MATH 246 — Probability and Random Processes

Solution to Final Examination

Fall 2003 **Course Instructor:** *Prof. Y. K. Kwok*

Time allowed: 100 minutes

1. (a) *Stationary increments*

   The increments of the Poisson process $N(t)$ over two time intervals of equal length have the same probability distribution, independent of the starting time of the interval. That is,

   $$P[N(t_1+\delta)-N(t_1)] = P[N(t_2+\delta)-N(t_2)]$$

   for any t_1, t_2 and δ.

   Independent increments

   The increments of the Poisson process over any two non-overlapping time intervals are independent.

   (b) (i) Assume $t_1 < t_2$,

   $$\begin{aligned} C_N(t_1,t_2) &= E[(N(t_1)-\lambda t_1)(N(t_2)-\lambda t_2)] \\ &= E[(N(t_1)-\lambda t_1)\{[N(t_2)-N(t_1)-\lambda(t_2-t_1)]+(N(t_1)-\lambda t_1)\}] \\ &= E[N(t_1)-\lambda t_1]E[N(t_2)-N(t_1)-\lambda(t_2-t_1)]+E[\{N(t_1)-\lambda(t_1)\}^2] \\ &= \text{var}(N(t_1)) = \lambda\min(t_1,t_2). \end{aligned}$$

   (ii) $$\begin{aligned} & P(N(t_1)=1|N(t_2)=1] \\ =\ & \frac{P[N(t_1)=1, \quad N(t_2)-N(t_1)=0]}{P[N(t_2)=1]} \\ =\ & \frac{P[N(t_1)=1]\ P[N(t_2-t_1)=0]}{P[N(t_2)=1]} \\ =\ & \frac{\lambda t_1 e^{-\lambda t_1} e^{-\lambda(t_2-t_1)}}{\lambda t_2 e^{-\lambda t_2}} = \frac{t_1}{t_2}. \end{aligned}$$

2. (a) $E[X_n] = E\left[\frac{Y_n+Y_{n-1}}{2}\right] = \frac{\lambda}{2}+\frac{\lambda}{2} = \lambda.$

   (b) Sum of two independent Poisson random variables remain to be Poisson so that $Y_n + Y_{n-1}$ is a Poisson random variable with parameter 2λ. Now

   $$P[X_n = k] = P[Y_n + Y_{n-1} = 2k] = \frac{(2\lambda)^{2k}}{(2k)!}e^{-2\lambda}, \quad k = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$$

   (c) $$\begin{aligned} R_X(i,j) &= \frac{1}{4}E[(Y_i+Y_{i-1})(Y_j+Y_{j-1})] \\ &= \frac{1}{4}\{E[Y_iY_j]+E[Y_iY_{j-1}]+E[Y_{i-1}Y_j]+E[Y_{i-1}Y_{j-1}]\} \end{aligned}$$

Note that $E[Y_iY_j] = \begin{cases} \text{var}(Y_i) + E[Y_i]^2 & \text{if } i = j \\ E[Y_i]E[Y_j] & \text{if } i \neq j \end{cases}$

$= \begin{cases} \lambda + \lambda^2 & \text{if } i = j \\ \lambda^2 & \text{if } i \neq j \end{cases}.$

(i) When $i = j$

$$\begin{aligned} R_X(i,i) &= \frac{1}{4}\{E[Y_i^2] + 2E[Y_iY_{i-1}] + E[Y_{i-1}^2]\} \\ &= \frac{1}{4}[\lambda + \lambda^2 + 2\lambda^2 + \lambda + \lambda^2) = \frac{\lambda(2\lambda+1)}{2}. \end{aligned}$$

(ii) When $i = j + 1$

$$\begin{aligned} R_X(i,i-1) &= \frac{1}{4}\{E[Y_iY_{i-1}] + E[Y_iY_{i-2}] + E[Y_{i-1}^2] + E[Y_{i-1}Y_{i-2}]\} \\ &= \frac{1}{4}(3\lambda^2 + \lambda + \lambda^2) = \frac{\lambda(4\lambda+1)}{4}. \end{aligned}$$

(iii) When $i = j - 1$, we also obtain $R_X(i,i+1) = \dfrac{\lambda(4\lambda+1)}{4}$.

(iv) When $i \neq j$ and $|i-j| \neq 1$, we have $R_X(i,j) = \lambda^2$.

3. (a) A random process is stationary if $X(t)$ and $X(t+\tau)$ have the same statistics for any τ.

A random process is wide sense stationary if

$$\begin{aligned} m_X(t) &= m && \text{for all } t \\ C_X(t_1,t_2) &= C_X(t_1 - t_2) && \text{for all } t_1 \text{ and } t_2. \end{aligned}$$

(b)

$$\begin{aligned} m_X(t) &= E[U \sin \omega_0 t] = \sin \omega_0 t E[U] = 0 \\ C_X(t_1,t_2) &= E[U^2 \sin \omega_0 t_1 \sin \omega_0 t_2] - m_X(t)^2 \\ &= \sin \omega_0 t_1 \sin \omega_0 t_2 (\text{Var}(U) + E[U]^2) \\ &= \sin \omega_0 t_1 \sin \omega_0 t_2. \end{aligned}$$

$X(t)$ is not wide sense stationary since $C_X(t_1,t_2)$ is not a function of $t_2 - t_1$.

