MATH 246 — Probability and Random Processes

Solution to Test Two

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1. (a) 

\[ P \left[ |X| > \frac{1}{2} \right] = 1 - P \left[ -\frac{1}{2} \leq X \leq \frac{1}{2} \right] = 1 - \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt 
- 1 - \left[ N \left( \frac{1}{2} \right) - N \left( -\frac{1}{2} \right) \right]. \]

(b) Consider

\[ P\{ |X| \leq x \} = P\{ -x \leq X \leq x \}, \quad x \geq 0 \]

so that

\[ f_{\mid X\mid}(x) = \frac{d}{dx} P\{ |X| \leq x \} = \frac{d}{dx} [N(x) - N(-x)] = 2n(x). \]

2. 

\[ f_Z(z) = \int_{-\infty}^{\infty} |y| f_{XY}(y, z) \, dy; \quad f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}, \quad f_Y(y) = \begin{cases} \frac{1}{2} & -2 < y < 0 \\ 0 & \text{otherwise} \end{cases} \]

and since \( X \) and \( Y \) are independent

\[ f_{XY}(y, z) = \begin{cases} \frac{1}{2} & 0 < yz < 1 \text{ and } -2 < y < 0 \\ 0 & \text{otherwise} \end{cases} \]

Consider the following cases

(i) when \( z > 0 \), \( yz \) is always negative, so \( 0 < yz < 1 \) is never satisfied;
(ii) when \( -\frac{1}{2} < z < 0 \), both \( 0 < yz < 1 \) and \( -2 < y < 0 \) are satisfied;
(iii) when \( z < -\frac{1}{2} \), we observe \( 0 < yz < 1 \Leftrightarrow \frac{1}{z} < y < 0. \)

We then have

(i) \( -\frac{1}{2} < z < 0 \), \( f_Z(z) = \int_{-\frac{1}{2}}^{0} \frac{1}{2} |y| \, dy = \int_{-\frac{1}{2}}^{0} -\frac{y}{2} \, dy = 1. \)

(ii) \( z < \frac{1}{2} \), \( f_Z(z) = \int_{1/z}^{0} \frac{y}{2} \, dy = \frac{-yz^2}{4} \bigg|_{1/z}^{0} = \frac{1}{4z^2}. \)

In summary, \( f_Z(z) = \begin{cases} \frac{1}{2} & -\frac{1}{2} < z < 0 \\ 0 & z < -\frac{1}{2} \end{cases} \).

As a check, consider

\[ \int_{-\infty}^{\infty} f_Z(z) \, dz = \int_{-\infty}^{-1/2} \frac{1}{4z^2} \, dz + \int_{-1/2}^{0} \frac{1}{2} \, dz = \int_{-\infty}^{-\frac{1}{2}} \frac{1}{4z^2} + \frac{1}{2} \bigg|_{-\frac{1}{2}}^{0} = \frac{1}{2} + \frac{1}{2} = 1. \]

3. (a)
\[ P[Y \leq y] = P[X \leq y]P[I = 1] + P[X \geq -y]P[I = -1] \\
= \frac{1}{2} \left[ \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt + \int_{-\infty}^{-y} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt \right]. \]

Since the Gaussian density function is symmetric, \( P[Y \leq y] = N(y) \) and \( f_Y(y) = n(y) \) so that \( Y \) is also a Gaussian random variable. The mean and variance of \( Y \) are zero and one, respectively.


\[ E[XY] = E[E[XY|\ell]] = \frac{1}{2} \{ E[X^2] + E[-X^2] \} = 0. \]

4. (a)

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<thead>
<tr>
<th>( P_{XY}(x, y) )</th>
<th>( x_1 = 0 )</th>
<th>( x_2 = 1 )</th>
<th>( x_3 = 2 )</th>
<th>( P_Y(y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y_1 = 0 )</td>
<td>( \frac{1}{4} )</td>
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<td>( y_1 = 1 )</td>
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<td>( y_2 = 2 )</td>
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Note that \( P_{XY}(x, y) \) is \( P_X(x)P_Y(y) \) for all \( x \) and \( y \), so \( X^2 \) and \( Y^2 \) are also independent.

(b) For each \( x \in S_X \), there corresponds to only one \( x_k \in S_X \) where \( x_k \) is the positive square root of \( x \). We then have \( P_{XY}(x, y) = P_X(\sqrt{x}) = P_X(x) \), and the same rule applies for the \( y_j \). So \( X^2 \) and \( Y^2 \) are independent, that is, \( P_{X^2Y^2}(x, y) = P_{X^2}(x)P_{Y^2}(y) \), then we observe that \( P_{XY}(x, y) = P_X(x)P_Y(y) \) so that \( X \) and \( Y \) must be independent.

5. (a) \( M(n, m) = E[X|Y = 1]P[Y = 1] + E[X|Y = 0]P[Y = 0] \) from the Law of Total Probability. It is seen that \( P[Y = 1] = \frac{n}{n+m}, P[Y = 0] = \frac{m}{n+m}, E[X|Y = 1] = 0, E[X|Y = 0] = 1 + M(n, m-1) \). The “one” comes in since one black ball has been drawn; after then there are \( m-1 \) black balls and \( n \) white balls remaining.

(b) \( M(n, 0) = 0 \) since there is no black ball remaining, \( M(n, 1) = \frac{1}{n+1}(1+0) = \frac{1}{n+1} \). \( M(n, 2) = \frac{2}{n+2} \left[ 1 + \frac{1}{n+1} \right] = \frac{2}{n+1} \). In general, \( M(n, m) = \frac{m}{n+1} \).