RMBI 4210 – Quantitative Methods for Risk Management

Topic One — Bond portfolio management and immunization

1.1 Bond yield and term structures of interest rates

1.2 Duration measures and convexity

1.3 Horizon rate of return: return from the bond investment over a time horizon

1.4 Immunization of bond investment
1.1 Bond yield and term structure of interest rates

A bond is a debt instrument requiring the issuer to repay to the lender/investor the amount borrowed (par or face value) plus interest over a specified period of time.

Specify (i) maturity date when the principal is repaid; (ii) coupon payments over the life of the bond.
• The coupon rate offered by the bond issuer represents the cost of raising capital (reflection of the creditworthiness of the bond issuer).

• Assume the bond issuer not to default or redeem the bond prior to maturity date, an investor holding this bond until maturity is assured of a known cash flow pattern.

*Corporate bonds*
Issued by corporations for the purpose of raising capital for operations and new ventures. Some bonds are traded on an exchange, but most are traded in the over-the-counter (OTC) markets in a network of bond dealers.
Convertible bond
The bondholder has the right to exchange the bond for a specified number of shares on a set of pre-specified dates or at maturity.

- Bondholders can take advantage of the future growth of the issuer’s company.
- Issuer can raise capital at a lower coupon rate.

Exchangeable bond
Allows bondholder to exchange the issue for a specified number of common stocks of another corporation.
Other features in bond indentures

1. Floating rate bond – coupon rates are reset periodically according to some predetermined financial benchmark.

2. Amortization feature – principal is repaid over the life of the bond.

3. Callable feature (callable bonds)
   The issuer has the right to buy back the bond at a specified price. Usually this call price falls with time, and often there is an initial call protection period wherein the bond cannot be called.

4. Put provision – grants the bondholder the right to sell back to the issuer at par value on designated dates.
Risk associated with investing in bonds

1. *Interest rate risk*

   The price of a typical bond will change in the opposite direction from a change in interest rates. As interest rates rise, the price of a bond will fall.

   • Sensitivity of a bond’s price to changes in market interest rates depends on the coupon stream, remaining life of the bond, and other factors.

2. *Default risk* (credit risk) – risk that the issuer of a bond may default on his payment obligation.

   • Bonds with default risk trade in the market at a price lower than comparable US Treasury securities (considered as default free).

   • Default risk is gauged by quality ratings.
3. *Inflation risk* (purchasing-power risk)

- Variation in the value of cash flows from a bond due to inflation, as measured in terms of purchasing power.
- For all but floating-rate bonds, an investor is exposed to inflation risk.


5. *Liquidity* or *marketability risk* – measured by the size of the bid-ask spreads on the bond price.
Time value of money

Future value  =  present value \times \text{compounding factor}

Present value  =  future value \times \text{discount factor}

Compounding factor  =  (1 + r)^n, where \( r \) is the interest rate over one time period and \( n \) is the number of interest earning periods;

discount factor  =  1/(1 + r)^n.

Continuous compounding

Let \( R \) be the interest rate per annum and \( t \) be the total interest earning time period (in years).
Suppose there are $m$ compounding intervals per year, then compounding factor $= \left(1 + \frac{R}{m}\right)^{mt}$.

In the limit $m \to \infty$, $\lim_{m \to \infty} \left(1 + \frac{R}{m}\right)^m = e^R$, and so

compounding factor $= e^{Rt}$.

For example, $R = 8\%$, $t = 2.5$ (years); the total interest earned for a principal $P_0$ is

$$P_0(e^{0.08 \times 2.5} - 1) = P_0(e^{0.2} - 1) \approx 0.2217P_0.$$  

If the interest is compounded semi-annually, then the total interest earned is

$$P_0(1.04^5 - 1) = 0.2167P_0.$$
Continuous compounding

\[
M_t = \text{money market account value at time } t;
\]

interest amount collected over the small time interval \( \Delta t = r_t M_t \Delta t \).
\[ M_{t+\Delta t} = M_t + \Delta M_t = (1 + r_t \Delta t)M_t \]

growth factor over \( \Delta t = \frac{M_{t+\Delta t}}{M_t} = 1 + r_t \Delta t. \)

Continuous compounding: \( dM_t = r_t M_t \, dt. \) Integrating the differential equation, we obtain

\[
\int_{M_0}^M \frac{dM}{M} = \int_0^t r_s \, ds
\]

or

\[
M_t = M_0 e^{\int_0^t r_s \, ds}.
\]

The result remains valid when \( r_t \) is a stochastic process. In this case, the growth factor becomes stochastic, so does the value of the money market account.
Numerical example

$A = $100, $n = 1, \ r = 0.1$ (constant interest rate)

Compound annually: terminal amount $= $100$(1 + 0.1 \times 1) = $110$.

Compound continuously: terminal amount $= $100$e^{0.1 \times 1} = $110.52$.

There is slight difference between $1 + r\Delta t$ and $e^{r\Delta t}$.

Mathematical aspect:

Growth factor over $\Delta t$ under continuous compounding

\[ e^{r\Delta t} \approx 1 + r\Delta t + \frac{r^2\Delta t^2}{2} + \frac{r^3\Delta t^3}{6} + \cdots. \]

In this numerical example

\[ e^{0.1 \times 1} \approx 1 + 0.1 \times 1 + \frac{(0.1 \times 1)^2}{2} + \frac{(0.1 \times 1)^3}{6} + \cdots \approx 1 + 0.1 + 0.005 + 0.000167 + \cdots \approx 1.1052. \]
Spot rates

The $n$-year spot rate is the rate of interest earned on a $n$-year zero-coupon riskless bond that starts today and lasts for $n$ years. This is also known as the $n$-year spot rate.

Example

Suppose a 5-year spot rate with continuous compounding is quoted as 5% per annum. That is, given $100, if invested for 5 years, it grows to

$$100 \times e^{0.05 \times 5} = 128.40.$$
Discount bonds

Discount bonds are bonds that do not pay any intermediate coupons. Most of them have short maturity, say, 3 months to 1 year.

- The spot rate $i_t$ may be inferred from the price of discount bond $Z_0$ and face value $F$. We have

$$Z_0(1 + i_t)^t = F \quad \text{or} \quad i_t = \left( \frac{F}{Z_0} \right)^{1/t} - 1.$$
Discount bond price and discount factor

The price of a 3-month (quarter of a year) unit par zero-coupon bond at the spot rate of 8% per annum is

\[ e^{-0.08/4} = 0.9802. \]

We may consider 0.9802 as the 3-month discount factor at the spot rate of 8% per annum.

\[
\begin{array}{ccc}
\text{time} & \text{money market account} & \text{discount bond} \\
0 & $1 & $0.9802 \\
3\text{-month later} & $1.0202 & $1
\end{array}
\]

Growth factor \[ = \frac{1}{\text{discount factor}} = \frac{1}{0.9802} = 1.0202. \]
Forward interest rates

\( f(0, t, T) \): Forward rate decided upon today, at time 0, for a loan starting at \( t \) and reimbursed at \( T \).

\( i(0, u) = f(0, 0, u) \): spot rate for a loan that starts at time 0 and is reimbursed at later time \( u \).

Arbitrage enforces

\[
[1 + i(0, t)]^t[1 + f(0, t, T)]^{T-t} = [1 + i(0, T)]^T, \quad t < T.
\]

Suppose \( RHS < LHS \), an arbitrageur can borrow $1 for a \( T \)-year loan based on \( i(0, T) \). At the same time, lend out this dollar for a \( t \)-year loan based on \( i(0, t) \). After \( t \) years, invest the proceeds received at \( t, [1 + i(0, t)]^t \), in the forward market for the remaining time period \( T - t \).
Numerical example – Forward rates from spot rates

For $T = 2$, suppose the one-year spot rate $i(0,1) = 5\%$ and the two-year spot rate $i(0,2) = 4\%$, then the forward rate between year one and year two should be

$$f(0,1,2) = \frac{1.04^2}{1.05} = 1.0301.$$

Several relations between the spot rates and forward rates

From the no-arbitrage relation

$$1 + i(0,T) = [1 + i(0,t)]^{t/T}[1 + f(0,t,T)]^{(T-t)/T},$$

the $T$-horizon spot dollar return, $1 + i(0,T)$, is the weighted geometric average of the dollar forward returns, the weights being the shares of the various trading periods for the forward rates in the total trading period $T$. 
**Generalization:** Let \( t_1, t_2, \ldots, t_n \) be an arbitrary partitioning of the time horizon \([0, T]\) into \( n \) intervals, with \( t_n = T \) and \( t_0 = 0 \).

\[
1 + i(0, t_n) = \prod_{j=1}^{n} [1 + f(0, t_{j-1}, t_j)]^{(t_j - t_{j-1})/t_n}
\]

Note that \( f(0, t_0, t_1) = f(0, 0, t_1) = i(0, t_1) \).
Creating synthetically a forward contract

Want to borrow $1 at the future date \( t \), to be reimbursed at \( T \). Using the following relation:

\[
[1 + f(0, t, T)]^{T-t} = \frac{[1 + i(0, T)]^T}{[1 + i(0, t)]^t},
\]

we borrow \( \frac{1}{[1+i(0,t)]^t} \) at time 0 on the \( T \)-horizon long market and lend out the same amount at time 0 on the \( t \)-horizon short market.

- These two combined trades are equivalent to entering into a forward contract that borrows $1 at time \( t \) and pay the forward rate \( f(0, t, T) \) for the period between \( t \) and \( T \).

- In this manner, the forward rate \( f(0, t, T) \) is implied by the current spot rates: \( i(0, t) \) and \( i(0, T) \). Why?
Cash flows in the synthetic construction

- Pays out zero dollar at $t = 0$.

- Receives $1$ at time $t$ since $\frac{1}{[1 + i(0,t)]^t}$ has been lent out at $t = 0$. This is equivalent to borrow $1$ at time $t$.

- Pays out $\frac{1}{[1 + i(0,t)]^t} \times [1 + i(0,t)]^T$ at time $T$.

This amount should be the same as $[1 + f(0, t, T)]^{T-t}$ at $T$ for a loan initiated at $t$ with unit notional and loan period $T - t$ based on the forward rate of $f(0, t, T)$. Therefore,

$$[1 + f(0, t, T)]^{T-t} = \frac{[1 + i(0, T)]^T}{[1 + i(0, t)]^t}.$$
Determinants of the term structure of the spot interest rates

In most of the times, the spot rate curves slope rapidly upward at short maturities and continues to slope upward but more gradually as maturities lengthen. Three theories are proposed to explain the evolution of the term structure of the spot rate curve:

1. Expectations;

2. Liquidity preference;

**Expectations theory**

From the spot rates $S_1, \cdots, S_n$ for the next $n$ years, we can deduce a set of implied forward rates $f_{1,2}, f_{1,3}, \cdots, f_{1,n}$. According to the expectations theory, these implied forward rates define the expected spot rate curves $S'_1, \cdots, S'_{n-1}$ for the next year.

