MATH 246 — Probability and Random Processes

Solution to Test One

Fall 2002 **Course Instructor:** *Prof. Y. K. Kwok*

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1. Label the four cards as follows

1^{st}	2^{nd}	3^{rd}	4^{th}
Black/Black	Red/Black	Red/Red	Black/Blue

Define $C_i = \{\text{the } i^{\text{th}} \text{ card is chosen}\}$

$B = \{\text{the upper side is black}\}$

(a) The required probability

$$= P[B] = \sum_{i=1}^{4} P[B|C_i]P[C_i]$$

$$= (1)\left(\frac{1}{4}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) + (0)\left(\frac{1}{4}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)$$

$$= \frac{1}{2}.$$

(b) Note that $\{C_1, C_2, C_3, C_4\}$ forms a partition of the sample space. By Bayes's theorem, the required probability

$$= P[C_4|B] = \frac{P[C_4 \cap B]}{P[B]}$$

$$= \frac{P[B|C_4]P[C_4]}{P[B]} = \frac{\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)}{\frac{1}{2}}$$

$$= \frac{1}{4}.$$

2. (a) $S_Y = \{2, 3, 4, \cdots, 11, 12\}$.

(b) $\{Y = 3\} = \{1^{\text{st}} \text{ time shown 1 and } 2^{\text{nd}} \text{ time shown 2}\} \cup \{1^{\text{st}} \text{ time shown 2 and } 2^{\text{nd}} \text{ time shown 1}\}$
$= \{(1, 2), (2, 1)\}$.

(c) $P[Y = 2] = P[\{(1, 1)\}] = \dfrac{1}{36}$

$P[Y = 3] = P[\{(1, 2), (2, 1)\}] = \dfrac{2}{36}$

$P[Y = 4] = P[\{(1, 3), (2, 2), (3, 1)\}] = \dfrac{3}{36}$

$$P[Y \le 4] = \sum_{k=2}^{4} P[Y = k] = \frac{1}{36} + \frac{2}{36} + \frac{3}{36} = \frac{1}{6}.$$

3. Given that N is a geometric random variable with probability of success p, we have

$$P[N = k] = (1 - p)^{k-1}p.$$

(a)
$$P[N > k] = \sum_{j=k+1}^{\infty} (1-p)^{j-1}p$$
$$= (1-p)^k p \sum_{j=0}^{\infty}(1-p)^j$$
$$= (1-p)^k p \cdot \frac{1}{1-(1-p)} = (1-p)^k.$$

(b) $P[N \text{ is an even number}]$

$$= \sum_{k=1}^{\infty} P[N=2k] = \sum_{k=1}^{\infty}(1-p)^{2k-1}p$$
$$= \frac{p}{1-p}\sum_{k=1}^{\infty}[(1-p)^2]^k$$
$$= \frac{p}{1-p} \cdot \frac{(1-P)^2}{1-(1-p)^2}$$
$$= \frac{1-p}{2-p}.$$

(c) $P[N=k|N \leq m] = \dfrac{P[N=k \cap N \leq m]}{P[N \leq m]} = \dfrac{P[N=k \cap N \leq m]}{1-P[N>m]}.$

When $k \leq m$, $\{N=k\} \cap \{N \leq m\} = \{N=k\}$;

when $k > m$, $\{N=k\} \cap \{N \leq m\} = \phi$.

Hence, $P[N=k|N \leq m] = \begin{cases} \frac{P[N=k]}{1-P[N>m]}, & k \leq m \\ P[\phi], & k > m \end{cases}$

$$= \begin{cases} \frac{(1-p)^{k-1}p}{1-(1-p)^m}, & k \leq m \\ 0, & k > m \end{cases}.$$

4. Let $N(t)$ = number of births over t-day period. Then the average number of birth over $[0,t]$ is $\alpha = 5.6t$ and $P[N(t)=k] = \dfrac{(5.6t)^k}{k!}e^{-5.6t}$.

(a) Note that 6 hours = 0.25 day and $5.6t = 5.6 \times 0.25 = 1.4$.

$$\begin{aligned} \text{The required probability} &= P[N(0.25) \geq 2] \\ &= 1 - P[N(0.25)=0] - P[N(0.25)=1] \\ &= 1 - e^{-1.4} - \frac{1.4}{1}e^{-1.4} \\ &= 1 - 2.4e^{-1.4} \\ &= 0.4082. \end{aligned}$$

(b) Over a 2-day period, $\alpha = 5.6 \times 2 = 11.2$. The mean number of births over 2 days $= E[N(2)] = \alpha = 11.2$.

(c) Over a 3-day period, $\alpha = 5.6 \times 3 = 16.8$. Since $P[N(3)=k]$ attains its maximum at $k = [\alpha] = 16$, so the most possible number of births over 3 days = 16.

5. (a) Finally, we have $F_T(x|T>t) = F_T(x-t)$, so $F_T(x|T>t) \neq F_T(x)$ for all $t > 0$.

(b) First, we need to show that

$$F_T(x|T > t) = \begin{cases} 0, & x < t \\ f_T(x)/[1 - F_T(t)], & x \geq t \end{cases}.$$

Now, $R(t) = P[T \geq t] \Rightarrow R(t) = 1 - F_T(t) \quad \text{and} \quad R'(t) = -f_T(t)$;
so $r(t) = f_T(t|T > t) = \dfrac{f_T(t)}{1 - F_T(t)} = \dfrac{-R'(t)}{R(t)}$.