RMBI 4210 – Quantitative Methods for Risk Management

Topic Two – Risk measures and economic capital

2.1 Types of financial risks and loss distributions

2.2 VAR (value at risk), expected shortfall and coherent risk measures

2.3 Economic capital and risk-adjusted return on capital
2.1 Types of financial risks and loss distributions

Risk can be defined as loss or exposure to mischance, or more quantitatively as the volatility of unexpected outcomes, generally related to the value of assets or liabilities of concern.

- While some firms may passively accept financial risks, others attempt to create a competitive advantage by judicious exposure to financial risk. In both cases, these risks must be monitored because of their potential for damage (or even ruin).

Risk management is the process by which various risk exposures are identified, measured, and controlled.
Major events / shocks in the financial markets

• On Black Monday, October 19, 1987, U.S. stocks collapsed by 23 percent, wiping out $1 trillion in capital.

• In the bond debacle of 1994, the Federal Reserve Bank, after having kept interest rates low for 3 years, started a series of six consecutive interest rate hikes that erased $1.5 trillion in global capital.

• The Japanese stock price bubble finally deflated at the end of 1989, sending the Nikkei index from 39,000 to 17,000 three years later. A total of $2.7 trillion in capital was lost, leading to an unprecedented financial crisis in Japan.
• The Asian turmoil of 1997 wiped off about three-fourth of the dollar capitalization of equities in Indonesia, Korea, Malaysia, and Thailand.

• The Russian default in August 1998 sparked a global financial crisis that culminated in the near failure of a big hedge fund, Long Term Capital Management.

• The bankruptcy of Lehman Brothers and failure of other major financial institutions, like AIG, CitiGroup, Merill Lynch, triggered the financial tsunami in 2008.

• The downgrade of US debt rating from AAA to AA+ by Standard & Poor in August 2011 triggered crashes in the financial markets around the globe.
Market risk

Market risk arises from movements in the level or volatility of market prices.

- Absolute risk, measured in dollar terms.
- Relative risk, measured relative to a benchmark index.

Directional risks: Exposures to the direction of movements in financial variables (linear approximations)

- Beta for exposure to stock market movement, $\beta_i = \frac{\sigma_{iM}}{\sigma_M^2}$.
- Duration for exposure to interest rate, $D = \sum c_i t_i / \sum c_i$.
- Delta for exposure of options to the underlying asset price, $\Delta = \frac{\partial V}{\partial S}$. 
Non-directional risks

Non-linear exposures and exposures to hedged positions or to volatilities.

1. Second order or quadratic exposures are measured by

   - convexity when dealing with interest rates
   - gamma when dealing with options, $\Delta = \frac{\partial^2 V}{\partial S^2}$.

2. Volatility risk measures exposure to movements in the actual or implied volatility.

3. Basis risk is the risk arising when offsetting investments in a hedging strategy do not experience price changes in entirely opposite directions from each other. The imperfect correlation between the two investments creates the potential for excess gains or losses in a hedging strategy.
Credit risk

The credit risk associated with investment on a financial instrument can be quantified by the spread, which is the yield above the risk-free Treasury rate.

Credit risk consists of two components: default risk and spread risk.

1. Default risk (違約風險): any non-compliance with the exact specification of a contract.

2. Spread risk: reduction in market value of the contract / instrument due to changes in the credit quality of the debtor / counterparty.
Event of default

1. Arrival risk – timing of the event, modeled by a stopping time $\tau$

   A stopping rule is defined such that one can determine whether a stochastic process stops or continues, given the information available at that time.

2. Magnitude risk – loss amount (exposure net of the recovery value)

   Loss amount = par value (possibly plus accrued interest) – market value of a defaultable bond
Risk elements

1. Exposure at default and recovery rate, both are random variables.

2. Default probability (characterization of the default criteria and the associated random time of default).

3. Credit migration – the process of changing the creditworthiness of an obligor as characterized by the transition probabilities from one credit state to other credit states.
Credit event

Occurs when the calculation agent is aware of publicly available information as to the existence of a credit condition.

- Credit condition means either a payment default or a bankruptcy (清算) event in respect of the issuer.
- Publicly available information means information that has been published in any two or more internationally recognized published or electronically displayed financial news sources.

Chapter 11 Bankruptcy Code

- It is a chapter of the US Bankruptcy code.
- A company is protected from creditors while it restructures its business, usually by downsizing and narrowing focus.
- Keep the organization intact while seeking protection from creditors.
Liquidity risk

Asset liquidity risk

This arises when a transaction cannot be conducted at prevailing market prices due to the size of the position relative to normal trading lots.

- Some assets, like Treasury bonds, have deep markets where most positions can be liquidated easily, with very little price impact.

- Other assets, like OTC (over-the-counter) derivatives or emerging market equities, any significant transaction can quickly affect prices.
Funding liquidity risk (cash-flow risk)

The inability to meet payment obligations, which may force early liquidation, thus transforming “paper” losses into realized losses.

– This is a problem for portfolios that are leveraged and subject to margin calls from the lender.

Cash-flow risk interacts with asset liquidity risk if the portfolio contains illiquid assets that must be sold at less than the fair market value.
Operational risk

Arising from human and technical errors or accidents.

- Fraud (traders intentionally falsify information), management failure, and inadequate procedures and controls.

- Back-office operations that record transactions.

Model risk

The mathematical model used to value positions is flawed.
Legal risk

Arises when a transaction proves unenforceable in law.

- Legal risk is generally related to credit risk, since counterparties that lose money on a transaction may try to find legal grounds for invalidating the transaction. It may take the form of shareholder law suits against corporations that suffer large losses.

Example

Two municipalities in Britain had taken large positions in interest rate swaps that turned out to produce large losses. The swaps were later ruled invalid by the British High Court. The court decreed that the city councils did not have the authority to enter into these transactions and so the cities were not responsible for the losses. Their bank counterparties had to swallow the losses.
Categorization of risks faced by a bank

The relative importance of different risks depends on the business mix.