4. (a)

$$P = \begin{pmatrix} (1-\beta)^2 & 2\beta(1-\beta) & \beta^2 \\ \alpha(1-\beta) & \alpha\beta + (1-\alpha)(1-\beta) & (1-\alpha)\beta \\ \alpha^2 & 2\alpha(1-\alpha) & (1-\alpha)^2 \end{pmatrix}$$

$P[X_1 = 0, X_0 = 1] = P[X_1 = 0 | X_0 = 1]P[X_0 = 1] = \alpha(1-\beta)/2.$

(b) $\boldsymbol{\pi}_\infty$ is obtained by solving

$$\boldsymbol{\pi}_\infty = \boldsymbol{\pi}_\infty P.$$

To verify that $\boldsymbol{\pi}_\infty = \begin{pmatrix} \frac{\alpha^2}{(\alpha+\beta)^2} & \frac{2\alpha\beta}{(\alpha+\beta)^2} & \frac{\beta^2}{(\alpha+\beta)^2} \end{pmatrix}$, consider

$$\begin{aligned} \text{LHS} &= \begin{pmatrix} \frac{\alpha^2}{(\alpha+\beta)^2} & \frac{2\alpha\beta}{(\alpha+\beta)^2} & \frac{\beta^2}{(\alpha+\beta)^2} \end{pmatrix} \begin{pmatrix} (1-\beta)^2 & 2\beta(1-\beta) & \beta^2 \\ \alpha(1-\beta) & \alpha\beta + (1-\alpha)(1-\beta) & (1-\alpha)\beta \\ \alpha^2 & 2\alpha(1-\alpha) & (1-\alpha)^2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{\alpha^2(1-\beta)^2 + 2\alpha^2\beta(1-\beta) + \alpha^2\beta^2}{(\alpha+\beta)^2} \\ \frac{2\alpha^2\beta(1-\beta) + 2\alpha\beta[\alpha\beta + (1-\alpha)(1-\beta)] + 2\beta^2\alpha(1-\alpha)}{(\alpha+\beta)^2} \\ \frac{\alpha^2\beta^2 + 2\alpha(1-\alpha)\beta^2 + \beta^2(1-\alpha)^2}{(\alpha+\beta)^2} \end{pmatrix}^T \\ &= \boldsymbol{\pi}_\infty = \text{RHS} \end{aligned}$$

$$P^{\infty} = \frac{1}{(\alpha+\beta)^2}\begin{pmatrix} \alpha^2 & 2\alpha\beta & \beta^2 \\ \alpha^2 & 2\alpha\beta & \beta^2 \\ \alpha^2 & 2\alpha\beta & \beta^2 \end{pmatrix}$$ since all the rows of P^{∞} are equal to $\boldsymbol{\pi}_{\infty}$.

(c) Take n = number of trials = 2 and p = probability of success in each trial = $\frac{\beta}{\alpha+\beta}$. Let X = number of successes in this binomial experiment. We then have

$$\begin{aligned}
P[X=0] &= {}_2C_0 \left(\frac{\beta}{\alpha+\beta}\right)^0 \left(\frac{\alpha}{\alpha+\beta}\right)^2 = \frac{\alpha^2}{(\alpha+\beta)^2} = \pi_{\infty,0} \\
P[X=1] &= {}_2C_1 \left(\frac{\beta}{\alpha+\beta}\right)^1 \left(\frac{\alpha}{\alpha+\beta}\right)^1 = \frac{2\alpha\beta}{(\alpha+\beta)^2} = \pi_{\infty,1} \\
P[X=2] &= {}_2C_2 \left(\frac{\beta}{\alpha+\beta}\right)^2 \left(\frac{\alpha}{\alpha+\beta}\right)^0 = \frac{\beta^2}{(\alpha+\beta)^2} = \pi_{\infty,2}.
\end{aligned}$$

5. (a) Y_n can assume the following values: $0, \frac{1}{2}, 1$. The respective probabilities are

$$\begin{aligned}
P[Y_n = 0] &= P[X_n = 0, X_{n-1} = 0] = \frac{4}{9} \\
P\left[Y_n = \frac{1}{2}\right] &= P[X_n = 1, X_{n-1} = 0] + P[X_n = 0, X_{n-1} = 1] \\
&= 2\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = \frac{4}{9} \\
P[Y_n = 1] &= P[X_n = 1, X_{n-1} = 1] = \frac{1}{9}.
\end{aligned}$$

(b)

$$P\left[Y_n = \frac{1}{2}\middle| Y_{n-1} = 1\right] = \frac{P\left[Y_n = \frac{1}{2}, Y_{n-1} = 1\right]}{P[Y_{n-1} = 1]}$$
$$= \frac{P[X_n = 0, X_{n-1} = 1, X_{n-2} = 1]}{P[Y_{n-1} = 1]} = \frac{\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)}{\frac{1}{9}} = \frac{2}{3}.$$

Note that $\{Y_{n-1} = 1, Y_{n-2} = 0\}$ is an impossible event. By convention

$$P\left[Y_n = \frac{1}{2}\middle| Y_{n-1} = 1, Y_{n-2} = 0\right] = 0.$$

(c) Since $\left[Y_n = \frac{1}{2}\middle| Y_{n-1} = 1\right] \neq P\left[Y_n = \frac{1}{2}\middle| Y_{n-1} = 1, Y_{n-2} = 0\right]$ so Y_n is *not* Markovian.