MATH 246 — Probability and Random Processes

Solution to Test Two

Fall 2002 **Course Instructor:** *Prof. Y. K. Kwok*

---

1. (a)

$$P\left[|X| > \frac{1}{2}\right] = 1 - P\left[-\frac{1}{2} \le X \le \frac{1}{2}\right] = 1 - \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt$$
$$= 1 - \left[N\left(\frac{1}{2}\right) - N\left(-\frac{1}{2}\right)\right].$$

(b) Consider

$$P[|X| \le x] = P[-x \le X \le x], \quad x \ge 0$$
$$= N(x) - N(-x)$$

so that

$$f_{|x|}(x) = \frac{d}{dx} P[|X| \le x] = \frac{d}{dx}[N(x) - N(-x)] = 2n(x).$$

2.
$$f_Z(z) = \int_{\infty}^{\infty} |y| f_{XY}(yz, y)\, dy; f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}, f_Y(y) = \begin{cases} \frac{1}{2} & -2 < y < 0 \\ 0 & \text{otherwise} \end{cases}$$

and since X and Y are independent

$$f_{XY}(yz, y) = \begin{cases} \frac{1}{2} & 0 < yz < 1 \text{ and } -2 < y < 0 \\ 0 & \text{otherwise} \end{cases}.$$

Consider the following cases

(i) when $z > 0, yz$ is always negative, so $0 < yz < 1$ is never satisfied;

(ii) when $-\frac{1}{2} < z < 0$, both $0 < yz < 1$ and $-2 < y < 0$ are satisfied;

(iii) when $z < -\frac{1}{2}$, we observe $\begin{cases} 0 < yz < 1 \\ -2 < y < 0 \end{cases} \Leftrightarrow \frac{1}{z} < y < 0.$

We then have

(i) $-\frac{1}{2} < z < 0, f_Z(z) = \int_{-2}^{0} \frac{1}{2}|y|\, dy = \int_{-2}^{0} -\frac{y}{2}\, dy = 1.$

(ii) $z < -\frac{1}{2}, f_Z(z) = \int_{1/z}^{0} -\frac{y}{2}\, dy = = \left.\frac{-y^2}{4}\right]_{1/z}^{0} = \frac{1}{4z^2}.$

In summary, $f_Z(z) = \begin{cases} 1 & -\frac{1}{2} < z < 0 \\ \frac{1}{4z^2} & z < -\frac{1}{2} \\ 0 & \text{otherwise} \end{cases}.$

As a check, consider

$$\int_{-\infty}^{\infty} f_Z(z)\, dz = \int_{-\infty}^{-1/2} \frac{1}{4z^2}\, dz + \int_{-\frac{1}{2}}^{0} \frac{1}{2}\, dz = -\left.\frac{1}{4z}\right]_{-\infty}^{-\frac{1}{2}} + \left. z\right]_{-\frac{1}{2}}^{0} = \frac{1}{2} + \frac{1}{2} = 1.$$

3. (a)

$$P[Y \leq y] = P[X \leq y]P[I = 1] + P[X \geq -y]P[I = -1]$$
$$= \frac{1}{2}\left[\int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt + \int_{-y}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt\right].$$

Since the Gaussian density function is symmetric, $P[Y \leq y] = N(y)$ and $f_Y(y) = n(y)$ so that Y is also a Gaussian random variable. The mean and variance of Y are zero and one, respectively.

(b) $\text{COV}(X, Y) = E[XY] - E[X]E[Y] = E[XY] = E[XY]$ since $E[X] = E[Y] = 0$.

$$E[XY] = E_I[E[XY|I]] = \frac{1}{2}\{E[X^2] + E[-X^2]\} = 0.$$

4. (a)

$P_{XY}(x_k, y_j)$	$x_1 = 0$	$x_2 = 1$	$x_3 = 2$	$P_Y(y_j)$
$y_1 = 0$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{2}$
$y_1 = 1$	$\frac{1}{18}$	$\frac{1}{9}$	$\frac{1}{6}$	$\frac{1}{3}$
$y_2 = 2$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{6}$
$P_X(x_k)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	

$P_{X^2Y^2}(\widetilde{x}_k, \widetilde{y}_j)$	$\widetilde{x}_1 = 0$	$\widetilde{x}_2 = 1$	$\widetilde{x}_3 = 4$	$P_{Y^2}(\widetilde{y}_j)$
$\widetilde{y}_1 = 0$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{2}$
$\widetilde{y}_2 = 1$	$\frac{1}{18}$	$\frac{1}{9}$	$\frac{1}{6}$	$\frac{1}{3}$
$\widetilde{y}_3 = 4$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{6}$
$P_{X^2}(\widetilde{x}_k)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	

Note that $P_{X^2Y^2}(\widetilde{x}_k, \widetilde{y}_j) = P_{X^2}(\widetilde{x}_k)P_{Y^2}(\widetilde{y}_j)$ for all k and j, so X^2 and Y^2 are also independent.

(b) For each $\widetilde{x}_k \in S_{X^2}$, there corresponds to only one $x_k \in S_X$ where x_k is the positive square root of $\widetilde{x}_k$. We then have $P_{X^2}(\widetilde{x}_k) = P_X(\sqrt{\widetilde{x}_k}) = P_X(x_k)$, and the same rule applied for $\widetilde{y}_j \in S_{Y^2}$ and $\sqrt{\widetilde{y}_k} = y_j \in S_Y$.

Suppose X^2 and Y^2 are independent, that is, $P_{X^2Y^2}(\widetilde{x}_k, \widetilde{y}_k) = P_{X^2}(\widetilde{x}_k)P_{Y^2}(\widetilde{y}_j)$, then we observe that $P_{XY}(x_k, y_j) = P_X(x_k)P_Y(y_j)$ so that X and Y must be independent.

5. (a) $M(n, m) = E[X|Y = 1]P[Y = 1] + E[X|Y = 0]P[Y = 0]$ from the Law of Total Probability. It is seen that $P[Y = 1] = \frac{n}{n + m}$, $P[Y = 0] = \frac{m}{n + m}$, $E[X|Y = 1] = 0, E[X|Y = 0] = 1 + M(n, m - 1)$. The "one" comes in since one black ball has been drawn; after then there are $m - 1$ black balls and n white balls remaining.

(b) $M(n, 0) = 0$ since there is no black ball remaining, $M(n, 1) = \frac{1}{n+1}(1 + 0) = \frac{1}{n+1}$, $M(n, 2) = \frac{2}{n+2}\left[1 + \frac{1}{n+1}\right] = \frac{2}{n+1}$. In general, $M(n, m) = \frac{m}{n+1}$.