- Banks taking deposits and making loans – credit risk.
- Investment banks – both credit risk and market risk.
- Asset managers – operational risk.
The Barings Bank Disaster

- Nicholas Leeson, an employee of Barings Bank in the Singapore office in 1995, had a mandate to look for arbitrage opportunities between the Nikkei 225 futures prices on the Singapore exchange and the Osaka exchange. Over time Leeson moved from being an arbitrageur to being a speculator without anyone in the Barings London head office fully understanding that he had changed the way he was using derivatives.

- In 1994, Leeson is thought to have made $20 million for Barings, one-fifth of the total firm’s profit. He drew a $150,000 salary with a $1 million bonus.

- He began to make losses, which he was able to hide. He then began to take bigger speculative positions in an attempt to recover the losses, but only resulted in making the losses worse.
• In the end, Leeson’s total loss was close to 1 billion dollars. As a result, Barings – a bank that had been in existence for 200 years – was wiped out.

• Lesson to be learnt: Both financial and nonfinancial corporations must set up controls to ensure that derivatives are being used for their intended purpose. Risk limits should be set and the activities of traders should be monitored daily to ensure that the risk limits are adhered to.
Long Term Capital Management

- Long Term Capital Management (LTCM) is a hedge fund formed in the mid 1990s. The hedge fund's investment strategy was known as convergence arbitrage.

- It would find two bonds, \( X \) and \( Y \), issued by the same company promising the same payoffs, with \( X \) being less liquid than \( Y \). The market always places a value on liquidity. As a result the price of \( X \) would be less than the price of \( Y \). LTCM would buy \( X \), short \( Y \) and wait, expecting the prices of the two bonds to converge at some future time.
• When interest rates increased (decreased), the company expected both bonds to move down (up) in price by about the same amount, so that the collateral it paid on bond $X$ would be about the same as the collateral it received on bond $Y$. It therefore expected that there would be no significant outflow of funds as a result of its collateralization agreements.

• In August 1998, Russia defaulted on its debt and this led to what is termed a “flight to quality” in capital markets. One result was that investors valued liquid instruments more highly than usual and the spreads between the prices of the liquid and illiquid instruments in LTCM’s portfolio increased dramatically (instead of convergence).
• The prices of the bonds LTCM had bought went down and the prices of those it had shorted increased. LTCM was required to post collateral on both.

• The company was highly leveraged and unable to make the payments required under the collateralization agreements. The result was that positions had to be closed out and there was a total loss of about $4 billion.

• If the company had been less highly leveraged, it would probably have been able to survive the flight to quality and could have waited for the prices of the liquid and illiquid bonds to become closer.
In 1996, Peter Young was a fund manager at Deutsche Morgan Grenfell, a subsidiary of Deutsche Bank. He was responsible for managing a fund called the European Growth Trust (EGT). It had grown to be a very large fund and Young had responsibilities for managing over £1 billion of investors’ money.

Certain rules were applied to EGT, one of these was that no more than 10% of the fund could be invested in unlisted securities. Peter Young violated this rule in a way that, it can be argued, benefited him personally.

When the facts were uncovered, he was fired and Deutsche Bank had to compensate investors. The total cost to Deutsche Bank was over £200 million.
Drivers of firm’s economic well-being

- future earnings and cashflows
- debts, short-term and long-term liabilities, and financial obligations
- capital structure (leverage)
- liquidity of the firm’s assets
- political situations
- industrial situations
- management quality, company structure, etc.

Rating on a company relies on the statistical analysis of financial variables plus soft factors.
Capital reserve for managing the credit risk of loans

- Charging an appropriate risk premium for every loan and collecting these risk premiums in an internal book account. The expected loss reserve creates a capital cushion for covering losses arising from defaulting loans. Financial variables to be considered include
  - default probability \((DP)\)
  - loss fraction called the loss given default \((LGD)\)
  - exposure at default \((EAD)\)

**Loss variable**

\[ \tilde{L} = EAD \times SEV \times L \]

where \(L = 1_D, \ E[1_D] = DP\). Here, \(D\) is the default event that the obligor defaults within a certain period of time.
We treat severity ($SEV$) of loss in case of default as a random variable with $E[SEV] = LGD$.

Based on the assumption that the exposure, severity and default event are independent, the expected loss ($EL$):

$$EL = E[\tilde{L}] = EAD \times LGD \times DP.$$  

Here, $EAD$ is assumed to be deterministic or the expectation of some underlying random variable.

**Goal:** Derive the portfolio risk based on the information of individual risks and their correlations.
Loss given default

\[ LGD = 1 - \text{recovery rate} \]

Driving factors

1. quality of collateral - how much the collateral can cover the loss

2. seniority of bank’s claim on borrower’s assets

- How do banks share knowledge about their practical \( LGD \) experience?

- How can we derive better techniques for estimating \( LGD \) from historical data?
Unexpected loss – standard deviation of $\tilde{L}$

Holding capital as a cushion against expected losses is not enough. As a measure of the magnitude of the deviation of losses from the $EL$, a natural choice is the standard deviation of the loss variable $\tilde{L}$.

$$\text{Unexpected loss (UL)} = \sqrt{\text{var}(\tilde{L})} = \sqrt{\text{var}(EAD \times SEV \times L)}.$$  

Under the assumption that the severity and the default event $D$ are independent, we have

$$UL = EAD \times \sqrt{\text{var}(SEV)} \times DP + LGD^2 \times DP(1 - DP).$$
Proof

We make use of \( \text{var}(X) = E[X^2] - E[X]^2 \), so that \( \text{var}(\mathbf{1}_D) = DP(1 - DP) \) since \( E[\mathbf{1}_D^2] = E[\mathbf{1}_D] = DP \). Assuming \( SEV \) and \( \mathbf{1}_D \) are independent, we have

\[
\text{var}(SEV \mathbf{1}_D) = E[SEV^2 \mathbf{1}_D^2] - E[SEV \mathbf{1}_D]^2 \\
= E[SEV^2]E[\mathbf{1}_D^2] - E[SEV]^2E[\mathbf{1}_D]^2 \\
= \{\text{var}(SEV) + E[SEV]^2\}DP - E[SEV]^2DP^2 \\
= \text{var}(SEV) \times DP + LGD^2 \times DP(1 - DP).
\]

