^2}, \beta_k = \frac{\sigma_k^2}{\mu_k}, p_k = \frac{\beta_k}{1 + \beta_k}, \text{ and } \mu_k = \sum_{k=1}^{m(k)} \epsilon_j^{(k)} v_j^{(k)}. \]

Now, we consider the limit where \( \sigma_k \to 0 \). Show that as \( \sigma_k \to 0 \), we recover the result in the static case (deterministic default probabilities) where

\[ G(z) = e^{\sum_{i} \frac{\epsilon_i}{\nu_i} (z^{\nu_i} - 1)}. \]

**Hint:** As \( \sigma_k \to 0 \) while \( \mu_k \) is finite, we have

\[ \beta_k \to 0, \quad p_k \to 0 \quad \text{and} \quad \alpha_k = \frac{\mu_k}{\beta_k} \to \frac{\mu_k}{p_k} \quad \text{since} \quad \beta_k \to p_k. \]

Also, the following mathematical results are useful.

(i) \( \lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f(x)}{g(x)} \), provided that both \( \lim_{x \to \infty} f(x) \) and \( \lim_{x \to \infty} g(x) \) exist.

(ii) \( \lim_{n \to \infty} (1 - \frac{\alpha}{n})^n = e^{-\alpha} \).

8. Suppose there are independent Poisson processes \( N_1, N_2, \) and \( N \) with respective intensity \( \lambda_1, \lambda_2 \) and \( \lambda \). Here, \( \lambda_i \) is the firm-specific shock intensity of firm \( i \), \( i = 1, 2 \), and \( \lambda \) is the intensity of a systemic shock affecting all firms simultaneously. Define the default time \( \tau_i \) of firm \( i \) by

\[ \tau_i = \inf\{t \geq 0 : N_i(t) + N(t) > 0\}, \quad i = 1, 2. \]

That is, a default occurs completely unexpectedly if either a firm-specific or a systemic shock strikes the firm for the first time. Since \( N_i \) and \( N \) are independent, firm \( i \) defaults with intensity \( \lambda_i + \lambda \) (see p.9 of Topic 3b in MATH246) so that the survival function is

\[ S_i(t) = P[\tau_i > t] = P[N_i(t) + N(t) = 0] = e^{-(\lambda_i + \lambda)t}, \quad i = 1, 2. \]

Define

\[ t \vee u = \max(u, t), \quad t \wedge u = \min(u, t), \]

so that

\[ t + u = \max(u, t) + \min(u, t) \]

\[ e^{\lambda(t \wedge u)} = \min(e^{\lambda t}, e^{\lambda u}). \]
The joint survival probability is found to be

\[ S(t, u) = P[\tau_1 > t, \tau_2 > u] \]
\[ = P[N_1(t) = 0, N_2(u) = 0, N(t \land u) = 0] \]
\[ = e^{-\lambda_1 t - \lambda_2 u - \lambda(t \land u)} \]
\[ = e^{-(\lambda_1 + \lambda)t - (\lambda_2 + \lambda)u + \lambda(t \land u)} \]
\[ = e^{-\lambda_1 t} e^{-\lambda_2 u} e^{-(\lambda_1 + \lambda) t \land u} \]
\[ = e^{-(\lambda_1 \theta_1 + \lambda_2 \theta_2)} \]
\[ = e^{-\theta_1 S_1(t) S_2(u)} \min(S_1(t), S_2(u)) \]

We define the survival copula \( C^\tau(\omega, \upsilon) \) of the default time vector \((\tau_1, \tau_2)\) so that the joint survival function \( S(t, u) \) can be represented by

\[ S(t, u) = C^\tau(S_1(t), S_2(u)) \]

(a) Let \( \theta_i = \frac{\lambda_i}{\lambda_1 + \lambda_2}, i = 1, 2 \). Show that the survival copula is given by

\[ C^\tau(\omega, \upsilon) = S(S_1^{-1}(\omega), S_2^{-1}(\upsilon)) = \min(\upsilon \omega^{1-\theta_1}, \omega \upsilon^{1-\theta_2}) \]

(b) Define the default copula \( K^\tau \) by

\[ K^\tau(P_1(t), P_2(u)) = P[\tau_1 \leq t, \tau_2 \leq u] = P(t, u) \]

where \( \rho(t, u) \) is the joint default function, and the marginal default probabilities are \( P_1(t) = P[\tau_1 \leq t] = 1 - S_1(t) \) and \( P_2(u) = P[\tau_2 \leq u] = 1 - S_2(u) \). Find \( K^\tau(\omega, \upsilon) \).

(c) Suppose the two firms default independent of each other. Show that the survival copula reduces to

\[ C^\tau(\omega, \upsilon) = \omega \upsilon \]