MATH 246, Fall 2001

Final Examination

Time allowed: 3 hours **Course Instructor:** *Prof. Y. K. Kwok*

[points]

1. (a) Suppose X and Y are a pair of discrete random variables. Show that

$$E[X] = \sum_{y_j} E[X|Y = y_j]P[Y = y_j],$$

where the summation is taken over all possible discrete values that can be assumed by Y.

Hint $\sum_{y_j} E[X|Y = y_j]P[Y = y_j] = \sum_{y_j}\sum_{x_i} x_i P[X = x_i|Y = y_j]P[Y = y_j]$, where the inner summation is taken over all possible discrete values that can be assumed by X. **[3]**

(b) A urn contains m white balls and n black balls. One ball at a time is randomly withdrawn without replacement until a white ball is drawn, then the drawing terminates. Let X denote the number of black balls withdrawn until the drawing experiment stops. We write $M(m,n)$ to denote the expectation $E[X]$, with dependence on m and n. Let Y denote the discrete random variable defined by

$$Y = \begin{cases} 1 & \text{if the first ball drawn is white} \\ 0 & \text{if the first ball drawn is black} \end{cases}.$$

(i) Using the result in part (a), show that

$$M(m,n) = E[X|Y = 1]P[Y = 1] + E[X|Y = 0]P[Y = 0].$$

[1]

(ii) Explain why $M(m,n)$ satisfies the following recursive relation:

$$M(m,n) = \frac{n}{m+n}[1 + M(m,n-1)].$$

[3]

(iii) Explain why $M(m,1) = \dfrac{1}{m+1}$, and use the recursive relation to show

$$M(m,2) = \frac{2}{m+1} \quad \text{and} \quad M(m,3) = \frac{3}{m+1}.$$

[2]

(iv) Using (ii) and (iii), find the solution for $M(m,n)$. **[1]**

2. Let X and Y be random variables that take on values from the set $\{-1, 0, 1\}$. Find a joint probability mass function assignment for which X and Y are *dependent*, but for which X^2 and Y^2 are *independent*. **[5]**

3. A point is chosen uniformly at random from the triangle that is formed by joining the three points $(-1, 0), (0, 1)$ and $(1, 0)$. Let X and Y be the co-ordinates of a randomly chosen point.
   (a) What is the joint density of X and Y? Also, compute $f_X(x|y)$ and $f_Y(y|x)$. **[4]**
   (b) Calculate the expected value of X and Y. Specify the domain of definition of each of these joint probability function and conditional probability functions. **[4]**
   (c) Find the correlation coefficient between X and Y. Are X and Y independent? **[3]**

4. Let $N(t), t \geq 0$ be a Poisson process with parameter $\lambda > 0$.
   (a) Describe the independence increments property and stationary increments property of a Poisson process. **[2]**
   (b) If $t_2 > t_1$, find the conditional expectation of $N(t_2)$, given $N(t_1) = k$. That is, find $E[N(t_2)|N(t_1) = k]$. **[4]**
   (c) Find the auto-covariance function $C_N(t_1, t_2)$. **[4]**
   (d) Find $P[N(t - d) = j|N(t) = k]$, with $d > 0$. **[4]**

5. Given a two-state Markov chain X_n taking the values 1 and 2 with the state transition matrix

$$P = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \end{array}.$$

   (a) Find the two-step transition matrix $P(2)$. **[1]**
   (b) Find $P[X_3 = 2, X_2 = 1, X_1 = 2|X_0 = 1]$. **[2]**
   (c) Show by mathematical induction that

$$P^n = \frac{1}{2} \begin{pmatrix} 1 + \left(\frac{1}{3}\right)^n & 1 - \left(\frac{1}{3}\right)^n \\ 1 - \left(\frac{1}{3}\right)^n & 1 + \left(\frac{1}{3}\right)^n \end{pmatrix}.$$

   Hence, find $\lim_{n\to\infty} P^n$. Explain why the rows in the matrix $\lim_{n\to\infty} P^n$ are identical. **[4]**
   (d) Determine the steady state probability mass functions for the Markov chain. **[3]**

6. Customers arrive at a soft drink dispensing machine according to a Poisson process with rate λ. Let $N(t)$ be the number of customer arrivals up to time t. Suppose that each time a customer deposits money, the machine dispenses a random number of soft drinks. This random number is a Poisson random variable with parameter μ. The number of soft drinks dispensed upon each money deposit is assumed to be independent and identically distributed. Let $X(t)$ denote the number of drinks dispensed up to time t. Assume that the machine holds an infinite number of soft drinks.