For example, suppose $S_1 = 7\%$, $S_2 = 8\%$, then

$$\begin{align*}
(1 + f_{1,2})1.07 &= 1.08^2 \\
 f_{1,2} &= \frac{(1.08)^2}{1.07} - 1 = 9.01\%.
\end{align*}$$

This value of 9.01\% is the market’s expected value of next year’s one-year spot rate $S'_1$. 
In general,

\[(1 + f_{1,j})^{j-1}(1 + S_1) = (1 + S_j)^j, \quad j = 2, 3, \ldots,\]

where we take the current forward rates \(f_{1,j}\) as the expectation of the spot rate \(S'_{j-1}\) in the next year. Hence, according to the Expectations Theory, we have

\[S'_{j-1} = f_{1,j} = \left[\frac{(1 + S_j)^j}{1 + S_1}\right]^{\frac{1}{j-1}} - 1.\]

**Example**

<table>
<thead>
<tr>
<th>current spot rates</th>
<th>(S_1)</th>
<th>(S_2)</th>
<th>(S_3)</th>
<th>(S_4)</th>
<th>(S_5)</th>
<th>(S_6)</th>
<th>(S_7)</th>
</tr>
</thead>
<tbody>
<tr>
<td>forecast rates one year later</td>
<td>6.00</td>
<td>6.45</td>
<td>6.80</td>
<td>7.10</td>
<td>7.36</td>
<td>7.56</td>
<td>7.77</td>
</tr>
<tr>
<td>(S'_2)</td>
<td>6.90</td>
<td>7.20</td>
<td>7.47</td>
<td>7.70</td>
<td>7.88</td>
<td>8.06</td>
<td></td>
</tr>
</tbody>
</table>

\[S'_2 = f_{1,3} = \left[\frac{(1.068)^3}{1.06}\right]^{\frac{1}{2}} - 1 = 0.072.\]
Let $f_t(T, T + 1)$ be the forward rate at time $t$ for the period from $T$ to $T + 1$, and $r_T$ be the one-period spot rate at time $T$. The expectation hypothesis states that

$$f_t(T, T + 1) = E_t[r_T].$$

Turn the view around: the expectation of next year’s curve determines what the current spot rate curve must be. That is, expectations about the future rates are part of today’s market.

**Weakness**

According to this hypothesis, one deduces that the market expects rates to increase whenever the spot rate curve slopes upward. Unfortunately, rates do not go up as often as expectations would imply.
Market segmentation hypothesis

The market for fixed income securities is segmented by maturity dates.

For example, the banking sector needs the short-dated bonds, while the pension funds require the longer-term ones.

- To the extreme, all points on the spot rate curves are mutually independent. Each sector of bonds with specific maturity is determined by the forces of supply and demand.

- A modification to the extreme view is that adjacent rates cannot become grossly out of line with each other. In other words, the spot rates of neighboring maturities should exhibit high correlation.
Liquidity preference hypothesis

For bank deposits, depositors usually prefer the short-term deposits over the long-term deposits since they do not like to tie up their capital (liquid rather than tied up). Hence, the long-term deposits should demand high rates. For bonds, the long-term bonds are more sensitive to interest rate changes. Hence, investors who anticipate to sell bonds shortly would prefer the short-term bonds.

- Compared to the Expectations Theory, there is an additional liquidity premium $\pi_t(T, T + 1)$. We have

$$f_t(T, T + 1) = E_t[r_T] + \pi_t(T, T + 1).$$

- Traders and speculators will borrow short and lend long in an effort to earn the liquidity premium, provided that they can absorb the liquidity risk at a lower cost than that of the market.
Evaluation of a coupon-bearing bond

What kind of discounting factor should be used to calculate the present value of any cash flow to be received $t$ years from now?

- $B_0 =$ the present value of the bond
- $B_T =$ the face (or the reimbursement) value of a bond (example: $1000)
- $c =$ the annual coupon of the bond (example: $70)
- $c/B_T =$ the coupon rate of the bond (in this example, the coupon rate is $70/1000 = 7\%$ per year)
- $i_t =$ the spot rate prevailing today for a contract of length equal to $t$ years; $i_t$ represents the term structure of interest rates. For example, $i_1 = 4.5\%; i_2 = 4.75\%; i_{10} = 5.5\%$
- $T =$ maturity of bond; this is the number of years remaining until maturity (example: 10 years).
An example of a term structure

<table>
<thead>
<tr>
<th>Term (years)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spot rate (%)</td>
<td>4.5</td>
<td>4.75</td>
<td>4.95</td>
<td>5.1</td>
<td>5.2</td>
<td>5.3</td>
<td>5.4</td>
<td>5.45</td>
<td>5.5</td>
<td>5.5</td>
</tr>
</tbody>
</table>

The present value of the bond is calculated as follows:

\[ B = \frac{c}{1 + i_1} + \frac{c}{(1 + i_2)^2} + \cdots + \frac{c}{(1 + i_t)^t} + \cdots + \frac{c + B_T}{(1 + i_T)^T}. \]

The present value of the bond, using the above term structure, would be

\[
B = \frac{70}{1.045} + \frac{70}{1.0475^2} + \frac{70}{1.0495^3} + \frac{70}{1.0514^4} + \frac{70}{1.0525^5} \\
+ \frac{70}{1.0536} + \frac{70}{1.0547} + \frac{70}{1.0545^8} + \frac{70}{1.0559} + \frac{1070}{1.055^{10}} \\
= \$1118.25.
\]

The bond value is above par since the rates of interest corresponding to the various terms of the bond are below the coupon rate.
Example (construction of a zero-coupon instrument from coupon bearing bonds)

Bond $\alpha$: 10-year bond with 10% coupon; $B_\alpha = $98.72.

Bond $\beta$: 10-year bond with 8% coupon; $B_\beta = $85.89.

Both bonds have the same par of $100.

Construct a portfolio of −0.8 unit of Bond $\alpha$ and one unit of Bond $\beta$. The resulting par value at maturity is $20, and the price is $B_\beta - 0.8B_\alpha = $6.914. Since the coupon payments are canceled, so this is a zero-coupon portfolio. The 10-year spot rate $S_{10}$ is obtained by

$$6.914(1 + S_{10})^{10} = 20$$

giving $S_{10} = 11.2\%.$
Construction of the spot rate curve from coupon-bearing bonds

- Use the prices of a series of zero-coupon bonds with varying maturity dates. However, "zero" with long maturities are rare.

- The spot rate curve can be determined from the prices of coupon-bearing bonds by beginning with short maturities and working forward to longer maturities.

Example

Consider a two-year US Treasury bond with coupon payments of amount $c$ at the end of each year. The price is $B_2$ and the par value is $P$. We deduce that

$$B_2 = \frac{c}{1 + S_1} + \frac{c + P}{(1 + S_2)^2}.$$
We determine $S_1$ by direct observation of the one-year zero-coupon Treasury bill rate. We can solve for $S_2$ algebraically from the above equation.

- The procedure is repeated with bond of longer maturities, say,

$$B_3 = \frac{c}{1 + S_1} + \frac{c}{(1 + S_2)^2} + \frac{c + P}{(1 + S_3)^3}.$$ 

- Note that the US Treasury bonds (assumed to be default free) are used to construct the benchmark spot rates.
Bond price under a flat term structure

As a simplification, we take \( i_t = i \) for all \( t = 1, \ldots, T \) so that

\[
B(i) = \frac{c}{1 + i} + \frac{c}{(1 + i)^2} + \cdots + \frac{c + B_T}{(1 + i)^T},
\]

so that

\[
B(i) = \frac{c}{i} \left[ 1 - \frac{1}{(1 + i)^T} \right] + \frac{B_T}{(1 + i)^T}.
\]

As \( T \to \infty \), corresponding to a perpetual bond or consol, we have

\[
\lim_{T \to \infty} B = \frac{c}{i}.
\]

With par equals \( \frac{c}{i} \), the coupon amount received per annum is given by \( \left( \frac{c}{i} \right)i = c \). This is simply the target coupon amount \( c \).
Alternatively, the bond value is quoted per unit of par value $B_T$

$$\frac{B}{B_T} = \frac{c/B_T}{i} \left[ 1 - \frac{1}{(1+i)^T} \right] + \frac{1}{(1+i)^T}.$$ 

- The normalized bond value $B/B_T$ is the weighted average between the normalized coupon rate $c/B_T$ divided by $i$ and the normalized par value (which is simply unity).

- The respective weights are $1 - \frac{1}{(1+i)^T}$ and $\frac{1}{(1+i)^T}$. As $T \to \infty$, the weight of the par tends to zero. The proportional contribution of the par payment $B_T$ to the bond value $B$ is seen to be $\frac{1}{(1+i)^T}$. 


Value of bond as a function of the rate of interest and of its maturity (coupon: 7%; maturities: 10, 7, 3, 1 and 0 year).

- All curves, each corresponding to a given maturity, exhibit a small convexity.

- The long-maturity bonds are more sensitive to change in interest rates.
• The more convexity a bond exhibits, the more valuable it will become in case of a drop in interest rates, and the less severe will be the loss for its owner if interest rates rise.

Convexity is favored by bond investors.
Evolution of the price of a bond through time for various interest rates (coupon 7%, maturity: 10 years).

- Should interest rates drop from 5% to 3%, our bond's trajectory would immediately jump up from 5% to 3%, and then follow that path until maturity.
Equation of arbitrage on interest rates (Fisher)

\[ i = \frac{q}{p_0} + \frac{\Delta p}{p_0} \]

\( q \) = rental rate generated from holding an asset

\( p_0 \) = initial price of the asset; \( i \) = rate of interest

\( \Delta p = p_1 - p_0 \) = change in asset value over one period

Coming back to the bond market, the above formula becomes

\[ i = \frac{c}{B} + \frac{\Delta B}{B}, \]

where \( \frac{c}{B} \) = direct yield of the bond.