Remark

It is common to have the situation where the severity of losses and the default events are random variables driven by a common set of underlying factors. Hence, one may question the assumption of independence between \( SEV \) and \( \mathbf{1}_D \).
Portfolio losses

Consider a portfolio of $m$ loans

$$
\bar{L}_i = EAD_i \times SEV_i \times 1_{D_i}, \quad i = 1, \ldots, m, \quad P[D_i] = E[1_{D_i}] = DP_i.
$$

The random portfolio loss $\bar{L}_p$ is given by

$$
\bar{L}_p = \sum_{i=1}^{m} \bar{L}_i = \sum_{i=1}^{m} EAD_i \times SEV_i \times L_i, \quad L_i = 1_{D_i}.
$$

Using the additivity of expectation, we obtain

$$
EL_p = \sum_{i=1}^{m} EL_i = \sum_{i=1}^{m} EAD_i \times LGD_i \times DP_i.
$$
In the case $UL$, additivity holds if the loss variable $\tilde{L}_i$ are pairwise uncorrelated. Unfortunately, correlations are “main part of the game” and a main driver of credit risk. In general, we have

$$UL_p = \sqrt{\text{var}(\tilde{L}_p)}$$

$$= \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{m} EAD_i \times EAD_j \times \text{cov}(SEV_i \times L_i, SEV_j \times L_j)}.$$

For a portfolio with constant severities, we have the following simplified formula

$$UL_p^2 = \sum_{i=1}^{m} \sum_{j=1}^{m} EAD_i \times EAD_j \times LGD_i \times LGD_j \times \sqrt{DP_i(1 - DP_i)DP_j(1 - DP_j)} \rho_{ij},$$

where

$$\rho_{ij} = \text{correlation coefficient between default events}$$

$$= \frac{\text{cov}(\mathbf{1}_{D_i}, \mathbf{1}_{D_j})}{\sqrt{\text{var}(L_i)\text{var}(L_j)}}.$$
Example

Take $m = 2$, $LGD_i = EAD_i = 1$, $i = 1, 2$, then

$$UL_p^2 = p_1(1 - p_1) + p_2(1 - p_2) + 2\rho \sqrt{p_1(1 - p_1)p_2(1 - p_2)},$$

where $p_i$ is the default probability of obligor $i$, $i = 1, 2$, and $\rho$ is the correlation coefficient.

(i) When $\rho = 0$, the two default events are uncorrelated.

(ii) When $\rho > 0$, the default of one counterparty increases the likelihood that the other counterparty may also default. Consider

$$P[L_2 = 1|L_1 = 1] = \frac{P[L_2 = 1, L_1 = 1]}{P[L_1 = 1]} = \frac{E[L_1 L_2]}{p_1} = \frac{p_1 p_2 + \text{cov}(L_1, L_2)}{p_1} = p_2 + \frac{\text{cov}(L_1, L_2)}{p_1}.$$  

Positive correlation leads to a conditional default probability higher than the unconditional default probability $p_2$ of obligor 2.
Relations between portfolio variance $UL^2_p$ and unexpected losses $UL_i$

Recall

$$UL^2_p = \text{var}(\tilde{L}_p) = \text{var}\left(\sum_{i=1}^{m} \tilde{L}_i\right)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} \text{cov}(\tilde{L}_i, \tilde{L}_j) = \sum_{i=1}^{m} \sum_{j=1}^{m} UL_iUL_j\rho_{ij},$$

where $UL^2_i = \text{var}(\tilde{L}_i)$, $i = 1, 2, \ldots, m$. Note that

$$\frac{\partial UL^2_p}{\partial UL_i} = 2UL_p \frac{\partial UL_p}{\partial UL_i} = 2 \sum_{j=1}^{m} UL_j\rho_{ij}.$$
Risk contribution

The risk contribution of a risky asset $i$ to the portfolio unexpected loss is defined to be the incrementa risk that the exposure of a single asset contributes to be the portfolio’s total risk, namely,

$$RC_i = UL_i \frac{\partial ULP}{\partial UL_i} = \frac{UL_i \sum_j UL_j \rho_{ij}}{UL_P},$$

Using the unexpected losses $UL_i$ and $UL_P$ as the quantifiers of risk, we expect that the risk contributions from the risky assets is simply the total portfolio risk. As a verification, it is seen mathematically that

$$UL_P = \sum_i RC_i.$$
Calculation of $EL$, $UL$ and $RC$ for a two-asset portfolio

$\rho$  
default correlation coefficient between the two exposures

$EL_p$  
portfolio expected loss

$EL_p = EL_1 + EL_2$

$UL_p$  
portfolio unexpected loss

$UL_p = \sqrt{UL_1^2 + UL_2^2 + 2\rho UL_1 UL_2}$

$RC_1$  
risk contribution from Exposure 1

$RC_1 = UL_1(UL_1 + \rho UL_2)/UL_p$

$RC_2$  
risk contribution from Exposure 2

$RC_2 = UL_2(UL_2 + \rho UL_1)/UL_p$

$UL_p = RC_1 + RC_2$
Third and fourth order moments of a distribution

- Skewness describes the departure from symmetry:

\[ \gamma = \frac{\int_{-\infty}^{\infty} (x - E[X])^3 f(x) \, dx}{\sigma^3}. \]

The skewness of a normal distribution is zero. Positive skewness indicates that the distribution has a long right tail and so entails large positive values.

- Kurtosis describes the degree of flatness of a distribution:

\[ \delta = \frac{\int_{-\infty}^{\infty} (x - E[X])^4 f(x) \, dx}{\sigma^4}. \]

The kurtosis of a normal distribution is 3. A distribution with kurtosis greater than 3 has the tails decay less quickly than that of the normal distribution, implying a greater likelihood of large value in both tails.
Characteristics of loss distributions for different risk types

<table>
<thead>
<tr>
<th>Type of risk</th>
<th>Second moment (standard deviation)</th>
<th>Third moment (skewness)</th>
<th>Fourth moment (kurtosis)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Market risk</td>
<td>High</td>
<td>Zero</td>
<td>Low</td>
</tr>
<tr>
<td>Credit risk</td>
<td>Moderate</td>
<td>Moderate</td>
<td>Moderate</td>
</tr>
<tr>
<td>Operational risk</td>
<td>Low</td>
<td>High</td>
<td>High</td>
</tr>
</tbody>
</table>

- The market risk loss distribution is symmetrical but not perfectly normally distributed.
• The credit risk loss distribution is quite skewed.

