1. The spread between the yield on a 3-year corporate bond and the yield on a similar risk-free bond is 50 basis points. The recovery rate is 30%. Estimate the average default intensity per year over the 3-year period. Next, suppose that the spread between the yield on a 5-year bond issued by the same company and the yield on a similar risk-free bond is 60 basis points. Assume the same recovery rate of 30%. Estimate the average default intensity per year over the 5-year period. What do your results indicate about the average default intensity in years 4 and 5?

2. A company has issued 3-year and 5-year bonds with a coupon of 4% per annum payable annually. The yields on the bonds (expressed with continuous compounding) are 4.5% and 4.75%, respectively. Risk-free rates are 3.5% with continuous compounding for all maturities. The recovery rate is 40%. Defaults can take place halfway through each year. The default rates per year are $Q_1$ for years 1 to 3 and $Q_2$ for years 4 and 5. Estimate $Q_1$ and $Q_2$.

3. Assume that a regular CDS pays $(1 - \text{recovery rate})$ of the notional while a binary CDS pays the full notional (independent of the recovery rate) upon default of the reference asset. Find the ratio of the corresponding CDS spread for the regular CDS for the two cases where the recovery rate is either 10% or 50%. Is the CDS spread of a binary CDS dependent on the recovery rate? Give an explanation to your answer.

4. Suppose that the risk-free zero-curve is flat at 6% per annum with continuous compounding. Consider a four-year plain vanilla credit default swap with annual payments on an underlying risky bond. Suppose that the recovery rate is 20% and the compensation payment is $(1 - \text{recovery rate})$ times notional. The forward probabilities of default of the bond during the first year, the second year, the third year, and the fourth year are assumed to be 0.01, 0.015, 0.02 and 0.025, respectively. Assume that the credit premium is paid by the Protection Seller at the end of each year (if the bond survives until then), and accrual premium from the last premium payment date to the time of default is paid when the bond defaults. If default does occur, it would take place either in mid-year or the end of the year. What is the credit default swap spread? What would be the credit default spread if the instrument were a binary credit default swap?

**Hint:** The probability of survival until the end of the second year = $100\% - (1\% + 1.5\%) = 97.5\%$, and the probability of survival until Year 1.5 is $100\% - (1.0\% + 0.5 \times 1.5\%) = 98.25\%$, and similar calculations for other survival probabilities can be performed. There are 4 swap premium payments if the bond survives throughout the life of the CDS. However, there are 8 possible dates at which the bond may default.

5. Suppose that:

   (a) The yield on a 5-year risk-free bond is 7%.
   (b) The yield on a 5-year corporate bond issued by company X is 9.5%.
   (c) A 5-year credit default swap providing insurance against the default of company X costs premium rate of 150 basis points per year.
What arbitrage opportunity is there in this situation? What arbitrage opportunity would there be if the credit default spread were 300 basis points instead of 150 basis points? Give two reasons why arbitrage opportunities such as those you identify may not be extracted fully by an arbitrageur.

6. Suppose we choose the mixture distribution in the Bernoulli mixture model to be the beta distribution whose density function is given by

\[
f(\tilde{p}) = \frac{1}{\beta(a, b)} \tilde{p}^{a-1}(1 - \tilde{p})^{b-1}, \quad a, b > 0, \quad 0 < \tilde{p} < 1,
\]

where the beta function \( \beta(a, b) \) is defined by

\[
\beta(a, b) = \int_0^1 x^{a-1}(1 - x)^{b-1} dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)}, \quad \Gamma(a) = \int_0^\infty e^{-x}x^{a-1} dx.
\]

Recall that the mean and variance of the beta distribution are given by

\[
\text{mean} = \frac{a}{a + b} \quad \text{and} \quad \text{variance} = \frac{ab}{(a + b)^2(a + b + 1)}.
\]

(a) Based on the Bernoulli mixture model, show that the probability of \( k \) defaults out of \( m \) (\( m \geq k \)) obligors is given by

\[
\mathbb{P}[M = k] = C_m^k \int_0^1 \tilde{p}^k(1 - \tilde{p})^{m-k} f(\tilde{p}) d\tilde{p} = C_m^k \frac{\beta(a + k, b + m - k)}{\beta(a, b)}.
\]

(b) Find the corresponding default-event correlation coefficient \( \rho(X_i, X_j) \). Look at various combinations of the two parameters for which \( \frac{a}{a + b} = \bar{p} \) for a given level of expected default probability \( \bar{p} \). Assuming \( \bar{p} \) to be fixed, as \( a \) increases, do we have higher or lower default-event correlation?