For example, over one year, with \( i = 5\% \), \( c = 7\% \), \( B = 1154.44; \)

\[ \frac{\Delta B}{B} = i - \frac{c}{B} = 5\% - \frac{70}{1154.44} = 5\% - 6.06\% = -1.06\%. \]

Fall in bond value over one year when coupon rate > interest rate.
Mathematical proof

\[ B_t = \frac{c}{i} \left[ 1 - \frac{1}{(1+i)^T} \right] + \frac{B_T}{(1+i)^T} = \frac{c}{i} + (1 + i)^{-T} (B_T - \frac{c}{i}) \]

\[ B_{t+1} = \frac{c}{i} + (1 + i)^{-(T-1)} (B_T - \frac{c}{i}) \]

(since the bond maturity is shortened to \(T - 1\))

\[ B_{t+1} - B_t = \left( B_T - \frac{c}{i} \right) \frac{i}{(1+i)^T} = iB_t - c \]

so that the proportional change of bond value over one year is

\[ \frac{\Delta B_t}{B_t} = \frac{B_{t+1} - B_t}{B_t} = i - \frac{c}{B_t}. \]
**Continuous limit**

Suppose the bond provides a continuous rate of coupon stream, we have

\[ i(t) = \frac{c(t)}{B(t)} + \frac{1}{B(t)} \frac{dB(t)}{dt} \]

or

\[ \frac{dB(t)}{dt} = i(t)B(t) - c(t). \]

This is an ordinary differential equation in \( B(t) \). The auxiliary condition is prescribed by the terminal condition: \( B(T) = B_T = \text{par value} \) (not bond par plus the last coupon payment as in the discrete model). A high-coupon bond falls in value over time since

\[ \frac{1}{B(t)} \frac{dB(t)}{dt} = i - \frac{c(t)}{B(t)} < 0. \]

This shows that \( \frac{dB(t)}{dt} \) increases with \( i(t)B(t) \) but decreases with \( c(t) \). Why?
Yield to maturity (YTM)

• The YTM of a bond is the internal rate of return of the bond investment.

• It is the discount rate that makes equal the cost of buying the bond at its market price and the present value of the future cash flows generated by the bond.

\[
B_0 = \frac{c}{1 + i^*} + \frac{c}{(1 + i^*)^2} + \cdots + \frac{c + B_T}{(1 + i^*)^T}
\]

or

\[
B_0(1 + i^*)^T = c(1 + i^*)^{T-1} + c(1 + i^*)^{T-2} + \cdots + (c + B_T);
\]

here \( i^* = YTM \)
• Future value of an investment $B_0$
  \[ B_0 = B_0(1 + i^*)^T \]
  = future value of all cash flows reinvested at rate $i^*$.

• Assumption that all future cash flows can be reinvested at the same rate $i^*$ is *hardly realistic*. Indeed, investors face “investment risk” when they receive cash flows earlier than scheduled (say, bonds are called back by issuers prior to maturity) and these cash flows are reinvested at a lower prevailing interest rate.
The yield offered on a particular bond depends on 4 major factors:

1. A nominal return required to induce people to save (commonly measured as the YTM on a US $T$-bond).

2. Compensation for default risk.

3. Compensation for various options embedded in the bond such as the right to call.

4. Tax and liquidity features.
Yield spread and risk premium

On Sept. 19, 1997, the yield on the Wal-Mart Stores bonds (rated AA) with 10 years to maturity was 6.476%. On the same date, the yield on the 10-year on-the-run (most recently issued) Treasury was 6.086%.

Yield spread = 6.476% − 6.086% = 0.39%.

This spread, called a risk premium, reflects the additional risks the investor faces by acquiring a bond security that is not issued by the US government. She demands a higher yield for bearing the risk.
1.2 Duration measures and convexity

Duration

The duration of a bond is the weighted average of the times of payment of all the cash flows generated by the bond, the weights being the proportional shares of the bond’s cash flows in the bond’s present value.

**Macauley’s duration**: internal rate of return is used

Let $i$ denote the yield to maturity of the bond. Bond duration is

$$D = 1 \frac{c/B}{1+i} + 2 \frac{c/B}{(1+i)^2} + \cdots + T \frac{(c + B_T)/B}{(1+i)^T}$$

$$= \frac{1}{B} \sum_{t=1}^{T} \frac{tc_t}{(1+i)^t}.$$  \hspace{1cm} (D1)
Measure of a bond’s sensitivity to change in yield to maturity

Starting from

\[ B = \sum_{t=1}^{T} c_t (1 + i)^{-t}, \]

\[ \frac{dB}{di} = \sum_{t=1}^{T} (-t)c_t (1 + i)^{-t-1} = -\frac{1}{1+i} \sum_{t=1}^{T} tc_t (1 + i)^{-t}; \]

\[ \frac{1}{B} \frac{dB}{di} = -\frac{1}{1+i} \sum_{t=1}^{T} \frac{tc_t (1 + i)^{-t}}{B} = -\frac{D}{1+i}. \]

Suppose a bond is at par, its coupon is 9%, so \( YTM = 9\% \). The duration is found to be 6.99.

Suppose that the yield increases by 1%, then the relative change in bond value is

\[ \frac{\Delta B}{B} \approx \frac{dB}{B} = -\frac{D}{1+i} di = -\frac{6.99}{1.09} \times 1\% = -6.4\%. \]
How good is the linear approximation?

\[
\frac{\Delta B}{B} = \frac{B(10\%) - B(9\%)}{B(9\%)} = \frac{93.855 - 100}{100} = -6.145\%.
\]

**Modified duration**

Modified duration \( D_M = \frac{\text{duration}}{1+i} \)

\[
\frac{\Delta B}{B} \approx \frac{dB}{B} = -D_m di
\]

\[
\text{var} \left( \frac{dB}{B} \right) = D_m^2 \text{var}(di).
\]

The standard deviation of the relative change in the bond price is a linear function of the standard deviation of the changes in interest rates, the coefficient of proportionality is the modified duration.
Example

A bond with annual coupon 70, par 1000, and yield 5%; duration was calculated at 7.7 years, modified duration $= \frac{77}{1.05} = 7.33$ yr.

A change in yield from 5% to 6% or 4% entails a relative change in the bond price approximately $-7.33\%$ or $+7.33\%$, respectively. The modified duration is seen to be the more accurate proportional factor.
Calculation of the duration of a bond with a 7% coupon rate for a yield to maturity $i = 5\%$

<table>
<thead>
<tr>
<th>Time of payment $t$</th>
<th>Cash flow in current value</th>
<th>Cash flows in present value ($i = 5%$)</th>
<th>Share of cash flows in present value in bond's price</th>
<th>Weighted time of payment (col. 1 ?col. 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>70</td>
<td>66.67</td>
<td>0.0577</td>
<td>0.0577</td>
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<td>2</td>
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<td>3</td>
<td>70</td>
<td>60.47</td>
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<td>0.1571</td>
</tr>
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<td>4</td>
<td>70</td>
<td>57.59</td>
<td>0.0499</td>
<td>0.1995</td>
</tr>
<tr>
<td>5</td>
<td>70</td>
<td>54.85</td>
<td>0.0475</td>
<td>0.2375</td>
</tr>
<tr>
<td>6</td>
<td>70</td>
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<td>0.0452</td>
<td>0.2715</td>
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<td>10</td>
<td>1070</td>
<td>656.89</td>
<td>0.5690</td>
<td>5.6901</td>
</tr>
</tbody>
</table>

Total 1700 1154.44 1 7.705 = duration

bond value = 1154.44.
**Relationship between duration and coupon rate**

Recall $D = -\frac{1 + i dB}{B} \frac{d B}{d i}$ and $B = \frac{c}{i} \left[1 - \frac{1}{(1+i)^T}\right] + \frac{B_T}{(1+i)^T}$,

we have

$$\frac{B}{B_T} = \frac{1}{i} \left\{ \frac{c}{B_T} \left[1 - \frac{1}{(1+i)^T}\right] + \frac{i}{(1+i)^T} \right\}.$$

$$d \ln \left( \frac{B}{B_T} \right) = \frac{d \ln B}{d i} = \frac{1}{B} \frac{d B}{d i} = -\frac{1}{i} + \frac{(c/B_T)T(1+i)^{-T-1} + (1+i)^{-T} + i(1+i)^{-T-1}(-T)}{(c/B_T)[1-(1+i)^{-T}] + i(1+i)^{-T}}$$

so that

$$D = -\frac{1 + i dB}{B} \frac{d B}{d i} = 1 + \frac{1}{i} + \frac{T \left(i - \frac{c}{B_T}\right) - (1+i)}{c/B_T[(1+i)^T - 1] + i}. \quad (D2)$$

- The impact of the coupon rate $c/B_T$ and maturity $T$ on duration $D$ can be deduced from the last term.
Duration of a bond as a function of its maturity for various coupon rates ($i = 10\%$).

- The last term in eq. (D2) indicates whether $D$ is an increasing function of maturity $T$ or otherwise. When coupon rate $\frac{c}{B_T}$ is less than $i$, the numerator may change sign at $T^*$ where

$$T^* \left( i - \frac{c}{B_T} \right) = 1 + i.$$
With an increase in coupon rate $c/B_T$, should there always be a definite increase in duration?

- Just looking at eq. (D1): $D = \frac{1}{B} \sum_{t=1}^{T} \frac{tc_t}{(1+i)^t}$, it is not apparent since the bond value $B$ also depends on the coupon rate.

- From eq. (D.2), it is seen that the numerator (denominator) in the last term decreases (increases) with increasing $c/B_T$. Hence, $D$ decreases with increasing $c/B_T$.

- Intuitively, when the coupon rate increases, the weights will be tilted towards the left, and the center of gravity will move to the left.
Duration of a bond as the center of gravity of its cash flows in present value (coupon: 7%; interest rate: 5%).
Adding one additional cash flow at $T + 1$ has two effects on the right end of the scale:
1. It *adds weight* at the end of the scale, first by adding a new coupon at $T + 1$, and second by increasing the length of the lever corresponding to the reimbursement of the principal.

2. It *removes weight* at the end of the scale by replacing $A$ with a smaller quantity, $A' = A/(1 + i)$.

Therefore, the total effect on the center of gravity (on duration) is ambiguous and requires more detailed analysis.
Perpetual bond – infinite maturity

- The last term in eq. (D2) gives the impact of maturity on $D$. When we consider a perpetual bond, where $T \to \infty$, the final par payment is immaterial.