• The operational risk distribution has a quite extreme shape. Most of the time, losses are modest, but occasionally they are very large.
Monte Carlo simulation of losses

In a Monte Carlo simulation, losses are simulated and tabulated in the form of a histogram. Assume that we have simulated $n$ runs of potential portfolio losses $\tilde{L}_p^{(1)}, \ldots, \tilde{L}_p^{(n)}$, hereby taking the driving distributions of the single loss variables and their correlations into account.

Define

$$1_{[0,x]}(y) = \begin{cases} 1 & y \leq x \\ 0 & y > x \end{cases}.$$

The empirical loss distribution function is given by

$$F(x) = \frac{1}{n} \sum_{j=1}^{n} 1_{[0,x]}(\tilde{L}_p^{(j)}).$$

That is, for a given value of $x$, we calculate the proportion of simulated portfolio losses out of $n$ simulations where $\tilde{L}_p^{(j)}$ falls within $[0, x]$. 
An empirical portfolio loss distribution obtained by Monte Carlo simulation. The histogram is based on a portfolio of 2,000 corporate loans.
Fitting of loss distribution

The two statistical measures about the credit portfolio are

1. mean, or called the portfolio expected loss;

2. standard deviation, or called the portfolio unexpected loss.

At the simplest level, we approximate the loss distribution of the original portfolio by a beta distribution through matching the first and second moments of the portfolio loss distribution.

The risk quantiles of the original portfolio can be approximated by the respective quantities of the approximating random variable $X$. The price for such convenience of fitting is model risk.

Reservation
A beta distribution with only two degrees of freedom is perhaps insufficient to give an adequate description of the tail events in the loss distribution.
**Beta distribution**

The density function of a beta distribution is

\[
f(x, \alpha, \beta) = \begin{cases} 
\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, & 0 < x < 1 \\
0 & \text{otherwise}
\end{cases} \quad \alpha > 0, \beta > 0,
\]

where \(\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1} \, dx\).

Mean \(\mu = \frac{\alpha}{\alpha+\beta}\) and variance \(\sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}\).
Monte Carlo simulation of loss distribution of a portfolio

1. \textit{Estimate default and losses}
   Assign risk ratings to loss facilities and determine their default probability. Also assign LGD and $\sigma_{LGD}$.

2. \textit{Estimate asset correlation between obligors}
   Determine pairwise asset correlation whenever possible or assign obligors to industry groupings, then determine industry pair correlation.

3. \textit{Generater random loss given default}
   Determine the stochastic loss given default.
4. *Generate correlated default events*
   - Decomposition of covariance matrix.
   - Simulate default point.

5. *Loss calculation*
   Calculate facility loss for each scenario and obtain the portfolio loss.

6. *Loss distribution*
   Construct the simulated portfolio loss distribution.
**Generation of correlated default events**

1. Generate a set of random numbers drawn from a standard normal distribution.

2. Perform a decomposition (Cholesky, SVD or eigenvalue) on the asset correlation matrix to transform the independent set of random numbers (stored in the vector $e$) into a set of correlated asset values (stored in the vector $e'$). Here, the transformation matrix is $M$, where

$$e' = Me.$$ 

The covariance matrix $\Sigma$ and $M$ are related by

$$M^T M = \Sigma.$$
Calculation of the default point

We assume the credit indexes to be standard normal random variables.

- The default point threshold, $DP$, of the $i^{th}$ obligor can be defined as default probability $= N(DP)$.

- For example, $N(-2.5) = 0.0062 = 0.62\%$; that is, the default point threshold equals $-2.5$ when the default probability is $0.62\%$. For obligor $i$, we take
  
  \[
  \text{default if } e_i' < DP_i \text{; no default if } e_i' \geq DP_i.
  \]
Generate loss given default

The LGD is a stochastic variable with an unknown distribution.

A typical example may be

<table>
<thead>
<tr>
<th>Recovery rate (%)</th>
<th>LGD (%)</th>
<th>$\sigma_{LGD}$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>secured</td>
<td>65</td>
<td>35</td>
</tr>
<tr>
<td>unsecured</td>
<td>50</td>
<td>50</td>
</tr>
</tbody>
</table>

$$LGD_i = LGD_s + f_i \times \sigma_{LGD}^s$$

where $f_i$ is drawn from a uniform distribution whose range is selected so that the resulting LGD has a standard deviation that is consistent with historical observation.
Calculation of loss

Summing all the simulated losses from one single scenario

\[
\text{Loss} = \sum_{\text{Obligors in default}} \text{Adjusted exposure}_i \times \text{LGD}
\]

Simulated loss distribution

The simulated loss distribution is obtained by repeating the above process sufficiently number of times.
Features of portfolio risk

- The variability of default risk within a portfolio is substantial.
- The correlation between default risks is generally low.
- The default risk itself is dynamic and subject to large fluctuations.
- Default risks can be effectively managed through diversification.
- Within a well diversified portfolio, the loss behavior is characterized by lower than expected default credit losses for much of the time but very large losses which are incurred infrequently.
2.2 VaR (value-at-risk), expected shortfall and coherent risk measure

In simple terminology, the value-at-risk measure can be translated as “I am $X \text{ percent}$ certain there will not be a loss of more than $V$ dollars in the next $N$ days.”

- The variable $V$ is the VaR of the portfolio. It is a function of (i) time horizon ($N$ days); (ii) confidence level ($X\%$).

- It is the loss level over $N$ days that has a probability of only $(100 - X)\%$ of being exceeded.

- Bank regulators require banks to calculate VaR for market risk with $N = 10$ and $X = 99$.

