MATH 246 — Probability and Random Processes

Final Examination

Fall 2003 **Course Instructor:** *Prof. Y. K. Kwok*

Time allowed: 100 minutes

[points]

1. (a) State the stationary increments and independent increments properties of a Poisson process. **[2]**

   (b) Let $N(t), t \geq 0$, be a Poisson process with parameter $\lambda > 0$.

   (i) Find the autocovariance $C_N(t_1, t_2)$ of $N(t)$. **[4]**

   (ii) Show that **[4]**

$$P[N(t_1) = 1 | N(t_2) = 1] = \frac{t_1}{t_2}, \quad 0 < t_1 < t_2.$$

Hint: $C_N(t_1, t_2) = E[N(t_1)N(t_2)] - E[N(t_1)]E[N(t_2)]$.

2. A discrete-time random process X_n is defined to be $\dfrac{Y_n + Y_{n-1}}{2}$, where Y_n is an independent and identically distributed sequence of Poisson random variables with parameter λ.

   (a) Find the mean of X_n. **[1]**

   (b) Find the pmf for X_n. **[2]**

   (c) Find the autocorrelation $R_X(i, j)$ of X_n. Distinguish between $i = j, |i - j| = 1$ or otherwise. **[5]**

Hint: $R_X(i, j) = E[X_i X_j]$ and observe that $E[Y_i Y_j] = \begin{cases} \lambda + \lambda^2 & \text{if } i = j \\ \lambda^2 & \text{if } i \neq j \end{cases}$.

3. (a) Give definition to each of the following terms: **[2]**

   (i) stationary random process

   (ii) wide sense stationary random process

   (b) Consider the random process **[4]**

$$X(t) = U \sin \omega_0 t$$

where ω_0 is a constant and U is the standard Gaussian random variable with zero mean and unit variance.

   (i) Compute mean $m_X(t)$ and autocovariance $C_X(t_1, t_2)$.

   (ii) Is $X(t)$ wide sense stationary? Why or why not?

4. A machine consists of *two* parts that fail and are repaired independently. A working part fails during any given day with probability α. A part that is not working is repaired by the next day with probability β. Let X_n be the number of working parts in day n. The set of values assumed by X_n is $\{0, 1, 2\}$, and let $\pi_{n,j} = P[X_n = j], j = 0, 1, 2$.

   (a) Find the one-step transition probability matrix P. If the initial state pmf vector is

   $$\boldsymbol{\pi}_0 = (\pi_{0,0} \quad \pi_{0,1} \quad \pi_{0,2}) = (0.5 \quad 0.5 \quad 0),$$

   find $P[X_1 = 0, X_0 = 1]$. **[5]**

   (b) Let $\boldsymbol{\pi}_\infty = (\pi_{\infty,0} \quad \pi_{\infty,1} \quad \pi_{\infty,2})$ denote the steady state pmf vector. Verify that

   $$\pi_{\infty,0} = \frac{\alpha^2}{(\alpha+\beta)^2}, \quad \pi_{\infty,1} = \frac{2\alpha\beta}{(\alpha+\beta)^2} \quad \text{and} \quad \pi_{\infty,2} = \frac{\beta^2}{(\alpha+\beta)^2}.$$

   Write down the expression for $\lim_{n\to\infty} P^n$. **[3]**

   (c) Show that the steady state pmf $\boldsymbol{\pi}$ is binomial and find the corresponding parameters. The two parameters in a binomial random variable are the number of trials and the probability of success in each trial. **[2]**

5. Consider the discrete process Y_n defined by

   $$Y_n = \frac{1}{2}(X_n + X_{n-1})$$

   where X_n's are members of an independent Bernoulli sequence with $P[X_n = 0] = \frac{2}{3}$ and $P[X_n = 1] = \frac{1}{3}$.

   (a) Compute the pmf for Y_n. **[2]**

   (b) Compute $P\left[Y_n = \frac{1}{2}\middle| Y_{n-1} = 1\right]$ and $P\left[Y_n = \frac{1}{2}\middle| Y_{n-1} = 1, Y_{n-2} = 0\right]$. **[3]**

   (c) Is Y_n a Markov process? Give your reasoning. **[1]**

— *End* —