- Recall that $D$ can be thought of as the average time that one has to wait to get back the money from the bond issuer. Interestingly,

$$D \to 1 + \frac{1}{i} \quad \text{as} \quad T \to \infty.$$  \hspace{1cm} (D3)

The inverse of the yield $i$ is the amount of time that one has to wait to recoup 100% of your money. Since the payment of coupons is not continuous, one has to wait one extra year, so this leads to $1 + \frac{1}{i}$ (see Qn 1 of HW 1).
Relationship between duration and maturity

1. For zero-coupon bonds, duration is always equal to maturity.

   For all coupon-bearing bonds,

   \[ \text{duration} \to 1 + \frac{1}{i} \text{ when maturity increases infinitely.} \]

   The limit is independent of the coupon rate.

2. Coupon rate \( \geq \) YTM (bonds above par)

   An increase in maturity entails an increase in duration towards
   the limit \( 1 + \frac{1}{i} \).

3. Coupon rate \(<\) YTM (bonds below par)

   When maturity increases, duration first increases, pass through
   a maximum and decrease toward the limit \( 1 + \frac{1}{i} \).
Relationship between duration and yield

Should the yield to maturity increases, the center of gravity will move to the left and duration will be reduced. Actually

$$\frac{dD}{di} = -\frac{S}{1+i},$$

where $S$ is the dispersion or variance of the payment times of the bond.
Proof

Starting from

\[ D = \frac{1}{B} \sum_{t=1}^{T} tc_t(1 + i)^{-t} \]

\[
\frac{dD}{di} = -\frac{1}{B^2} \left[ \sum_{t=1}^{T} t^2 c_t(1 + i)^{-t-1} B(i) + \sum_{t=1}^{T} tc_t(1 + i)^{-t} B'(i) \right]
\]

\[
= -\frac{1}{1+i} \left[ \sum_{t=1}^{T} \frac{t^2 c_t(1 + i)^{-t}}{B(i)} + (1 + i) \frac{B'(i)}{B(i)} \frac{\sum_{t=1}^{T} tc_t(1 + i)^{-t}}{D} \right]
\]

\[
= -\frac{1}{1+i} \left[ \frac{\sum_{t=1}^{T} t^2 c_t(1 + i)^{-t}}{B(i)} - D^2 \right].
\]
If we write

\[ w_t = \frac{c_t(1 + i)^{-t}}{B(i)} \]

so that

\[ \sum_{t=1}^{T} w_t = 1 \quad \text{and} \quad D = \sum_{t=1}^{T} tw_t. \]

Here, \( w_t \) is the share of the bond’s cash flow (in the present value) in the bond’s value. The bracket term becomes

\[ \sum_{t=1}^{T} t^2 w_t - D^2 = \sum_{t=1}^{T} w_t(t - D)^2, \]

which is equal to the weighted average of the squares of the difference between the times \( t \) and their average \( D \).
We obtain

\[
\frac{dD}{di} = -\frac{1}{1 + i} \sum_{t=1}^{T} w_t(t - D)^2
\]

which is always negative.

- Intuitively, since the discount factor is \((1 + i)^{-t}\), an increase in \(i\) will move the center of gravity to the left, and the duration is reduced.
Fisher-Weil’s duration

- Fisher-Weil’s duration: weights of the times of payment make use of the spot rates pertaining to each term.

- Time structure of the spot interest rates over successive years:

\[ i(0,1), i(0,2), \ldots, i(0,t), \ldots, i(0,T). \]

\[
\begin{align*}
\text{growth factor over } [z, z + dz] &= e^{i(z)dz}.
\end{align*}
\]
• Let $i(z)$ denote the instantaneous forward rate

$$e^{\int_0^t i(z) \, dz} = e^{i(0,t)t}$$

so that

$$i(0,t) = \frac{1}{t} \int_0^t i(z) \, dz.$$  

• The spot rate $i(0,t)$ is the average of all implicit forward rates.

• Since

$$[i(0,t) + \alpha]t = \int_0^t i(z) \, dz + \int_0^t \alpha \, dz = \int_0^t [i(z) + \alpha] \, dz$$

so increasing $i(0,t)$ by $\alpha$ is equivalent to displacing vertically by $\alpha$ the implicit forward rates structure.
Mystery behind duration

• We would like to understand the financial intuition why duration is the multiplier that relates relative change in bond value and interest rate.

• Under the continuous framework, the bond value $B(\vec{i})$ is given by

$$B(\vec{i}) = \int_0^T c(t)e^{-i(0,t)t} dt$$

where $c(t)$ is the cash flow received at time $t$. 
This is considered as a *functional* since this is a relation between a function \( \vec{i} \) (term structure of interest rates) and a number \( B(\vec{i}) \).

Naturally, duration of the bond with the initial term structure is

\[
D(\vec{i}) = \frac{1}{B(\vec{i})} \int_0^T tc(t)e^{-i(0,t)t} dt,
\]

where \( \frac{c(t)e^{-i(0,t)t}}{B(\vec{i})} dt \) represents the weighted present value of cash flow within \((t, t + dt)\).

Suppose the whole term structure of interest rates move by \( \Delta \alpha \), then

\[
B(\vec{i} + \Delta \alpha) = \int_0^T c(t)e^{-i(0,t)t} e^{-t\Delta \alpha} dt.
\]
Note that when $\Delta \alpha$ is infinitesimally small, we have

$$e^{-t\Delta \alpha} \approx 1 - t\Delta \alpha$$

so that the discounted cash flow $e^{-i(0,t)}tc(t) \, dt$ within $(t, t + dt)$ falls in proportional amount $t\Delta \alpha$.

The corresponding contribution to the relative change in value as normalized by $B(i)$ is

$$\frac{tc(t)e^{-i(0,t)} \, dt}{B(i)} \Delta \alpha.$$ 

This is the payment time weighted by discounted cash flow within $(t, t + dt)$ multiplied by the change in interest rate $\Delta \alpha$. 
The total relative change in bond value

\[ \frac{\Delta B}{B} = -\Delta \alpha \left[ \text{weighted average of payment times that are weighted according to present value of cash flow} \right] = -D \Delta \alpha. \]

In the limit, we obtain

\[ \frac{dB}{B} = -D(\tilde{i}) \, di. \]

- In the discrete case, we need to modify the duration multiplier by multiplying the discount factor \( \frac{1}{1+i} \) over one year so that

\[ \frac{1}{B} \frac{dB}{di} = -\frac{D}{1+i}. \]
1.3 Horizon rate of return: return from the bond investment over a time horizon

**Horizon rate of return**, $r_H$ – bond is kept for a time horizon $H$

It is the return that transforms an investment bought today at price $B_0$ by its owner into a future value at horizon $H$, which is $F_H$.

\[ B_0(1 + r_H)^H = F_H \quad \text{or} \quad r_H = \left( \frac{F_H}{B_0} \right)^{1/H} - 1. \]

- All coupons to be paid are supposed to be regularly reinvested until horizon $H \ (< T)$. As a result, we need to evaluate the future rates of interest at which it will be possible to reinvest these coupons. This requires a forecast of the future rates of interest.
Suppose a bond investor bought a bond valued at $B(i_0)$ when the interest rate common to all maturities was $i_0$ (flat rate). On the following day the interest rate moves up to $i$ (parallel shift). Given $F_H = B(i)(1 + i)^H$, where $B(i)$ is the bond value at interest rate $i$.

Combining the results, we obtain

$$r_H = \left[ \frac{B(i)}{B_0} \right]^{1/H} (1 + i) - 1.$$  

The impact on the bond value on changing interest rate is spread out in $H$ years. Future cash flows from the bond are compounded annually at the new interest rate $i$. 
• As a function of $i$, the horizon rate of return $r_H$ is a product of a decreasing function $B(i)$ and an increasing function $(1 + i)$. This represents a counterbalance between an immediate capital gain/loss and rate of return on the cashflows from now till $H$.

• Whatever the horizon, the rate of return will always be $i_0$ if $i$ does not move away from this value. In this case,

$$F_H = B_0(1 + i_0)^H = B_0(1 + r_H)^H$$

so that $r_H = i_0$ for any $H$.

• If $H \to \infty$, $r_H = \left[ \frac{B(i)}{B_0} \right]^{1/H} (1 + i) - 1 \to i$. 
The table shows the horizon return (in percentage per year) on the investment in a 7% coupon, 10-year maturity bond bought at 1154.44 when interest rates were at 5%, should interest rates move immediately either to 6% or 4%. At \( H = 7.7 \), \( r_H \) increases when \( i \) either increases or decreases.

<table>
<thead>
<tr>
<th>Horizon (years)</th>
<th>Interest rates</th>
<th>4%</th>
<th>5%</th>
<th>6%</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>12.01</td>
<td>5</td>
<td>-1.42</td>
</tr>
<tr>
<td>2, increasing</td>
<td></td>
<td>7.93</td>
<td>5</td>
<td>2.22</td>
</tr>
<tr>
<td>3</td>
<td>( r_H )</td>
<td>6.60</td>
<td>5</td>
<td>3.47</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>5.95</td>
<td>5</td>
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</tr>
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<td>5</td>
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<td>10</td>
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<td>5</td>
<td>5.23</td>
</tr>
<tr>
<td>( \infty )</td>
<td></td>
<td>4.00</td>
<td>5</td>
<td>6.00</td>
</tr>
</tbody>
</table>
Example – Calculation of $r_H$

A 10-year bond with coupon rate of 7% was bought when the interest rates were at 5%. We have $B(i_0) = $1154.44.

Suppose in the next day, the interest rates move up to 6%. The bond drops in value to $1073.60. If he holds his bond for 5 years, and if interest rates stay at 6%, then

$$r_H = \left( \frac{1073.60}{1154.44} \right)^{1/5} \times (1.06) - 1 = 4.47\%.$$
Observation

Though the rate of interest at which the investor can reinvest his coupons (which is now 6%) is higher, his overall performance will be lower than 5% ($r_H$ is only 4.47%).

Counterbalance between

- Capital loss on the bond value due to an increase in the interest rate.
- Gain from the reinvestment of coupons at a higher rate.
• The longer the horizon and the longer the reinvestment of the coupons at a higher rate, the greater the chance that the investor will outperform the initial yield of 5%.

• If the horizon is 7.7 years, then the horizon rate of return will be slightly above 5% (5.006%) whether the interest rate falls to 4% or increases to 6%.

• Comparing the one-year horizon and four-year horizon, if interest rates rise, the four-year horizon return is higher than the one-year return. Here, we assume that bond’s maturity is longer than four years.
1. The capital loss will be less severe with the four-year horizon since the bond’s value will have come back closer to par after four years than after one year.

