MATH 246 — Probability and Random Processes

Test Two

Fall 2002 **Course Instructor:** *Prof. Y. K. Kwok*

Time allowed: 75 minutes

[points]

1. Let X be the standard Gaussian random variable with zero mean and unit standard deviation. Find
   (a) $P\left[|X| > \frac{1}{2}\right]$; **[3]**
   (b) the density function of the random variable $|X|$. **[3]**
   Express your answers in terms of $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt$ and $n(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

2. Let X and Y be a pair of *independent* random variables, where X is uniformly distributed over $(0, 1)$ and Y is uniformly distributed over $(-2, 0)$. Find the probability density function of $Z = X/Y$. **[6]**

3. Let X be the standard Gaussian random variable with zero mean and unit standard deviation. Let I, independent of X, be such that
   $$P[I = 0] = P[I = 1] = \frac{1}{2}.$$
   Define
   $$Y = \begin{cases} X & \text{if } I = 1 \\ -X & \text{if } I = 0 \end{cases},$$
   that is, Y is equally likely to equal either X or $-X$.
   (a) Is Y a Gaussian random variable? Find its mean and variance.
   Hint: $P[Y \le y] = P[X \le y]P[I = 1] + P[X \ge -y]P[I = 0]$
   $$= \frac{1}{2}\left[\int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt + \int_{-y}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt\right]$$
   and make use of the symmetry property of the integrand function. **[3]**
   (b) Compute $\mathrm{COV}(X, Y)$.
   Hint: $\mathrm{COV}(X, Y) = E[XY] - E[X]E[Y]$ and $E[XY] = E[E[XY|I]]$. **[3]**

4. Let X and Y be random variables that take on values from the set $\{0, 1, 2\}$.
   (a) Find a joint probability mass assignment for which X and Y are independent, and illustrate that X^2 and Y^2 are also independent. **[3]**
   (b) Can we find a joint pmf assignment for which X and Y are not independent, but for which X^2 and Y^2 are independent? If yes, find an example; if not, explain why. **[3]**
5. An urn contains n white and m black balls. One ball is drawn randomly at a time until the first white ball is drawn.
   (a) Let X denote the number of black balls that are drawn before the first white ball appears. We write $M(n, m)$ to be the expected value of X (showing its dependence on n and m). Explain why

$$M(n, m) = E[X] = E[X|Y = 1]P[Y = 1] + E[X|Y = 0]P[Y = 0]$$

   where Y is the discrete random variable defined by

$$Y = \begin{cases} 1 & \text{if the first ball selected is white} \\ 0 & \text{if the first ball selected is black} \end{cases}$$

   then show that $M(n, m) = \dfrac{m}{n+m}[1 + M(n, m-1)]$. **[3]**
   (b) Explain why $M(n, 0) = 0$, show that $M(n, 1) = \dfrac{1}{n+1}$, $M(n, 2) = \dfrac{2}{n+1}$, $M(n, 3) = \dfrac{3}{n+1}$; then deduce the value of $M(n, m)$. **[3]**

— *End* —