- In essence, it asks the simple question: “How bad can things get?”.
Calculation of VaR from the probability distribution of the change in the portfolio value; confidence level is $X\%$. Gains in portfolio value are positive; losses are negative.
Alternative situation where VaR is the same, but the potential loss is larger.
Formal definition

VaR is defined for a probability measure $P$ and some confidence level $\alpha$ as the $\alpha$-quantile of a loss random variable $X$

$$\text{var}_\alpha(X) = \inf\{x \geq 0 | P[X \leq x] \geq \alpha\}.$$ 

Banks should hold some capital cushion against unexpected losses. Using $UL$ is not sufficient since there might be a significant likelihood that losses will exceed portfolio’s $EL$ by more than one standard deviation of the portfolio loss. That is, $X \geq EL + UL$. 
Example 1

Suppose that the gain from a portfolio during six months is normally distributed with a mean of $2 million and a standard deviation of $10 million.

Recall the cumulative normal distribution:

\[ N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt, \]

and \( N(-2.33) = 0.01 = 1\% \).

- From the properties of the normal distribution, the one-percentile point of this distribution is \( 2 - 2.33 \times 10 \), or -$21.3 million.

- The VaR for the portfolio with a time horizon of six months and confidence level of 99% is therefore $21.3 million.
Example 2

Suppose that for a one-year project all outcomes between a loss $50 million and a gain of $50 million are considered equally likely.

- The loss from the project has a uniform distribution extending from $-50$ million to $+50$ million. There is a 1\% chance that there will be a loss greater than $49$ million.

- The VaR with a one-year time horizon and a 99\% confidence level is therefore $49$ million.
Cumulative loss distribution for Example 3
Example 3

A one-year project has a 98% chance of leading to a gain of $2 million, a 1.5% chance of leading to a loss of $4 million and a 0.5% chance of leading to a loss of $10 million. The point on this cumulative distribution that corresponds to a cumulative probability of 99% and a one-year time horizon is $4 million.

- VaR with a confidence level of 99% and a one-year time horizon is $4 million.

Suppose that we are interested in calculating a VaR using a confidence level of 99.5%. The figure shows that all losses between $4 and $10 million have a cumulative probability of 99.5% of not being exceeded. There is a probability of 0.5% of a loss equal to $V$ dollars being exceeded for this range of values of $V$.

- VaR is therefore not uniquely defined. A sensible convention in this type of situation is to set VaR equal to the midpoint of the range of possible VaR value; so VaR would be set equal to $7 million.
Expected Shortfall

The *tail conditional expectation*, and *expected shortfall*, with respect to a confidence level $\alpha$ is defined as

$$TCE_\alpha(X) = \mathbb{E}[X | X \geq VaR_\alpha(X)].$$

From an insurance point of view, expected shortfall is a very reasonable measure:

Define by $c = VaR_\alpha(X)$, a *critical loss threshold* corresponding to some confidence level $\alpha$, expected shortfall capital provides a cushion against the mean value of losses exceeding the critical threshold $c$.

TCE focusses on the expected loss in the tail, starting at $c$, of the portfolio’s loss distribution.
Expected Shortfall $\mathbb{E}[X | X \geq VaR_\alpha(X)]$. 
Coherent risk measures

Denote by $L(\Omega, P)$ the space of bounded real random variables, defined on a probability space $(\Omega, P)$. A mapping $\gamma : L(\Omega, P) \rightarrow \mathbb{R}$ is called a coherent measure if the following properties hold:

1. monotonicity

$$\forall X, Y \in L, \text{with } X \leq Y, \quad \gamma(X) \leq \gamma(Y)$$

2. translation invariance

$$\forall x \in \mathbb{R}, \forall X \in L, \quad \gamma(X + x) = \gamma(X) + x.$$ This would imply $\gamma(X - \gamma(X)) = 0$ for every loss $X \in L$.

3. positive homogeneity

$$\forall \lambda > 0, \forall X \in L, \quad \gamma(\lambda X) = \lambda \gamma(X)$$

4. subadditivity

$$\forall X, Y \in L, \quad \gamma(X + Y) \leq \gamma(X) + \gamma(Y)$$
Financial interpretation of the four properties

1. **Monotonicity**: If a portfolio produces a worse result than another portfolio for every state of the world, its risk measure should be greater.

2. **Translation invariance**: If an amount of cash $K$ is added to a portfolio, its risk measure should go down by $K$.

3. **Positive homogeneity**: Changing the size of a portfolio by a positive factor $\lambda$, while keeping the relative amounts of different items in the portfolio the same, should result in the risk measure being multiplied by $\lambda$.

4. **Subadditivity**: The risk measure for two portfolio after they have been merged should be no greater than the sum of their risk measures before they were merged.

VaR satisfies the first three conditions. However, it does not always satisfy the fourth one.
Example 4

Suppose each of two independent projects has a probability of 0.02 of loss of $10 million and a probability of 0.98 of a loss of $1 million during a one-year period.

- The one-year, 97.5% VaR for each project is $1 million. When the projects are put in the same portfolio, there is a $0.02 \times 0.02 = 0.0004$ probability of a loss of $20$ million, a $2 \times 0.02 \times 0.98 = 0.0392$ probability of a loss of $11$ million, and a $0.98 \times 0.98 = 0.9604$ probability of a loss of $2$ million. The one-year 97.5% VaR for the portfolio is $11$ million.

- The total of the VaRs of the projects considered separately is $2$ million. The VaR of the portfolio is therefore greater than the sum of the VaRs of the projects by $9$ million. This violates the subadditivity condition.
Example 5

A bank had two $10 million one-year loans, each of which has a 1.25% chance of defaulting. If a default occurs, all losses between 0% and 100% of the principal are equally likely. If the loan does not default, a profit of $0.2 million is made. To simplify matters, we suppose that if one loan defaults it is certain that the other loan will not default.

1. Consider first a single loan. This has a 1.25% chance of defaulting. When a default occurs, the loss experienced is evenly distributed between zero and $10 million. This means that there is a 1.25% chance that a loss greater than zero will be incurred; there is a 0.625% chance that a loss greater than $10 million. The loss level that has a probability of 1% of being exceeded is $2 million.
• Conditional on a loss being made, there is an 80% (0.8) chance that the loss will be greater than $2 million. Because the probability of a loss is 1.25% (0.0125), the unconditional probability of a loss greater than $2 million is $0.8 \times 0.0125 = 0.01$ or 1%.