2. The four-year investor will be able to benefit from the coupon reinvestment at a higher rate.

• For the extreme case of $H \to \infty$, $r_H = i$. The immediate capital gain/loss is immaterial since all cash flows from the bonds remain the same while they can be reinvested at the rate of return $i$. 
• The horizon rate of return is a decreasing function of $i$ when the horizon is short and an increasing one for long horizons. For a horizon equal to the duration, the horizon rate of return first decreases, goes through a minimum for $i = i_0$ then increases by $i$.

• There is a critical value for $H$ such that $r_H$ changes from a decreasing function of $i$ to an increasing function of $i$. 
Dependence of $r_H$ on $i$ with varying $H$
The critical $H$ happens to be $D$. Why?

Relative change in bond value due to change of interest rate of amount $\Delta i$:

$$\frac{\Delta B}{B} \approx -D \times \Delta i$$

(neglecting the effect of the discount factor over one year).

- The immediate capital loss is spread over $H$ years, which is of amount $D\Delta i$.

- The gain in a higher rate of return of the future cash flows is $H\Delta i$ over $H$ years of horizon of investment.

- These two effects are counterbalanced if $H = D$. 
Stronger mathematical result

There exists a horizon $H$ such that $r_H$ always increases when the interest rate moves up or down from the initial value $i_0$.

Theorem

There exists a horizon $H$ such that the rate of return for such a horizon goes through a minimum at point $i_0$. 
Proof

Minimizing $r_H$ is equivalent to minimizing any positive transformation of it, and so it is equivalent to minimizing $F_H$. Note that

$$\frac{dF_H}{di} = \frac{d}{di}[B(i)(1 + i)^H] = B'(i)(1 + i)^H + HB(i)(1 + i)^{H-1} = 0$$

We want $\frac{dF_H}{di} = 0$ to hold at $i = i_0$ so that

$$B'(i_0)(1 + i_0) + HB(i_0) = 0$$

and

$$H = -\frac{1 + i_0}{B(i_0)}B'(i_0) = \text{duration.}$$

The horizon must be equal to duration at the initial rate of return for $F_H$ to run through a minimum.
Checking the second order condition

We have the first order condition for $r_H$ to go through a minimum. We would like to prove that $\ln F_H(i)$ has a positive 2\textsuperscript{nd} derivative.

\[
\frac{d}{di} \ln F_H = \frac{d}{di} \ln B(i) + \frac{H}{1+i} = \frac{1}{B(i)} dB(i) + \frac{H}{1+i} = -D + H
\]

\[
\frac{d^2}{di^2} \ln F_H = \frac{1}{(1+i)^2} \left[ -\frac{dD}{di} (1+i) + D - H \right]
\]

and since $D = H$, we obtain

\[
\frac{d^2}{di^2} \ln F_H = -\frac{1}{1+i} \frac{dD}{di} = \frac{S}{(1+i)^2} > 0.
\]

Therefore, $F_H$ and $r_H$ go through a global minimum at point $i = i_0$ whenever $H = D$. 
**Continuous case**

We determine the bond’s value at horizon $H$ in order to immunize its future value as well as the rate of return at horizon $H$. Define $F_H(\vec{i})$ to be the future value when the term structure of interest rates is $\vec{i}$.

Recall

$$F_H(\vec{i}) = B(\vec{i}) e^{[i(0,H)]H}$$

so that

$$F_H(\vec{i} + \alpha) = B(\vec{i} + \alpha) e^{[i(0,H) + \alpha]H}.$$
How to determine a horizon $H$ such that this future value is equal, at a minimum, to the value it would have if the term structure did not change, that is, if $\alpha = 0$. We have to find $H$ such that $F_H(\vec{i} + \alpha)$ goes through a minimum in space $(\alpha, F_H)$.

$$\frac{d}{d\alpha} \ln F_H(\vec{i} + \alpha) = \frac{d}{d\alpha} \ln B(\vec{i} + \alpha) \bigg|_{\alpha=0} + H = 0$$

so that

$$H = -\frac{1}{B(\vec{i})} \frac{dB(\vec{i})}{d\alpha} = D(\vec{i}).$$

The immunizing horizon must be equal to the duration.
First order condition for immunization

Value of the bond at horizon $H$, when the term structure of interest rates is $\tilde{i} + \alpha$. 
Second order condition for immunization

First, we consider

\[
\frac{d^2 \ln F_H}{d\alpha^2} = \frac{d}{d\alpha} \left[ \frac{1}{B(\vec{i} + \alpha)} \frac{dB(\vec{i} + \alpha)}{d\alpha} \right] = \frac{d}{d\alpha} [-D(\vec{i} + \alpha)].
\]

The derivative of \(D(\vec{i} + \alpha)\) with respect to \(\alpha\) is equal to

\[
\frac{dD(\vec{i} + \alpha)}{d\alpha} = \frac{d}{d\alpha} \left\{ \frac{1}{B(\vec{i} + \alpha)} \int_0^T tc(t)e^{-[i(0,t)+\alpha]t} \, dt \right\}
= \frac{1}{[B(\vec{i} + \alpha)]^2} \left\{ \int_0^T -t^2 c(t)e^{-[i(0,t)+\alpha]t} \, dt \cdot B(\vec{i} + \alpha)
- \int_0^T tc(t)e^{-[i(0,t)+\alpha]t} \, dt \cdot B'(\vec{i} + \alpha) \right\}
= - \left\{ \int_0^T t^2 c(t)e^{-[i(0,t)+\alpha]t} \, dt \frac{B(\vec{i} + \alpha)}{B(\vec{i} + \alpha)} + \int_0^T tc(t)e^{-[i(0,t)+\alpha]t} \, dt \frac{B'(\vec{i} + \alpha)}{B(\vec{i} + \alpha)} \right\}.
\]
The expression in the second term of the last brace is recognized as $-D^2(\vec{i} + \alpha)$. Simplifying the notation, we may thus write

$$w(t) = \frac{c(t)e^{-[i(0,t)+\alpha]t}}{B(\vec{i} + \alpha)}$$

with $\int_0^T w(t) \, dt = 1$,

and $D(\vec{i} + \alpha) = D(\alpha)$. We obtain

$$\frac{dD}{d\alpha} = -\left\{ \int_0^T t^2w(t) \, dt - D^2 \right\}$$

$$= -\left\{ \int_0^T t^2w(t) \, dt - 2D \int_0^T tw(t) \, dt + D^2 \int_0^T w(t) \, dt \right\}$$

$$= -\int_0^T w(t)[t^2 - 2tD + D^2] \, dt = -\int_0^T w(t)(t - D)^2 \, dt.$$
As a happy surprise, the last expression is none other than minus the dispersion (or variance) of the terms of the bond, and such a variance is of course always positive. Denoting the bond’s variance as $S(\vec{i}, \alpha)$, we may write

$$\frac{d^2 \ln F_H}{d\alpha^2} = -\frac{dD}{d\alpha} = S(\vec{i}, \alpha) > 0.$$  

Consequently, we may conclude that $\ln F_H$ is indeed convex.

- Together with the first order condition, this condition is sufficient for $\ln F_H$ to go through a global minimum at point $\alpha = 0$.  

Initial term structure of interest rates $i(0, t)$, variation of the structure $\alpha \eta(0, t)$, and new structure $i(0, t) + \alpha \eta(0, t)$. 
What about immunization when the term structure undergoes any kind of variation?

**Result:** We cannot be sure that we can immunize our portfolio.

To each structure, \( i(0, t) + \alpha \eta(0, t) \) corresponds a value of bond, where

\[
B(i(0, t) + \alpha \eta(0, t)) = \int_0^T c(t) e^{-[i(0, t) + \alpha \eta(0, t)]t} dt = B(\alpha).
\]

The logarithmic derivative is

\[
-\frac{1}{B} \frac{dB}{d\alpha} \bigg|_{\alpha=0} = \frac{1}{B} \int_0^T t\eta(0, t)c(t)e^{-i(0,t)t} dt,
\]

which is not equal to the duration. The trouble is the presence of \( \eta(0, t) \).
The value of the bond at horizon $H$ is

$$F_H = B(\alpha)e^{[i(0,H) + \alpha \eta(0,H)]H}$$

so that

$$\frac{d \ln F_H}{d\alpha} = \frac{d}{d\alpha} \ln B(\alpha) + \eta(0, H)H.$$  

Unfortunately, we cannot choose a horizon $H^*$ that would minimize $F_H$. It is not possible to choose $H^*$ such that

$$H^* = -\left.\frac{d}{d\alpha} \ln B(\alpha)\right|_{\alpha=0} \frac{1}{\eta(0, H)},$$

since the variation $\eta(0, t)$ is unknown at the time of the purchase.
Main formulas

• Value of a bond in continuous time, with $\vec{i} \equiv i(0,t)$ being the term structure or interest rates:

$$B(\vec{i}) = \int_0^T c(t)e^{-i(0,t)t} \, dt$$

• Duration of the bond:

$$D(\vec{i}) = \frac{1}{B(\vec{i})} \int_0^T tc(t)e^{-i(0,t)t} \, dt$$

• Duration of the bond when $\vec{i}$ receives a (constant) variation $\alpha$:

$$D(\vec{i} + \alpha) = \frac{1}{B(\vec{i} + \alpha)} \int_0^T tc(t)e^{-[i(0,t)+\alpha]t} \, dt$$
• Fundamental property of duration:

\[ -\frac{1}{B(i)} \frac{dB(i)}{d\alpha} = \frac{1}{B(i)} \int_0^T tc(t)e^{-i(0,t)t} \, dt = D \]

• First order condition for immunization:

\[ H = -\frac{1}{B(i)} \frac{dB(i)}{d\alpha} = D(i) \]

• Second order condition for immunization:

\[ \frac{d^2 \ln F_H}{d\alpha^2} > 0 \Rightarrow -\frac{d}{d\alpha} [D(i + \alpha)] = S(i, \alpha) > 0 \]
1.4 Immunization of bond investment

- In the case of either a drop or a rise in interest rates, when the horizon was properly chosen, the *horizon rate of return* for the bond’s owner was about the same as if interest rates had not moved. **This horizon is the duration of the bond.**

- Immunization is the set of *bond management procedures* that aim at protecting the investor against changes in interest rates.

- It is dynamic since the passage of time and changes in interest rates will modify the portfolio’s duration by an amount that will not necessarily correspond to the steady and natural decline of the investor’s horizon.
• Even if interest rates do not change, the simple passage of one year will reduce duration of the portfolio by less than one year. The money manager will have to change the composition of the portfolio so that the duration is reduced by a whole year.

• Changes in interest rates will also modify the portfolio’s duration.

