MA246
Homework Three

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1. A die is tossed twice; let X_1 and X_2 denote the outcome of the first and second toss, respectively.

   a. What is the joint pmf for (X_1, X_2) if the tosses are independent and if the outcomes of each toss are equiprobable?

   b. Let $X = \min(X_1, X_2)$ and $Y = \max(X_1, X_2)$. Find the joint pmf for (X, Y).

   c. Find the marginal pmf's for X and Y in part b.

2. The general form of the joint pdf for two jointly Gaussian random variables is

$$f_{X,Y}(x,y) = \frac{\exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x-m_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-m_1}{\sigma_1}\right)\left(\frac{y-m_2}{\sigma_2}\right) + \left(\frac{y-m_2}{\sigma_2}\right)^2\right]\right\}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}$$

   for $-\infty < x < \infty$ and $-\infty < y < \infty$. Show that the marginal pdf's of X and Y are those of Gaussian random variables with means m_1 and m_2 and variances σ_1^2 and σ_2^2, respectively.

3. Let X be the input to a communication channel. X takes on the values ± 1 with equal probability. Suppose that the output of the channel is $Y = X + N$, where N is a Laplacian random variable with pdf

$$f_N(z) = \frac{1}{2}\alpha e^{-\alpha|z|}, \quad -\infty < z < \infty.$$

   a. Find $P[X = k, Y \leq y]$ for $k = \pm 1$.

   b. Find the marginal pdf of Y.

   c. Suppose we are given that $Y > 0$. Which is more likely, $X = 1$ or $X = -1$?

4. Let X and Y be independent random variables. Find an expression for the probability of the following events in terms of $F_X(x)$ and $F_Y(y)$:

   a. $\{a < X \leq b\} \cap \{Y \leq d\}$.

   b. $\{a \leq X \leq b\} \cap \{c \leq Y \leq d\}$.

   c. $\{|X| > a\} \cap \{c \leq Y \leq d\}$.

5. Consider a sequence of $n + m$ independent Bernoulli trials with probability of success p in each trial. Let N be the number of successes in the first n trials and let M be the number of successes in the remaining m trials.

   a. Why are N and M independent random variables?

   b. Find the joint pmf of N and M and the marginal pmf's of N and M.

   c. Find the pmf for the total number of successes in the $n + m$ trials.

6. Let $X = \cos\Theta$ and $Y = \sin\Theta$, where Θ is an angle that is uniformly distributed in the interval $(0, 2\pi)$. Find $f(y|x)$ and $E[Y|X]$.

7. The random variables X and Y have the joint pdf

$$f_{X,Y}(x,y) = 2e^{-(x+y)}, \qquad 0 \leq y \leq x < \infty.$$

   Find the pdf of $Z = X + Y$. Note that X and Y are not independent.