• The one-year, 99% VaR is therefore $2 million.

• Consider next the portfolio of two loans. Each loan defaults 1.25% of the time and they never default together. There is therefore a 2.5% probability that a default will occur. As before, the loss experienced on a defaulting loan is evenly distributed between zero and $10 million.
• The VaR in this case turns out to be $5.8 million. This is because there is a 2.5% (0.025) chance of one of the loans defaulting and conditional on this event there is an 40% (0.4) chance that the loss on the loan that defaults is greater than $6 million. The unconditional probability of a loss from a default being greater than $6 million is therefore $0.4 \times 0.025 = 0.01$ or 1%.

• In the event that one loan defaults, a profit of $0.2 million is made on the other loan, showing that the one-year 99% VaR is $5.8 million.

The total VaR of the loans considered separately is $2 + 2 = $4 million. The total VaR after they have been combined in the portfolio is $1.8 million greater at $5.8 million. This shows that the subadditivity condition is violated.
**Example 6 – Expected shortfall**

In the above example, the VaR for one of the projects considered on its own is $1 million. To calculate the expected shortfall for a 97.5% confidence level we note that, of the 2.5% tail of the loss distribution, 2% corresponds to a loss is $10 million and a 0.5% to a loss of $1 million.

- Note that the other 97.5% of the distribution also corresponds to a loss of $1 million.

- Conditional that we are in the 2.5% tail of the loss distribution, there is therefore an 80% probability of a loss of $10 million and a 20% probability of a loss of $1 million. The expected loss is $8.1 million.
• When the two projects are combined, of the 2.5% tail of the loss distribution, 0.04% corresponds to a loss of $20 million and 2.46% corresponds to a loss of $11 million. Conditional that we are in the 2.5% tail of the loss distribution, the expected loss is therefore 
\[(0.04/2.5) \times 20 + (2.46/2.5) \times 11,\] or $11.144 million.

• Since $8.1 + 8.1 > 11.144$, the expected shortfall measure does satisfy the subadditivity condition for this example.
We showed that the VaR for a single loan is $2 million. The expected shortfall from a single loan when the time horizon is one year and the confidence level is 99% is therefore the expected loss on the loan conditional on a loss greater than $2 million is halfway between $2 million and $10 million, or $6 million.

The VaR for a portfolio consisting of the two loans was calculated in Example 5 as $5.8 million. The expected shortfall from the portfolio is therefore the expected loss on the portfolio conditional on the loss being greater than $5.8 million.
• When one loan defaults, the other (by assumption) does not and outcomes are uniformly distributed between a gain of $0.2 million and a loss of $9.8 million.

• The expected loss, given that we are in the part of the distribution between $5.8 million and $9.8 million, is $7.8 million. This is therefore the expected shortfall of the portfolio.

• Because $7.8 million is less than $2 \times$ $6$ million, the expected shortfall measure does satisfy the subadditivity condition.

The subadditivity condition is not of purely theoretical interest. It is not uncommon for a bank to find that, when it combines two portfolios (e.g., its equity portfolio and its fixed income portfolio), the total VaR goes up.
Spectral risk measure

A risk measure can be characterized by the weights it assigns to quantiles of the loss distribution.

- VaR gives a 100% weighting to the $X$th quantile and zero to other quantiles.

- Expected shortfall gives equal weight to all quantiles greater than the $X$th quantile and zero weight to all quantiles below $X$th quantile.

We can define what is known as a **spectral risk measure** by making other assumptions about the weights assigned to quantiles. A general result is that a spectral risk measure is coherent (i.e., it satisfies the subadditivity condition) if the weight assigned to the $q$th quantile of the loss distribution is a nondecreasing function of $q$. Expected shortfall satisfies this condition.
Calculation of VaR using historical simulation

Suppose that VaR is to be calculated for a portfolio using a 1-day time horizon, a 99% confidence level, and 501 days of data.

- The first step is to identify the market variables affecting the portfolio. These are typically exchange rates, equity prices, interest rates, etc.

- Data is then collected on the movements in these market variables over the most recent 501 days. This provides 500 alternative scenarios for what can happen between today and tomorrow.

- Scenario 1 is where the percentage changes in the values of all variables are the same as they were between Day 0 and Day 1, scenario 2 is where they are the same as they were between Day 1 and Day 2, and so on.
Data for VaR historical simulation calculation.

<table>
<thead>
<tr>
<th>Day</th>
<th>Market variable 1</th>
<th>Market variable 2</th>
<th>...</th>
<th>Market variable n</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>20.33</td>
<td>0.1132</td>
<td></td>
<td>65.37</td>
</tr>
<tr>
<td>1</td>
<td>20.78</td>
<td>0.1159</td>
<td></td>
<td>64.91</td>
</tr>
<tr>
<td>2</td>
<td>21.44</td>
<td>0.1162</td>
<td></td>
<td>65.02</td>
</tr>
<tr>
<td>3</td>
<td>20.97</td>
<td>0.1184</td>
<td></td>
<td>64.90</td>
</tr>
<tr>
<td>...</td>
<td>:</td>
<td>:</td>
<td>...</td>
<td>:</td>
</tr>
<tr>
<td>498</td>
<td>25.72</td>
<td>0.1312</td>
<td></td>
<td>62.22</td>
</tr>
<tr>
<td>499</td>
<td>25.75</td>
<td>0.1323</td>
<td></td>
<td>61.99</td>
</tr>
<tr>
<td>500</td>
<td>25.85</td>
<td>0.1343</td>
<td></td>
<td>62.10</td>
</tr>
</tbody>
</table>
Define $v_i$ as the value of a market variable on Day $i$ and suppose that today is Day $m$. The $i$th scenario assumes that the value of the market variable tomorrow will be $v_m \frac{v_i}{v_{i-1}}$.