• *Immunization* may be defined as the process by which an investor can protect himself against interest rate changes by suitably choosing a bond or a portfolio of bonds such that its duration is kept equal to his horizon dynamically.
**Numerical example**

A company has an obligation to pay $1 million in 10 years. It wishes to invest money now that will be sufficient to meet this obligation. The purchase of a single zero-coupon bond would provide one solution, but such discount bonds are not always available in the required maturities.

<table>
<thead>
<tr>
<th></th>
<th>coupon rate</th>
<th>maturity</th>
<th>price</th>
<th>yield</th>
<th>duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>bond 1</td>
<td>6%</td>
<td>30 yr</td>
<td>69.04</td>
<td>9%</td>
<td>11.44</td>
</tr>
<tr>
<td>bond 2</td>
<td>11%</td>
<td>10 yr</td>
<td>113.01</td>
<td>9%</td>
<td>6.54</td>
</tr>
<tr>
<td>bond 3</td>
<td>9%</td>
<td>20 yr</td>
<td>100.00</td>
<td>9%</td>
<td>9.61</td>
</tr>
</tbody>
</table>

- The above 3 bonds all have the yield of 9%. Present value of obligation at 9% yield is $414,643.
Since bond 2 and bond 3 have their duration shorter than 10 years, it is not possible to attain a portfolio with duration 10 years using these two bonds. A bond with a longer maturity is required (say, bond 1).

Suppose we use bond 1 and bond 2 of amount $V_1$ and $V_2$ in the portfolio,

\[
V_1 + V_2 = PV = 414,643
\]
\[
D_1 V_1 + D_2 V_2 = 10 \times PV = 4,146,430
\]

giving

\[
V_1 = 292,788.64, V_2 = 121,854.78.
\]
<table>
<thead>
<tr>
<th>Bond 1</th>
<th>Price</th>
<th>9.0</th>
<th>8.0</th>
<th>10.0</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>shares</td>
<td>4241</td>
<td>4241</td>
<td>4241</td>
</tr>
<tr>
<td></td>
<td>value</td>
<td>292,798.64</td>
<td>328,168.58</td>
<td>263,535.74</td>
</tr>
<tr>
<td>Bond 2</td>
<td>Price</td>
<td>113.01</td>
<td>120.39</td>
<td>106.23</td>
</tr>
<tr>
<td></td>
<td>shares</td>
<td>1078</td>
<td>1078</td>
<td>1078</td>
</tr>
<tr>
<td></td>
<td>value</td>
<td>121,824.78</td>
<td>129,780.42</td>
<td>114,515.94</td>
</tr>
<tr>
<td></td>
<td>Obligation value</td>
<td>414,642.86</td>
<td>456,386.95</td>
<td>376,889.48</td>
</tr>
<tr>
<td></td>
<td>Surplus</td>
<td>-19.44</td>
<td>1,562.05</td>
<td>1,162.20</td>
</tr>
</tbody>
</table>

- Surplus at 8% yield = 328,168.58 + 129,780.42 − 456,386.95 = 1,562.05.
Observation: At different yields (8% and 10%), the value of the portfolio almost agrees with that of obligation (at the new yield).

Difficulties

• It is quite unrealistic to assume that both the long- and short-duration bonds can be found with identical yields. Usually longer-maturity bonds have higher yields.

• When yields change, it is unlikely that the yields on all bonds will change by the same amount.
Convexity and its uses in bond portfolio management

<table>
<thead>
<tr>
<th></th>
<th>Coupon rate</th>
<th>maturity</th>
<th>price</th>
<th>yield to maturity</th>
<th>duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bond A</td>
<td>9%</td>
<td>10 years</td>
<td>$1,000</td>
<td>9%</td>
<td>6.99 years</td>
</tr>
<tr>
<td>Bond B</td>
<td>3.1%</td>
<td>8 years</td>
<td>$673</td>
<td>9%</td>
<td>6.99 years</td>
</tr>
</tbody>
</table>

Bond B is found such that it has the same duration and YTM as Bond A. Bond B has coupon rate 3.1% and maturity equals 8 years. Its price is $673.

- Portfolio α consists of 673 unit of Bond A ($673,000)
- Portfolio β consists of 1,000 units of Bond B ($673,000)
Would an investor be indifferent to these two portfolios since they are worth exactly the same, offer the same YTM and have the same duration (same interest rate risk).

Recall the formula:

$$\text{Convexity} = \frac{1}{B(i) \frac{d}{di}} \left( \frac{dB}{di} \right) = \frac{1}{B} \frac{d^2B}{di^2}.$$  

What makes a bond more convex than the other one if they have the same duration? The key is the dispersion of payment times.

$$\text{Convexity} = \frac{\text{dispersion} + \text{duration} (\text{duration} + 1)}{(1+i)^2}.$$  

The convexity has the second order effect on bond portfolio management.
The effect of a greater convexity for bond A than for bond B enhances an investment in A compared to an investment in B in the event of change in interest rates. Investment in A will gain more value than investment in B if interest rates drop and it will lose less value if interest rates rise.
Calculation of the convexity of bond $A$ (coupon: 9%; yield to maturity: 10 years)

<table>
<thead>
<tr>
<th>Time of payment $t$</th>
<th>$t(t+1)$</th>
<th>Cash flow in nominal value $c_t$</th>
<th>Share of the discounted cash flows in bond’s value $c_t(1+i)^{-t}/B$</th>
<th>$t(t+1)$ times share of discounted cash flows $= (2) \times (4)$ $t(t+1)c_t(1+i)^{-t}/B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>9</td>
<td>0.0826</td>
<td>0.165</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>9</td>
<td>0.0758</td>
<td>0.456</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
<td>9</td>
<td>0.0695</td>
<td>0.834</td>
</tr>
<tr>
<td>4</td>
<td>20</td>
<td>9</td>
<td>0.0638</td>
<td>1.275</td>
</tr>
<tr>
<td>5</td>
<td>30</td>
<td>9</td>
<td>0.0585</td>
<td>1.755</td>
</tr>
<tr>
<td>6</td>
<td>42</td>
<td>9</td>
<td>0.0537</td>
<td>2.254</td>
</tr>
<tr>
<td>7</td>
<td>56</td>
<td>9</td>
<td>0.0492</td>
<td>2.757</td>
</tr>
<tr>
<td>8</td>
<td>72</td>
<td>9</td>
<td>0.0452</td>
<td>3.252</td>
</tr>
<tr>
<td>9</td>
<td>90</td>
<td>9</td>
<td>0.0414</td>
<td>3.729</td>
</tr>
<tr>
<td>10</td>
<td>110</td>
<td>109</td>
<td>0.4604</td>
<td>50.647</td>
</tr>
</tbody>
</table>

Convexity: total of $(5) \times \frac{1}{(1.09)^2} = 56.5$ (years$^2$)
\[ C = \frac{1}{B} \frac{d^2 B}{d i^2} = \frac{1}{B(1 + i)^2} \sum_{t=1}^{T} t(t + 1)c_t(1 + i)^{-t} \]

How can we increase the convexity of a portfolio of bonds, given its duration?

*Improvement in the measurement of a bond's price change by using convexity*

<table>
<thead>
<tr>
<th>Change in the rate of interest</th>
<th>Change in the bond's price in linear approximation</th>
<th>Change in the bond's price in quadratic approximation (using both duration and convexity)</th>
<th>Change in the bond's price in exact value</th>
</tr>
</thead>
<tbody>
<tr>
<td>+1%</td>
<td>-6.4%</td>
<td>-6.135%</td>
<td>-6.14%</td>
</tr>
<tr>
<td>-1%</td>
<td>+6.4%</td>
<td>+6.700%</td>
<td>+6.71%</td>
</tr>
</tbody>
</table>
By Taylor series, the relative increase in the bond’s value is given, in quadratic approximation, by

\[
\frac{\Delta B}{B} \approx \frac{1}{B} \frac{dB}{di} di + \frac{1}{2} \frac{d^2B}{di^2} (di)^2
\]

\[
= -6.4176 + 0.2825 = -6.135\%.
\]

On the other hand, suppose \(i\) decreases by 1%. With \(di\) equal to \(-1\%\), we obtain

\[
\frac{\Delta B}{B} \approx +6.4176 + 0.2825 = 6.700\%.
\]
Linear and quadratic approximations of a bond's value
The linear approximation is given by the following equation:

$$\frac{\Delta B}{B_0} = \frac{1}{B_0} \frac{dB}{di} di = -D_m di, \quad D_m = \frac{\text{duration}}{1 + i}.$$ 

Replacing \( \Delta B \) by \( B - B_0 \) and \( di \) by \( i - i_0 \), we have

$$B(i) = B_0[(1 + D_m i_0) - D_m i].$$

The quadratic approximation is given by

$$\frac{\Delta B}{B_0} = \frac{1}{B_0} \frac{dB}{di} di + \frac{1}{2} \frac{1}{B_0} \frac{d^2B}{di^2} (di)^2 = -D_m di + \frac{1}{2} C (di)^2$$
or, equivalently

\[ B(i) = B_0 \left[ \frac{C}{2} i^2 - (D_m + c_i 0)i + 1 + D_m i_0 + \frac{C}{2} i_0^2 \right]. \]

- Most financial services provide the value of convexity for bonds by dividing the value of \( \frac{1}{\overline{B}} \frac{d^2 B}{d\overline{B}^2} \) by 200. [In the example, \( \overline{d} = 0.01, \frac{1}{\overline{B}} \frac{d^2 B}{d\overline{B}^2} \frac{1}{200} = 0.28 \). When this is added or subtracted from the modified duration, we obtain \(-6.12\%\) and \(+6.68\%\).]
What makes a bond convex?

To make the convexity \( \frac{1}{B} \frac{d^2B}{di^2} \) appear, we consider the derivative of \( D \) and equate the result to \(-\frac{S}{1+i}\). Now

\[
D = -\frac{1+i}{B(i)} B'(i) \quad \text{so} \quad \frac{dD}{di} = -\left[ \frac{B - (1+i)B'}{B^2} \right] B'(i) - \frac{1+i}{B} B''.
\]

Recall \( \frac{dD}{di} = -\frac{S}{1+i} \) so that

\[
\frac{1}{B} (1 + D) B'(i) + \frac{1+i}{B} B'' = \frac{S}{1+i}.
\]
Writing $\frac{B'}{B} = -\frac{D}{1+i}$ and $\frac{B''}{B} = C$ so that

$$-D(D + 1) + (1 + i)^2 C = S.$$ 

Finally, we obtain

$$C = \frac{S + D(D + 1)}{(1 + i)^2}.$$ 

Convexity depends on both dispersion and duration. A portfolio will be highly convex for any given duration when the variance of its times of payments is high.
Yield curve strategies

- Seek to capitalize on expectations based on the short-term movements in yields.