   (a) Show that $P[N(t_1) = 1|N(t_2) = 1] = t_1/t_2, 0 < t_1 < t_2$. **[4]**

   Hint This result is equivalent to say that given that one arrival has occurred in the interval $[0, t_2]$, then the customer arrival time is uniformly distributed in the interval $[0, t_2]$. Explain how $P[N(t_1) = 1|N(t_2) = 1]$ is related to $P[N(t_1) = 1]$, $P[N(t_2) - N(t_1) = 0]$ and $P[N(t_2) = 1]$.

   (b) Find $P[X(t) = j|N(t) = n]$. **[3]**

   Hint Conditional on $N(t) = n$, $X(t)$ can be considered as a sum process of n iid Poisson random variables.

   (c) Find $P[X(t) = j]$. Express your answer in terms of the function

$$f(x; j) = \sum_{n=1}^{\infty} \frac{x^n n^j}{n!}$$

[3]

List of useful formulae

Binomial Random Variable

$S_X = \{0, 1, \ldots, n\}$ $\qquad$ $p_k = C_k^n p^k (1-p)^{n-k} \quad k = 0, 1, \ldots, n$

$E[X] = np \qquad \text{VAR}[X] = np(1-p)$

Poisson Random Variable

$S_X = \{0, 1, 2, \ldots\}$ $\qquad$ $p_k = \dfrac{\alpha^k}{k!} e^{-\alpha} \qquad k = 0, 1, \ldots$ and $\alpha > 0$

$E[X] = \alpha \qquad \text{VAR}[X] = \alpha$

Uniform Random Variable

$S_X = [a, b]$ $\qquad$ $f_X(x) = \dfrac{1}{b-a} \quad a \le x \le b$

$E[X] = \dfrac{a+b}{2} \qquad \text{VAR}[X] = \dfrac{(b-a)^2}{12}$

Exponential Random Variable

$S_X = [0, \infty)$ $\qquad$ $f_X(x) = \lambda e^{-\lambda x} \quad x \ge 0$ and $\lambda > 0$

$E[X] = \dfrac{1}{\lambda} \qquad \text{VAR}[X] = \dfrac{1}{\lambda^2}$

Marginal pdf's

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y')\, dy' \quad \text{and} \quad f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x', y)\, dx'$$

Independence of X and Y

X and Y are independent if and only if $f_{XY}(x, y) = f_X(x) f_Y(y)$, for all x, y

Conditional pdf of Y given $X = x$

$$f_Y(y|x) = \frac{f_{XY}(x, y)}{f_X(x)}$$

Conditional expectation of Y given $X = x$

Continuous $\qquad E[Y|x] = \displaystyle\int_{-\infty}^{\infty} y f_Y(y|x)\, dy$

discrete $\qquad F[Y|x] = \displaystyle\sum_{y_j} y_j P_Y(y_j|x)$

Correlation and covariance of two random variables

$\text{COV}(X, Y) = E[(X - m_X)(Y - m_Y)]$, where m_X and m_Y are $E[X]$ and $E[Y]$, respectively.

$\rho_{XY} = \dfrac{\text{COV}(X, Y)}{\sigma_X \sigma_Y} = \dfrac{E[XY] - E[X]E[Y]}{\sigma_X \sigma_Y}$

autocovariance $C_X(t_1, t_2)$ of a random process $X(t)$

$$C_X(t_1, t_2) = E[\{X(t_1) - m_X(t_1)\}\{X(t_2) - m_X(t_2)\}]$$

— *End* —