For the first variable, the value today, $v_{500}$, is 25.85. Also $v_0 = 20.33$ and $v_1 = 20.78$. It follows that the value of the first market variable in the first scenario is $25.85 \times \frac{20.78}{20.33} = 26.42$. 
Scenarios generated for tomorrow (Day 501) using data in the last table.

<table>
<thead>
<tr>
<th>Scenario number</th>
<th>Market variable 1</th>
<th>Market variable 2</th>
<th>...</th>
<th>Market variable n</th>
<th>Portfolio value ($ millions)</th>
<th>Change in value ($ millions)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>26.42</td>
<td>0.1375</td>
<td>...</td>
<td></td>
<td>61.66</td>
<td>23.71</td>
</tr>
<tr>
<td>2</td>
<td>26.67</td>
<td>0.1346</td>
<td>...</td>
<td></td>
<td>62.21</td>
<td>23.12</td>
</tr>
<tr>
<td>3</td>
<td>25.28</td>
<td>0.1368</td>
<td>...</td>
<td></td>
<td>61.99</td>
<td>22.94</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>499</td>
<td>25.88</td>
<td>0.1354</td>
<td>...</td>
<td></td>
<td>61.87</td>
<td>23.63</td>
</tr>
<tr>
<td>500</td>
<td>25.95</td>
<td>0.1363</td>
<td>...</td>
<td></td>
<td>62.21</td>
<td>22.87</td>
</tr>
</tbody>
</table>
How to estimate the 1-percentile point of the distribution of changes in the portfolio value?

Since there are a total of 500 scenarios, we can estimate this as the fifth-worst number in the final column of the table. The $N$-day VaR for a 99% confidence level is calculated as $\sqrt{N}$ times the 1-day VaR.
2.3 Economic capital and risk-adjusted return on capital

Definition of economic capital (also called risk capital)

- This is the amount of capital a financial institution needs in order to absorb losses over a certain time horizon (usually one year) with a certain confidence level.

- The confidence level depends on financial institutions’ objectives. Corporations rated AA have a one-year probability of default less than 0.1%. This suggests that the confidence level should be 99.9%, or even higher.
Take a *target level of statistical confidence* into account. For a given level of confidence $\alpha$,

\[
q_\alpha = \alpha - \text{quantile of } \tilde{L}_p \\
= \inf\{q > 0 | P[\tilde{L}_p \leq q] \geq \alpha \}.
\]

$q_\alpha$ is sometimes called the credit VaR. Define

\[
EC = \text{economic capital} = q_\alpha - EL_P.
\]

Say, $\alpha = 99.98\%$, this would mean $EC_\alpha$ will be sufficient to cover unexpected losses in 9,998 out of 10,000 years, assuming a planning horizon of one year.
Why reducing the quantile $q_\alpha$ by the $EL$? This is the usual practice of decomposing the total risk capital into (i) expected loss (ii) cushion against unexpected losses.

Note that $EL$ charges are *portfolio independent* (diversification has no impact) while $EC$ charges are *portfolio dependent*. New loans may add a lot or little risk contributions (risk concentration).
When lending in a certain region of the world a AA-rated bank estimates its losses as 1% of outstanding loans per year on average. The 99.9% worst-case loss (i.e., the loss exceeded only 0.1% of the time) is estimated as 5% of outstanding loans. The economic capital required per $100 of loans made is therefore $4.0 (this is difference between the 99.9% worst-case loss and the expected loss).
Bottom-up approach

The loss distributions are estimated for different types of risk and different business units and then aggregated.

- The first step in the aggregation is to calculate the probability distributions for losses by risk type or losses by business unit.

- A final aggregation gives a probability distribution of total losses for the whole financial institution.

Operational risk as "the risk of loss resulting from inadequate or failed internal processes, people, and systems or from external events". Operational risk includes model risk and legal risk, but it does not include risk arising from strategic decisions or reputational risk. This type of risk is collectively referred as business risk. Regulatory capital is not required for business risk under Basel II, but some banks do assess economic capital for business risk.
Aggregating economic capital

Suppose a financial institution has calculated market, credit, operational, and (possibly) business risk loss distributions for a number of different business units.

**Question**

How to aggregate the loss distributions to calculate a total economic capital for the whole enterprise?

According to Basel II,

\[
E_{\text{total}} = \sum_{i=1}^{n} E_i.
\]

This is based on the assumption of perfect correlation between the different types of risks – too conservative.
Hybrid approach

\[ E_{\text{total}} = \sqrt{ \sum_{i=1}^{n} \sum_{j=1}^{n} E_i E_j \rho_{ij} }, \]

where \( \rho_{ij} \) is the correlation between risk \( i \) and risk \( j \).

- When the distributions are normal, this approach is exact.

Economic capital estimates

<table>
<thead>
<tr>
<th>Type of risk</th>
<th>Business Unit 1</th>
<th>Business Unit 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Market risk</td>
<td>30</td>
<td>40</td>
</tr>
<tr>
<td>Credit risk</td>
<td>70</td>
<td>80</td>
</tr>
<tr>
<td>Operational risk</td>
<td>30</td>
<td>90</td>
</tr>
</tbody>
</table>
Correlations between losses:
MR, CR, and OR refer to market risk, credit risk, and operational risk; 1 and 2 refer to the business units.