- Source of return depends on the maturity of the securities in the portfolio. For example, the price of the one-year securities will not be sensitive to one-year yield, but the price of 30-year securities will be highly sensitive.

How to choose the spacing of the maturity of bonds?
**Bullet strategy**

Maturity of the securities are highly concentrated on one maturity date.

**Barbell strategy**

Maturity of the securities are concentrated at two extreme maturities.

**Ladder strategy**  Equal amount of each maturity.
Example

<table>
<thead>
<tr>
<th>Bond</th>
<th>Coupon</th>
<th>Maturity</th>
<th>Price</th>
<th>YTM</th>
<th>Duration</th>
<th>Convexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>8.5%</td>
<td>5</td>
<td>100</td>
<td>8.50</td>
<td>4.005</td>
<td>19.8164</td>
</tr>
<tr>
<td>B</td>
<td>9.5%</td>
<td>20</td>
<td>100</td>
<td>9.50</td>
<td>8.882</td>
<td>124.1702</td>
</tr>
<tr>
<td>C</td>
<td>9.25%</td>
<td>10</td>
<td>100</td>
<td>9.25</td>
<td>6.434</td>
<td>55.4506</td>
</tr>
</tbody>
</table>

- Bullet portfolio: 100% bond C

- Barbell portfolio: 50.2% bond A and 49.8% bond B

  duration of barbell portfolio = \(0.502 \times 4.005 + 0.498 \times 8.882\)  
  \[= 6.434\]

  convexity of barbell portfolio = \(0.502 \times 19.8164 + 0.498 \times 124.1702\)  
  \[= 71.7846.\]
Yield

portfolio yield for the barbell portfolio

\[= 0.502 \times 8.5\% + 0.498 \times 9.5\% = 8.998\% < \text{yield of bond C} \]

Duration

Both strategies have the same duration.

Convexity

convexity of barbell > convexity of bullet

The barbell strategy gives up yield in order to achieve a higher convexity.
Assume a 6-month investment horizon

1. Yield curve shifts in a parallel fashion

   When the change in yield $\Delta \lambda < 100$ basis points, the bullet portfolio outperforms the barbell portfolio; vice versa if otherwise.

   If $\lambda$ shifts parallel in a small amount, the portfolio with less convexity provides a better total return.

2. Non-parallel shift (flattening of the yield curve)

   $\Delta \lambda$ of bond $A = \Delta \lambda$ of bond $C + 25$ bps
   $\Delta \lambda$ of bond $B = \Delta \lambda$ of bond $C - 25$ bps

   The barbell strategy always outperforms the bullet strategy.
3. Non-parallel shift (steepening of the yield curve)

\[ \Delta \lambda \text{ of bond } A = \Delta \lambda \text{ of bond } C - 25 \text{ bps} \]
\[ \Delta \lambda \text{ of bond } B = \Delta \lambda \text{ of bond } C + 25 \text{ bps} \]

The bullet portfolio outperforms the barbell portfolio as long as the yield on bond \( C \) does not rise by more than 250 bps or fall by more than 325 bps.

**Conclusion**

The performance depends on the magnitude of the change in yields and how the yield curve shifts.
Convexity in bond management

- Search for convexity that improves the investor’s performance.

- **Active management**
  Bonds were bought with a short-term horizon, expecting fall in interest rates. When interest rates fall, the rate of return will be higher with more convex bonds or portfolio. If interest rates move in the opposite direction, the loss will be smaller for highly convex bonds.

- **Defensive management (immunization)**
  Any movement in interest rates that occur immediately after the purchase of the portfolio will translate into higher returns if the portfolio is highly convex.
Comparing two coupon-bearing bonds with differing maturities

- Coupon rate is set from zero to 15%.
- Yield to maturity is set at 9%.
- Increasing the coupon rate decrease both the duration and convexity.

**Characteristics of bonds** *A* and *B*

<table>
<thead>
<tr>
<th></th>
<th>Bond <em>A</em></th>
<th>Bond <em>B</em></th>
</tr>
</thead>
<tbody>
<tr>
<td>Maturity</td>
<td>10 years</td>
<td>20 years</td>
</tr>
<tr>
<td>Coupon</td>
<td>1</td>
<td>13.5</td>
</tr>
<tr>
<td>Duration</td>
<td>9.31 years</td>
<td>9.31 years</td>
</tr>
<tr>
<td>Convexity</td>
<td>84.34 years²</td>
<td>115.97 years²</td>
</tr>
</tbody>
</table>

Bond *A* is way below par (48.66) and Bond *B* is above par (141.08). They have the same duration but differing convexities.
### Duration and convexity for two types of bonds

<table>
<thead>
<tr>
<th>Coupon (c)</th>
<th>Type I bond</th>
<th>Type II bond</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Maturity: 10 years</td>
<td>Maturity: 20 years</td>
</tr>
<tr>
<td></td>
<td>Duration (years)</td>
<td>Convexity (years$^2$)</td>
</tr>
<tr>
<td>0</td>
<td>10.00</td>
<td>92.58</td>
</tr>
<tr>
<td>1</td>
<td>9.31</td>
<td>84.34</td>
</tr>
<tr>
<td>2</td>
<td>8.79</td>
<td>78.02</td>
</tr>
<tr>
<td>3</td>
<td>8.37</td>
<td>73.02</td>
</tr>
<tr>
<td>4</td>
<td>8.03</td>
<td>68.96</td>
</tr>
<tr>
<td>5</td>
<td>7.75</td>
<td>65.60</td>
</tr>
<tr>
<td>6</td>
<td>7.52</td>
<td>62.79</td>
</tr>
<tr>
<td>7</td>
<td>7.32</td>
<td>60.38</td>
</tr>
<tr>
<td>8</td>
<td>7.15</td>
<td>58.31</td>
</tr>
<tr>
<td>9</td>
<td>6.99</td>
<td>56.50</td>
</tr>
<tr>
<td>10</td>
<td>6.86</td>
<td>54.90</td>
</tr>
<tr>
<td>11</td>
<td>6.76</td>
<td>53.49</td>
</tr>
<tr>
<td>12</td>
<td>6.64</td>
<td>52.23</td>
</tr>
<tr>
<td>13</td>
<td>6.55</td>
<td>51.10</td>
</tr>
<tr>
<td>13.5</td>
<td>6.50</td>
<td>50.58</td>
</tr>
<tr>
<td>14</td>
<td>6.46</td>
<td>50.08</td>
</tr>
<tr>
<td>15</td>
<td>6.38</td>
<td>49.16</td>
</tr>
</tbody>
</table>
Suppose that initial rates are 9% and that they quickly move up by 1 or 2% or drop by the same amount. The rate of return over 9% for the more convex bond is tenfold that of the bond with lower convexity.

Rates of return for A and B with horizon $D = 9.31$ years when rates move quickly from 9% to another value and stay there

<table>
<thead>
<tr>
<th>Scenario</th>
<th>$i = 7%$</th>
<th>$i = 8%$</th>
<th>$i = 9%$</th>
<th>$i = 10%$</th>
<th>$i = 11%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bond A</td>
<td>9.008%</td>
<td>9.002%</td>
<td>9%</td>
<td>9.002%</td>
<td>9.008%</td>
</tr>
<tr>
<td>Bond B</td>
<td>9.085%</td>
<td>9.021%</td>
<td>9%</td>
<td>9.020%</td>
<td>9.079%</td>
</tr>
</tbody>
</table>

- investment in $A$: $1,000,000(1 + 0.9008)^{9.31} = 2,232,222$
- investment in $B$: $1,000,000(1 + 0.9085)^{9.31} = 2,246,245$

which implies a difference of $14,023$ for no trouble at all, except looking up the value of convexity.
Active Management

- Three short horizons have been chosen.

- Shorter horizon, the gain of horizon rate of return of Bond $B$ is more significant.

- With an increase in interest rates from 9% to 11%, the convexity of $B$ will cushion the loss from 6.2% to 5.7%.
Rates of return when \( i \) takes a new value immediately after the purchase of bond A and bond B (in percentage per year)

<table>
<thead>
<tr>
<th>Horizon (in years) and rates of return for A and B</th>
<th>Scenario</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( i = 7% )</td>
</tr>
<tr>
<td>( H = 1 )</td>
<td></td>
</tr>
<tr>
<td>( R^A )</td>
<td>27.2</td>
</tr>
<tr>
<td>( R^B )</td>
<td>28.1</td>
</tr>
<tr>
<td>( H = 2 )</td>
<td></td>
</tr>
<tr>
<td>( R^A )</td>
<td>16.7</td>
</tr>
<tr>
<td>( R^B )</td>
<td>17.1</td>
</tr>
<tr>
<td>( H = 3 )</td>
<td></td>
</tr>
<tr>
<td>( R^A )</td>
<td>8.9</td>
</tr>
<tr>
<td>( R^B )</td>
<td>8.9</td>
</tr>
</tbody>
</table>
(a) Characteristics of bonds $A$ and $B$

<table>
<thead>
<tr>
<th></th>
<th>Bond $A$ (zero-coupon)</th>
<th>Bond $B$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Coupon</strong></td>
<td>0</td>
<td>9</td>
</tr>
<tr>
<td><strong>Maturity</strong></td>
<td>10.58 years</td>
<td>25 years</td>
</tr>
<tr>
<td><strong>Duration</strong></td>
<td>10.58 years</td>
<td>10.58 years</td>
</tr>
<tr>
<td><strong>Convexity</strong></td>
<td>103.12 years$^2$</td>
<td>159.17 years$^2$</td>
</tr>
</tbody>
</table>

(b) Rates of return of $A$ and $B$ when $i$ moves from $i = 9\%$ to another value after the bond has been bought (horizon is set to be the same as the duration)

<table>
<thead>
<tr>
<th>Scenario</th>
<th>$i = 7%$</th>
<th>$i = 8%$</th>
<th>$i = 9%$</th>
<th>$i = 10%$</th>
<th>$i = 11%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bond $A$ (zero-coupon)</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>Bond $B$</td>
<td>9.14</td>
<td>9.04</td>
<td>9</td>
<td>9.02</td>
<td>9.08</td>
</tr>
</tbody>
</table>
Rates of return when $i$ takes a new value immediately after the purchase of bond A or bond B (in percentage per year)

<table>
<thead>
<tr>
<th>Horizon and rates of return for A and B</th>
<th>Scenario</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$i = 7%$</td>
</tr>
<tr>
<td></td>
<td>$R^A$</td>
</tr>
<tr>
<td></td>
<td>30.2</td>
</tr>
<tr>
<td></td>
<td>31.9</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$R^B$</td>
</tr>
<tr>
<td></td>
<td>18.0</td>
</tr>
<tr>
<td></td>
<td>18.8</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$R^A$</td>
</tr>
<tr>
<td></td>
<td>8.9</td>
</tr>
<tr>
<td></td>
<td>8.9</td>
</tr>
</tbody>
</table>
Asset and liabilities management

- How should a pension fund, or an insurance company, set up its asset portfolio in such a way as to be practically certain it will be able to meet its payment obligations in the future?

