MATH 246 — Probability and Random Processes

Final Examination

Fall 2002 **Course Instructor:** *Prof. Y. K. Kwok*

Time allowed: 2 hours

[points]

1. Consider a cosine wave with random amplitude and random phase. Let $X(t)$ be defined by

$$X(t) = A\cos(\omega t + \Phi)$$

where ω is a constant, A is a non-negative random variable with finite expected value and finite variance, Φ is assumed to be uniformly distributed over $[0, 2\pi]$ and independent of A.

(a) Find the mean (or called the trend function) of $X(t)$. **[2]**

(b) Show that the autocovariance function $C(s,t)$ depends only on the difference $\tau = t - s$.
Hint: $C(s,t) = E[(X(s) - m(s))(X(t) - m(t))] = E[X(s)X(t)] - m(s)m(t)$
$\cos(\omega s + \phi)\cos(\omega t + \phi) = \frac{1}{2}\{\cos\omega(t-s) + \cos[\omega(t+s) + 2\phi]\}$. **[3]**

2. Let $N(t)$ be a Poisson process with parameter λ, where λ is the average number of event occurrences per unit time.

(a) Explain why inter-event times are independent and identically distributed exponential random variables. Find the probability density of the inter-event random time variable. **[4]**

(b) Compute $P[N(s) = k | N(t) = n]$, $s > t$. **[2]**

3. A Markov chain $\{X_0, X_1, \cdots\}$ has the state space $Z = \{0, 1, 2\}$ and transition matrix

$$P = \begin{pmatrix} 0.5 & 0 & 0.5 \\ 0.4 & 0.2 & 0.4 \\ 0 & 0.4 & 0.6 \end{pmatrix}.$$

(a) Show that $P[X_0 = j_0, X_1 = j_1, X_2 = j_2]$
$= P[X_0 = j_0]P[X_1 = j_1 | X_0 = j_0]P[X_2 = j_2 | X_1 = j_1]$.
Hence compute $P[X_0 = 0, X_1 = 1, X_2 = 1]$, given that $P[X_0 = 0] = 0.4$. **[3]**

(b) Determine $P[X_{n+1} = 2, X_n = 0 | X_{n-1} = 0]$ for $n > 1$. **[3]**

4. Let $\{Y_0, Y_1, \cdots\}$ be a sequence of independent, identically distributed binary random variables with $P[Y_i = 0] = P[Y_i = 1] = \frac{1}{2}$, $i = 0, 1, 2, \cdots$. Define a sequence of random variables $\{X_1, X_2, \cdots\}$ by

$$X_n = \frac{1}{2}(Y_n - Y_{n-1}), \quad n = 1, 2, \cdots.$$

Determine whether the random sequence $\{X_1, X_2, \cdots\}$ has the Markovian property. Give your reasoning in details. **[5]**

5. (a) Explain why a Poisson process is a continuous time Markov chain. **[2]**

(b) Let X_t be a Poisson process with parameter λ, show that the autocovariance $C_X(t_1, t_2)$ of the Poisson process X_t is given by

$$C_X(t_1, t_2) = \lambda \min(t_1, t_2).$$

[4]

6. Suppose on the zeroth day, a house has three new light bulbs in reserve. Let the probability that a light bulb in use fails on a given day be 0.4. When the light bulb fails, the house replace it by a new light bulb. This is a Markov chain with state space $Z = \{0, 1, 2, 3\}$, where the state gives the number of new light bulbs in reserve.

   (a) Find the one-step transition probability matrix. **[2]**

   (b) Find the probability that the house has one new bulb in reserve after ten days. **[2]**

   (c) Explain why the number of new bulb in reserve becomes zero when the number of days become infinite. **[2]**

7. Let $\{X_1, X_2, \cdots\}$ be a sequence of independent and identically distributed Bernoulli random variables. Define $S_n = \sum_{i=1}^{n} X_i$ be a binomial counting process and let p be the probability of success.

   (a) Compute $P[S_{n_2} = j | S_{n_1} = i]$, where $n_2 > n_1$. Distinguish between $j \geq i$ and $j < i$. **[2]**

   (b) For $n_2 > n_1 > n_0$, show that

$$P[S_{n_2} = j | S_{n_1} = i, S_{n_0} = k] = P[S_{n_2} = j | S_{n_1} = i].$$

[4]

— *End* —