<table>
<thead>
<tr>
<th></th>
<th>MR-1</th>
<th>CR-1</th>
<th>OR-1</th>
<th>MR-2</th>
<th>CR-2</th>
<th>OR-2</th>
</tr>
</thead>
<tbody>
<tr>
<td>MR-1</td>
<td>1.0</td>
<td>0.5</td>
<td>0.2</td>
<td>0.4</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>CR-1</td>
<td>0.5</td>
<td>1.0</td>
<td>0.2</td>
<td>0.0</td>
<td>0.6</td>
<td>0.0</td>
</tr>
<tr>
<td>OR-1</td>
<td>0.2</td>
<td>0.2</td>
<td>1.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>MR-2</td>
<td>0.4</td>
<td>0.0</td>
<td>0.0</td>
<td>1.0</td>
<td>0.5</td>
<td>0.2</td>
</tr>
<tr>
<td>CR-2</td>
<td>0.0</td>
<td>0.6</td>
<td>0.0</td>
<td>0.5</td>
<td>1.0</td>
<td>0.2</td>
</tr>
<tr>
<td>OR-2</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.2</td>
<td>0.2</td>
<td>1.0</td>
</tr>
</tbody>
</table>

- Correlation between 2 different risk types in 2 different business units = 0.
- Correlation between market risks across business units = 0.4.
- Correlation between credit risks across business units = 0.6.
- Correlation between operational risks across business units = 0.
• Total market risk economic capital is

$$\sqrt{30^2 + 40^2 + 2 \times 0.4 \times 30 \times 40} = 58.8.$$ 

• Total credit risk economic capital is

$$\sqrt{70^2 + 80^2 + 2 \times 0.6 \times 70 \times 80} = 134.2.$$ 

• Total operational risk economic capital is

$$\sqrt{30^2 + 90^2} = 94.9.$$ 

• Total economic capital for Business Unit 1 is

$$\sqrt{30^2 + 70^2 + 30^2 + 2 \times 0.5 \times 30 \times 70 + 2 \times 0.2 \times 30 \times 30 + 2 \times 0.2 \times 70 \times 30} = 100.0.$$ 

• Total economic capital for Business Unit 2 is

$$\sqrt{40^2 + 80^2 + 90^2 + 2 \times 0.5 \times 40 \times 80 + 2 \times 0.2 \times 40 \times 90 + 2 \times 0.2 \times 80 \times 90} = 153.7.$$
The total enterprise-wide economic capital is the square root of
\[30^2 + 40^2 + 70^2 + 80^2 + 30^2 + 90^2 + 2 \times 0.4 \times 30 \times 40 + 2 \times 0.5 \times 30 \times 70 + 2 \times 0.2 \times 30 \times 30 + 2 \times 0.5 \times 40 \times 80 + 2 \times 0.2 \times 40 \times 90 + 2 \times 0.6 \times 70 \times 80 + 2 \times 0.2 \times 70 \times 30 + 2 \times 0.2 \times 80 \times 90\]
which is 203.224.

There are significant diversification benefits. The sum of the economic capital estimates for market, credit, and operational risk is
\[58.8 + 134.2 + 94.9 = 287.9,\]
and the sum of the economic capital estimates for two business units is
\[100 + 153.7 = 253.7.\]
Both of these are greater than the total economic capital estimate of 203.2.
**Risk-adjusted return on capital (RAROC)**

Risk-adjusted performance measurement (RAPM) has become an important part of how business units are assessed. There are many different approaches, but all have one thing in common. They compare return with capital employed in a way that incorporates an adjustment for risk.

The most common approach is to compare expected return with economic capital. The formula for RAROC is

\[
RAROC = \frac{\text{Revenues} - \text{Costs} - \text{Expected losses}}{\text{Economic capital}}.
\]

The numerator may be calculated on a pre-tax or post-tax basis. Sometimes, a risk-free rate of return on the economic capital is calculated and added to the numerator.
Example

When lending in a certain region of the world, a AA-rated bank estimates its losses as 1% of outstanding loans per year on average. The 99.9% worst-case loss (i.e., the loss exceeded only 0.1% of the time) is 5% of outstanding loans.

- The economic capital required per $100 of loans is $4, which is the difference between the 99.9% worst-case loss and the expected loss. This ignores diversification benefits that would in practice be allocated to the lending unit.

- The spread between the cost of funds and the interest charged is 2.5%. Subtracting from this the expected loan loss of 1%, the expected contribution per $100 of loans is $1.50.
• Assuming that the lending unit’s administrative costs total 0.7% of the amount loaned, the expected profit is reduced to $0.80 per $100 in the loan portfolio. RAROC is therefore

\[
\frac{0.80}{4} = 20\%.
\]

• An alternative calculation would add the interest on the economic capital to the numerator. Suppose that the risk-free interest rate is 2%. Then \(0.02 \times 4 = 0.08\) is added to the numerator, so that RAROC becomes

\[
\frac{0.88}{4} = 22\%.
\]
Deutsche Bank’s economic capital and regulatory capital
(millions of euros).

<table>
<thead>
<tr>
<th>Risk Category</th>
<th>Amount (in millions of euros)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Credit risk</td>
<td>8,506</td>
</tr>
<tr>
<td>Market risk</td>
<td>3,481</td>
</tr>
<tr>
<td>Operational risk</td>
<td>3,974</td>
</tr>
<tr>
<td>Diversification benefit across credit, market, and operational risk</td>
<td>(2,651)</td>
</tr>
<tr>
<td>Business risk</td>
<td>301</td>
</tr>
<tr>
<td><strong>Total economic capital</strong></td>
<td><strong>13,611</strong></td>
</tr>
</tbody>
</table>

Allocation of Deutsche Bank’s economic capital (millions of euros).

<table>
<thead>
<tr>
<th>Business Segment</th>
<th>Amount (in millions of euros)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Corporate banking and securities</td>
<td>10,533</td>
</tr>
<tr>
<td>Global transaction banking</td>
<td>430</td>
</tr>
<tr>
<td>Asset and wealth management</td>
<td>871</td>
</tr>
<tr>
<td>Private business clients</td>
<td>1,566</td>
</tr>
<tr>
<td>Corporate investments</td>
<td>207</td>
</tr>
<tr>
<td>Consolidation and adjustments</td>
<td>5</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>13,611</strong></td>
</tr>
</tbody>
</table>
RAROC can be calculated *ex-ante* (before the start of the year) or *ex-post* (after the end of the year). *Ex-ante* calculations are based on estimates of expected profit. *Ex-post* calculations are based on actual profit results. *Ex-ante* calculations are typically used to decide whether a particular business unit should be expanded or contracted. *Ex-post* calculations are typically used for performance evaluation and bonus calculations.

It is usually not appropriate to base a decision to expand or contract a particular business unit on an *ex-post* analysis (although there is a natural temptation to do this). It may be that results were bad for the most recent year because credit losses were much larger than average or because there was an unexpectedly large operational risk loss. Key strategic decisions should be based on expected long-term results.