**Redington conditions**

Assume that $L_t, t = 1,\ldots, T$ and $A_t, t = 1,\ldots, T$ are known. Interest term structure is flat, equal to $i$.

$$L = \sum_{t=1}^{T} \frac{L_t}{(1 + i)^t} \text{ and } A_t = \sum_{t=1}^{T} \frac{A_t}{(1 + i)^t}.$$  

$N = A - L = 0$ initially.
1. How should one choose the structure of the assets such that this net value does not change in the event of a change in interest rate?

1st order condition (first Redington property):

\[ N = A - L \] to be insensitive to \( i \).

Set

\[
\frac{dN}{di} = \frac{1}{1+i} \sum_{t=1}^{T} t(L_t - A_t)(1+i)^{-t} = \frac{1}{1+i} (D_{LL}L - D_{AA})
\]

\[
= \frac{A}{1+i} (D_L - D_A) = 0, \text{ (since } L = A),
\]

where

\[
D_L = \sum_{t=1}^{T} \frac{t L_t}{L} \frac{1}{(1+i)^t} \quad \text{ and } \quad D_A = \sum_{t=1}^{T} \frac{t A_t}{L} \frac{1}{(1+i)^t}.
\]
2. Sufficient (not necessary) condition such that for all possible values within an interval of \( i_0 \), \( N \) remains positive. Here, \( N(i) \) is a convex function of \( i \) within that interval.

Second Redington condition: \[ \frac{d^2 A}{di^2} > \frac{d^2 L}{di^2}. \]

Once the duration is given, convexity depends positively on the dispersion \( S \) of the cash flows. Therefore, a necessary condition for the second Redington condition is that the dispersion of the inflows be larger than that of the outflows.
Example (Savings and Loan Associations in US in early 1980s)

They had inflows (deposits) with short maturities (duration) while outflows (their loans) had very long durations, since they financed mainly housing projects.

- When the interest rates climbed sharply, the net worth of the savings and loans associations fall drastically.

In this case, even the first Redington condition was not met. This spelled disaster.
Example (Net initial position of the financial firm is sound)

- Investing $1 million in a 20-year, 8.5% coupon bond.
- Financed with a 9-year loan carrying an 8% interest rate.

Same initial value; Recall the duration formula:

\[
D = \frac{1}{i} + \theta + \frac{N}{m} \left( i - \frac{c}{B_T} \right) - \left( 1 + \frac{i}{m} \right) \frac{c}{B_T} \left[ (1 + \frac{i}{m})^N - 1 \right] + \frac{i}{i} ,
\]

where

\[
\theta = \text{time to wait for the next coupon to be paid } (0 \leq \theta \leq 1)
\]
\[
m = \text{number of times a payment is made within one year}
\]
\[
N = \text{total number of coupons remaining to be paid.}
\]

Here, \( \theta = \frac{1}{2} \), \( m = 2 \); \( N = 40 \) for the 20-year bond and \( N = 18 \) for the 9-year loan.
We obtain the durations as

\[ D_A = 9.944 \text{ years} \quad \text{and} \quad D_L = 6.583 \text{ years} \]

so that the modified durations are

\[ D_{mA} = \frac{D_A}{1 + i_A} = 9.165 \text{ years} \quad \text{and} \quad D_{mL} = \frac{D_L}{1 + i_L} = 6.095 \text{ years}. \]

Note that

\[ \Delta V_A \approx \frac{dV_A}{V_A} = -D_{mA} \, di_A \quad \text{and} \quad \Delta V_B \approx \frac{dV_L}{V_L} = -D_{mL} \, di_L \]

so that

\[ \frac{\Delta V_P}{V_P} = \frac{\Delta V_A}{V_A} - \frac{\Delta V_L}{V_L} \approx -(D_{mA} \, di_A - D_{mL} \, di_L); \quad V = V_A - V_L. \]
Suppose $i_A$ and $i_L$ receive the same increment, we have

$$\frac{\Delta V_P}{V_P} \approx -(D_{mA} - D_{mL})di = -3.070di.$$ 

Based on the linear approximation, if interest rates increase by 1%, the net value of the project diminishes by 3.070%. Its risk exposure presents a net duration of $D_A - D_L = 3.361$. 

Remarks

1. Immunization is a short-term series of measures destined to match sensitivities of assets and liabilities. As time passes, these sensitivities continue to change since the duration does not generally decrease in the same amount as the planning horizon.

2. Whenever interest rates change, the duration also changes.

3. So far we have considered flat term structures and parallel displacements of them. More refined duration measures and analysis are required if we do not face such flat structures.

4. Financial manager may also want to pay special attention to the convexity of his assets and liabilities as well.
Cash matching problem

- A known sequence of future monetary obligations over $n$ periods.
  \[ y = (y_1 \ldots y_n) \]

- Purchase bonds of various maturities and use the coupon payments and redemption values to meet the obligations.

Suppose there are $m$ bonds, and the cash stream associated with one unit of bond $j$ is $c_j = (c_{1j} \ldots c_{nj})$.

- $p_j =$ price of bond $j$
- $x_j =$ amount of bond $j$ held in the portfolio

Minimize

\[
\sum_{j=1}^{m} p_j x_j
\]

subject to

\[
\sum_{j=0}^{m} c_{ij} x_j \geq y_j \quad i = 1, 2, \ldots, n
\]

\[
x_j \geq 0 \quad j = 1, 2, \ldots, m.
\]
### Numerical example – Six-year match

To match cash obligations over a 6-year period using 10 bonds.

<table>
<thead>
<tr>
<th>Yr</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>Req’d</th>
<th>Actual</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>7</td>
<td>8</td>
<td>6</td>
<td>7</td>
<td>5</td>
<td>10</td>
<td>8</td>
<td>7</td>
<td>100</td>
<td>100</td>
<td>171.74</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>7</td>
<td>8</td>
<td>6</td>
<td>7</td>
<td>5</td>
<td>10</td>
<td>8</td>
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<td>200</td>
<td>200.00</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>7</td>
<td>8</td>
<td>6</td>
<td>7</td>
<td>5</td>
<td>110</td>
<td>108</td>
<td>800</td>
<td>800</td>
<td>800.00</td>
<td></td>
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<tr>
<td>4</td>
<td>10</td>
<td>7</td>
<td>8</td>
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<td></td>
<td>100</td>
<td>119.34</td>
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<td>8</td>
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<td>107</td>
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<td></td>
<td>800</td>
<td>800.00</td>
<td></td>
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<tr>
<td>6</td>
<td>110</td>
<td>107</td>
<td>108</td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td>1,200</td>
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</tr>
<tr>
<td>p</td>
<td>109</td>
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<td>110</td>
<td>104</td>
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<tr>
<td>x</td>
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<td>6.81</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>6.3</td>
<td>0.28</td>
<td>0</td>
<td>Cost</td>
<td></td>
</tr>
</tbody>
</table>

In two of the years, extra cash is generated beyond what is required.
Difficulties and weaknesses

• There are high requirements in some years so a larger number of bonds must be purchased that mature at those dates. These bonds generate coupon payments in earlier years and only a portion of these payments is needed to meet obligations in these early years. Such problem is alleviated with a smoother set of cash requirements.

• The extra surpluses should be reinvested. This requires the estimation of future interest rate movements.
• Given typical liability schedules and bonds available for cash flow matching, perfect matching is unlikely.

• Strike the tradeoff between (i) avoidance of the risk of not satisfying the liability stream (ii) lower cost.

• How to combine immunization with cash matching?
Rebalancing an immunized portfolio

Since the market yield will fluctuate over the investment horizon, how often should the portfolio be rebalanced in order to adjust its duration?

- Immunization involves minimizing the initial portfolio cost subject to the constraint of having sufficient cash to satisfy the liabilities.

- Transaction cost must be included in the optimization framework such that a tradeoff between transaction costs and risk minimization is considered.
The above immunization works perfectly only when the yield curve is flat (no convexity) and any changes in the yield curve are parallel changes. Duration matched portfolios are not unique.

**Goal**  How to construct an immunized portfolio that has the lowest risk of not realizing the target yield?

**Rules of thumb**

- Immunization risk is reinvestment risk.

- When the cash flows are concentrated around the horizon date, the portfolio is subject to less reinvestment risk.
**Immunization risk measure**

Immunization risk measure

\[ \text{Immunization risk measure} = \sum \text{PVCF} \times (\text{time to receipt} - \text{horizon date})^2 / \text{initial investment value} \]

where PVCF is the present value of cash flow.

Zero immunization risk portfolio is a portfolio consisting of zero-coupon bonds maturing on the horizon dates.
Bankruptcy of Orange County, California (see Qn 9 in HW 1)

“A prime example of the interest rate risk incurred when the duration of asset investments is not equal to the duration of fund needs.”

Orange County (like most municipal governments) maintained an operating account of cash from which operating expenses were paid. During the 1980’s and early 1990s, interest rates in US had been falling.

Seeing the larger returns being earned on long-term securities, the treasurer of Orange County decided to invest in long-term fixed income securities.
• Between 1991 and 1993, the county enjoyed more than a 8.5% return on investments.

• Started in February 1994, the Federal Reserve Board raised the interest rate in order to cool an expanding economy. All through the year, paper losses on the fund led to margin calls from Wall Street brokers that had provided short-term financing.

• In December 1994, as news of the loss spread, investors tried to pull out their money. Finally, as the fund defaulted on collateral payments, brokers started to liquidate their collateral.

• Bankruptcy caused the County to have difficulties to meet payrolls, 40% cut in health and welfare benefits and school employees were laid off.
• County officials blamed the county treasurer, Bob Citron, for undertaking risky investments. He claimed that there was no risk in the portfolio since he was holding to maturity.

• Since government accounting standards do not require municipal investment pools to report "paper" gains or losses, Citron did not report the market value of the portfolio.

• Investors, in touch with monthly fluctuations in values, also may have refrained from the "run on the bank" that happened in December 1994.