7. Consider a portfolio of \( m \) risky bonds (of equal face value) with uniform default probability \( p \). Let \( L_i \) denote the default event indicator of bond \( i \), where \( L_i \sim B(1; p) \). Let \( \rho \) be the uniform correlation coefficient between pairwise defaults of any two bonds. In this problem, we use the Binomial Expansion Technique where the defaults in a comparison portfolio are assumed to be independent. By matching the second order moment of the original portfolio and the comparison portfolio consisting of \( n(\rho) \) independent bonds, show that the diversification score \( n(\rho) \) is given

\[
n(\rho) = \frac{m}{1 + \rho(m - 1)}.
\]

Show that the above diversification score is bounded from above by \( 1/\rho \).

8. This problem is an extension of the mixture approach to the Poisson model of default. Consider the Poisson mixture model where the loss statistics is a random vector \( \mathbf{L} = (L_1, \cdots, L_m) \) of Poisson random variables \( L_i \sim \text{Pois}(\Lambda_i) \), where \( \Lambda = (\Lambda_1, \cdots, \Lambda_m) \) is a random vector with some distribution function \( \mathbf{F} \) with support in \( [0, \infty)^m \). Note that the default probability of obligor \( i \) is given by \( p_i = \mathbb{P}[L_i = 1] \). We assume that conditional on a realization \( \mathbf{\lambda} = (\lambda_1, \cdots, \lambda_m) \) of \( \Lambda \), the variables \( L_1, L_2, \cdots, L_m \) are independent:

\[
L_i|\Lambda_i = \lambda_i \sim \text{Pois}(\lambda_i), \quad (L_i|\Lambda = \mathbf{\lambda})_{i=1,\cdots,m} \text{ are independent}.
\]
The (unconditional) joint distribution of the variables $L_i$ is given by

$$
P[L_1 = \ell_1, \ldots, L_m = \ell_m] = \int_{[0,1)^m} e^{-\lambda_1 + \cdots + \lambda_m} \prod_{i=1}^m \frac{\lambda_i^{\ell_i}}{\ell_i!} d\mathcal{F}(\lambda_1, \ldots, \lambda_m),
$$

where $\ell_i \in \{0,1\}$. Show that the correlation coefficient between pairwise default events is given by

$$
\rho(L_i, L_j) = \frac{\text{cov}(\Lambda_i, \Lambda_j)}{\sqrt{\text{var}(\Lambda_i) + E[\Lambda_i] \sqrt{\text{var}(\Lambda_j) + E[\Lambda_j]}}).
$$

Hint:

$$
P[L_i = \ell_i] = e^{-\lambda_i \ell_i^{\ell_i}} \frac{\ell_i!}{\ell_i!}, \quad \ell_i = 0 \text{ or } 1,$$

$$
E[L_i] = E[\Lambda_i],$$

$$
\text{var}(L_i) = \text{var}(E[L_i|\mathbf{A})] + E[\text{var}(L_i|\mathbf{A})] = \text{var}(\Lambda_i) + E[\Lambda_i].
$$

9. Suppose there are Poisson processes $N_1, N_2, \ldots, N_m$ and $N$ with respective intensity $\lambda_1$, $\lambda_2$, $\ldots$, $\lambda_m$ and $\lambda$. Here, $\lambda_i$ is the idiosyncratic shock intensity of firm $i$, $i = 1, 2, \ldots, m$, and $\lambda$ is the intensity of a macro-economic shock that affects all $m$ firms simultaneously. Define the default time $\tau_i$ of firm $i$ by

$$
\tau_i = \inf\{t \geq 0 : N_i(t) + N(t) > 0\}, \quad i = 1, 2, \ldots, m.
$$

The survival function of firm $i$ is given by

$$
S_i(t) = P[\tau_i > t] = e^{-(\lambda_i + \lambda)t}, \quad i = 1, 2, \ldots, m.
$$

(a) Find the joint survival function as defined by

$$
S(t_1, t_2, \ldots, t_m) = P[\tau_1 > t_1, \tau_2 > t_2, \ldots, \tau_m > t_m].
$$

(b) The exponential survival copula associated with the survival function $S(t_1, t_2, \ldots, t_m)$ can be found via

$$
C^*(u_1, u_2, \ldots, u_m) = S(S_1^{-1}(u_1), S_2^{-1}(u_2), \ldots, S_m^{-1}(u_m)).
$$

Fixing some $i, j \in \{1, 2, \ldots, m\}$ with $i \neq j$, show that the two-dimensional marginal copula is given by

$$
C^*(u_i, u_j) = C^*(1, \ldots, 1, u_i, 1, \ldots, 1, u_j, 1, \ldots, 1) = \min(u_j u_i^{1-\theta_i}, u_i u_j^{1-\theta_j}).
$$

Find $\theta_i$ and $\theta_j$ in terms of $\lambda_i$, $\lambda_j$ and $\lambda$. 

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