MA246
Homework Two

Course Instructor: *Prof. Y. K. Kwok*

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1. Let X be the exponential random variable.
   a. Find and plot $F_X(x|X > t)$. How does $F_X(x|X > t)$ differ from $F_X(x)$.
   b. Find and plot $f_X(x| > t)$.
   c. Show that $P[X > t + x|X > t] = P[X > x]$. Explain why this is called the *memoryless property*.

2. Let X be the geometric random variable. Find and plot $F_X(x|A)$ if: $A = \{X > k\}$ where k is a positive integer; $A = \{X < k\}$; and $\{X$ is an even number$\}$.

3. Messages arrive at a computer at an average rate of 15 messages per second. The number of messages that arrive in 1 second is known to be a Poisson random variable.
   a. Find the probability that no messages arrive in 1 second.
   b. Find the probability that more than 10 messages arrive in a 1-second period.

   Hint: Use $p_{k+1} = \dfrac{\lambda}{k+1} p_k$ to compute the probabilities.

4. It is possible for a computer to pick up an erroneous signal that does not show up as an error on the screen. The error is called a silent error. A particular terminal is defective and when using the system word processor it introduces a silent paging error with probability 0.1. The word processor is used 20 times during a given week.
   a. Find the probability that no silent paging errors occurs.
   b. Find the probability that at least one such error occurs.
   c. Would it be unusual for more than three such errors to occur? Explain, based on the probability involved.

5. Let X be a Gaussian random variable with mean m and variance σ^2. Find the following probabilities:

$$\begin{array}{lll} P[X < m] \quad \text{and} \quad P[|X - m| > k\sigma] & \text{for} \quad k = 1, 2, 3, & \text{and} \\ P[X > m + k\sigma] & \text{for} \quad k = 1.28, 3.09. & \end{array}$$

6. Let $Y = |X|$ be the output of a full-wave rectifier with input voltage X.
   a. Find the cdf of Y by finding the equivalent event of $\{Y \leq y\}$. Find the pdf of Y by differentiation of the cdf.
   b. Find the pdf of Y by finding the equivalent event of $\{y < Y \leq y + dy\}$. Does the answer agree with part a?
   c. What is the pdf of Y if $f_X(x)$ is an even function of x?

7. Let $Y = e^X$.
   a. Find the cdf and pdf of Y in terms of cdf and pdf of X.
   b. Find the pdf of Y when X is a Gaussian random variable. In this case Y is said to have a *lognormal pdf*.

8. (*An application to sort algorithms.*) In studying various sort algorithms in computer science, it is of interest to compare their efficiency by estimating the average number of interchanges needed to sort random arrays of various sizes. It is also of interest to compare these estimated averages to the "ideal" average where by "ideal" we mean the expected minimum number of interchanges needed to sort the array. In this exercise you will derive this ideal average.

   a. Consider a random array of length n. When the positions of exactly two elements of the array are exchanged, we say that an "interchange" has taken place. Let X_n denote the minimum number of interchanges necessary to sort an array of size n. Note that

   $$X_n = X_{n-1} + I$$

   where $I = 0$ if the last element of the array is in the correct position and $I = 1$ otherwise. Argue that $P[I = 0] = 1/n$ and $P[I = 1] = 1 - (1/n)$.

   b. Show that

   $$E[I] = 1 - \frac{1}{n}.$$

   c. Argue that

   $$\begin{aligned}
   E[X_n] &= E[X_{n-1}] + 1 - \frac{1}{n} \\
   E[X_{n-1}] &= E[X_{n-2}] + 1 - \frac{1}{n-1} \\
   E[X_{n-2}] &= E[X_{n-3}] + 1 - \frac{1}{n-2} \\
   &\vdots \\
   E[X_3] &= E[X_2] + 1 - \frac{1}{3} \\
   E[X_2] &= E[X_1] + 1 - \frac{1}{2}.
   \end{aligned}$$

   d. Use a recursive argument to show

   $$E[X_n] = (n-1) - \sum_{i=2}^{n} \frac{1}{i}.$$

   e. Illustrate the expression given in part (d) by find $E[X_5]$.

   f. Elementary calculus can be used to approximate $E[X_n]$ by noting that

   $$\sum_{i=2}^{n} \frac{1}{i} \doteq \int_{1.5}^{n+0.5} \frac{1}{t}\, dt.$$

   Use this idea to approximate $E[X_5]$ and compare the result to the exact solution found in part (e).

   g. A random digit generator is used to generate sets of 100 different three-digit numbers lying between 0 and 1. What is the ideal average number of interchanges needed to sort such an array?