MATH 246 — Probability and Random Processes

Solution to Mid-term Test

Fall 2001 **Course Instructor:** *Prof. Y. K. Kwok*

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1. Define E_A, E_B and E_C be the event that the chips are from manufacturer A, B and C, respectively. Define $D = \{\text{a selected chip is defective}\}$. Assume E_A, E_B and E_C are equiprobable, i.e.,

$$P[E_A] = P[E_B] = P[E_C] = \frac{1}{3}.$$

Given $P[D|E_A] = 0.1$ then $P[D^C|E_A] = 0.9$
$P[D|E_B] = 0.2$ then $P[D^C|E_B] = 0.8$
$P[D|E_C] = 0.3$ then $P[D^C|E_C] = 0.7$.

Note that $\{E_A, E_B, E_C\}$ forms a partition of the sample space. By Bayes's theorem,

$$\begin{aligned}
\text{required probability} &= P[E_B^C|D^C] \\
&= 1 - P[E_B|D^C] \\
&= 1 - \frac{P[D^C|E_B]P[E_B]}{P[D^C|E_A]P[E_A] + P[D^C|E_B]P[E_B] + P[D^C|E_C]P[E_C]} \\
&= 1 - \frac{\frac{1}{3}(0.8)}{\frac{1}{3}(0.9) + \frac{1}{3}(0.8) + \frac{1}{3}(0.7)} \\
&= \frac{2}{3}.
\end{aligned}$$

3. (b) Given that the lifetime T is exponentially distributed, the average lifetime $= 10$ hours $\Rightarrow \lambda = \frac{1}{10}$. We then have $P[T > t] = e^{-\frac{1}{10}t}$. Hence

$$\begin{aligned}
\text{required probability} &= P[T > 6 + 2|T > 6] \\
&= P[T > 2] \quad \text{by (a)} \\
&= e^{-\frac{1}{10}\times 2} \\
&= 0.8187.
\end{aligned}$$

4. (a) $R(t) = \exp\left(-\int_0^t r(s)\ ds\right)$.

(b) First, compute $\int_0^t r(s)\ ds$.

(i) when $0 \leq t < 10$, $\int_0^t r(s)\ ds = \int_0^t 1\ ds = t$;

(ii) when $t \geq 10$,

$$\begin{aligned}
\int_0^t r(s)\ ds &= \int_0^{10} d\ ds + \int_{10}^t [1 + 10(s-10)]\ ds \\
&= 10 + \left[s + 10\left(\frac{1}{2}s^2 - 10s\right)\right]\Bigg|_{10}^t \\
&= 5t^2 - 99t + 500
\end{aligned}$$

so that

$$
\begin{aligned}
R(t) = & \exp\left(-\int_0^t r(s)\ ds\right) \\
= & \begin{cases} e^{-t}, & 0 \le t < 10 \\ e^{-5t^2+99t-500}, & t \ge 10 \end{cases}.
\end{aligned}
$$

$$
\begin{aligned}
f_T(t) = & r(t)R(t) \\
= & \begin{cases} e^{-t}, & 0 \le t < 10 \\ [1+10(t-10)]e^{-5t^2+99t-500}, & t \ge 10 \end{cases}.
\end{aligned}
$$

5. Given $X = 4 - 5T$,

$x = 4 - 5t \quad \Rightarrow \quad t = \dfrac{4-x}{5}$ and $\dfrac{dx}{dt} = -5$.

Note that $t \ge T_0 \quad \Longleftrightarrow \quad x \le 4 - 5T_0$

$t < T_0 \quad \Longleftrightarrow \quad x < 4 - 5T_0$

so that

$$
\begin{aligned}
f_X(x) = & \left.\frac{f_T(t)}{|dx/dt|}\right|_{t=\frac{4-x}{5}} \\
= & \begin{cases} \left.\frac{\lambda e^{-\lambda(t-T_0)}}{|-5|}\right|_{t=\frac{4-x}{5}}, & x \le 4 - 5T_0 \\ 0, & x > 4 - 5T_0 \end{cases} \\
= & \begin{cases} \frac{\lambda}{5} e^{-\lambda\left(\frac{4-x}{5} - T_0\right)}, & x \le 4 - 5T_0 \\ 0, & x > 4 - 5T_0 \end{cases}.
\end{aligned}
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