1. Consider a cosine wave with random amplitude and random phase. Let $X(t)$ be defined by

$$X(t) = A \cos (\omega t + \Phi)$$

where $\omega$ is a constant, $A$ is a non-negative random variable with finite expected value and finite variance, $\Phi$ is assumed to be uniformly distributed over $[0, 2\pi]$ and independent of $A$.

(a) Find the mean (or called the trend function) of $X(t)$. [2]

(b) Show that the autocovariance function $C(s, t)$ depends only on the difference $\tau = t - s$.

*Hint:* $C(s, t) = E[(X(s) - m(s))(X(t) - m(t))] = E[X(s)X(t)] - m(s)m(t)$

$$\cos(\omega s + \phi) \cos(\omega t + \phi) - \frac{1}{2} \{ \cos(\omega(t - s)) + \cos(\omega(t + s) + 2\phi) \}.$$ [3]

2. Let $N(t)$ be a Poisson process with parameter $\lambda$, where $\lambda$ is the average number of event occurrences per unit time.

(a) Explain why inter-event times are independent and identically distributed exponential random variables. Find the probability density of the inter-event random time variable. [4]

(b) Compute $P[N(s) - k|N(t) - n], s > t.$ [2]

3. A Markov chain $\{X_0, X_1, \cdots\}$ has the state space $Z = \{0, 1, 2\}$ and transition matrix

$$P = \begin{pmatrix} 0.5 & 0 & 0.5 \\ 0.4 & 0.2 & 0.4 \\ 0 & 0.4 & 0.6 \end{pmatrix}.$$ [points]

(a) Show that $P[X_0 - j_0, X_1 - j_1, X_2 - j_2] - P[X_0 - j_0]P[X_1 - j_1|X_0 - j_0]P[X_2 - j_2|X_1 - j_1]$.

Hence compute $P[X_0 = 0, X_1 = 1, X_2 = 1]$ given that $P[X_0 = 0] = 0.4$. [3]

(b) Determine $P[X_{n+1} = 2, X_n = 0|X_{n-1} = 0]$ for $n > 1$. [3]

4. Let $\{Y_0, Y_1, \cdots\}$ be a sequence of independent, identically distributed binary random variables with $P[Y_i = 0] = P[Y_i = 1] = \frac{1}{2}, i = 0, 1, 2, \cdots$. Define a sequence of random variables $\{X_1, X_2, \cdots\}$ by

$$X_n = \frac{1}{2}(Y_n - Y_{n-1}), \quad n = 1, 2, \cdots.$$ Determine whether the random sequence $\{X_1, X_2, \cdots\}$ has the Markovian property. Give your reasoning in details. [5]

5. (a) Explain why a Poisson process is a continuous time Markov chain. [2]
(b) Let $X_t$ be a Poisson process with parameter $\lambda$, show that the autocovariance $C_X(t_1, t_2)$ of the Poisson process $X_t$ is given by

$$C_X(t_1, t_2) = \lambda \min(t_1, t_2).$$

[4]

6. Suppose on the zeroth day, a house has three new light bulbs in reserve. Let the probability that a light bulb in use fails on a given day be 0.4. When the light bulb fails, the house replace it by a new light bulb. This is a Markov chain with state space $Z = \{0, 1, 2, 3\}$, where the state gives the number of new light bulbs in reserve.

(a) Find the one-step transition probability matrix.

(b) Find the probability that the house has one new bulb in reserve after ten days.

(c) Explain why the number of new bulb in reserve becomes zero when the number of days become infinite.

[2]

7. Let $\{X_1, X_2, \cdots\}$ be a sequence of independent and identically distributed Bernoulli random variables. Define $S_n = \sum_{i=1}^{n} X_i$ be a binomial counting process and let $p$ be the probability of success.

(a) Compute $P[S_{n_2} = j | S_{n_1} = i]$, where $n_2 > n_1$. Distinguish between $j \geq i$ and $j < i$.

(b) For $n_2 > n_1 > n_0$, show that

$$P[S_{n_2} = j | S_{n_1} = i, S_{n_0} = k] = P[S_{n_2} = j | S_{n_1} = i].$$

[4]

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