1. Let $X$ be the standard Gaussian random variable with zero mean and unit standard deviation. Find

(a) $P\left[|X| > \frac{1}{2}\right]$; \hspace{1cm} [3]

(b) the density function of the random variable $|X|$. \hspace{1cm} [3]

Express your answers in terms of $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2} \, dt$ and $n(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

2. Let $X$ and $Y$ be a pair of independent random variables, where $X$ is uniformly distributed over $(0, 1)$ and $Y$ is uniformly distributed over $(-2, 0)$. Find the probability density function of $Z = X/Y$. \hspace{1cm} [6]

3. Let $X$ be the standard Gaussian random variable with zero mean and unit standard deviation. Let $I$, independent of $X$, be such that

$$ P[I = 0] = P[I = 1] = \frac{1}{2}. $$

Define

$$ Y = \begin{cases} X & \text{if } I = 1 \\ -X & \text{if } I = 0 \end{cases}, $$

that is, $Y$ is equally likely to equal either $X$ or $-X$.

(a) Is $Y$ a Gaussian random variable? Find its mean and variance.

**Hint:** $P[Y \leq y] = P[X \leq y|P[I = 1] + P[X \geq -y|P[I = 0]$

$$ = \frac{1}{2} \left[ \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt + \int_{-\infty}^{-y} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt \right] $$

and make use of the symmetry property of the integrand function. \hspace{1cm} [3]

(b) Compute $\text{COV}(X, Y)$.

**Hint:** $\text{COV}(X, Y) = E[XY] - E[X]E[Y]$ and $E[XY] = E[E[XY|I]]$. \hspace{1cm} [3]
4. Let $X$ and $Y$ be random variables that take on values from the set $\{0, 1, 2\}$.
   (a) Find a joint probability mass assignment for which $X$ and $Y$ are independent, and illustrate that $X^2$ and $Y^2$ are also independent. 
   \[ \text{[3]} \]
   (b) Can we find a joint pmf assignment for which $X$ and $Y$ are not independent, but for which $X^2$ and $Y^2$ are independent? If yes, find an example; if not, explain why. 
   \[ \text{[3]} \]

5. An urn contains $n$ white and $m$ black balls. One ball is drawn randomly at a time until the first white ball is drawn.
   (a) Let $X$ denote the number of black balls that are drawn before the first white ball appears. We write $M(n,m)$ to be the expected value of $X$ (showing its dependence on $n$ and $m$). Explain why
   \[ M(n, m) = E[X] = E[X|Y = 1]P[Y = 1] + E[X|Y = 0]P[Y = 0] \]
   where $Y$ is the discrete random variable defined by
   \[
   Y = \begin{cases} 
   1 & \text{if the first ball selected is white} \\
   0 & \text{if the first ball selected is black} 
   \end{cases}
   \]
   then show that $M(n, m) = \frac{m}{n+m}[1 + M(n, m - 1)]$.
   \[ \text{[3]} \]
   (b) Explain why $M(n, 0) = 0$, show that $M(n, 1) = \frac{1}{n+1}, M(n, 2) = \frac{2}{n+1}, M(n, 3)$
   \[ \text{[3]} \]
   \[ = \frac{3}{n+1}; \text{ then deduce the value of } M(n, m). \]

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