Multiple Random Variables

Vector random variable X is a function that assigns a vector of real numbers to each outcome ξ in S (sample space of the random experiment).

Example The random experiment of selecting one student from a class, define the following functions:

$$\begin{aligned} H(\xi) &= \text{height of student } \xi \text{ in inches} \\ W(\xi) &= \text{weight of student } \xi \text{ in pounds} \\ A(\xi) &= \text{age of student } \xi \text{ in years.} \end{aligned}$$

$(H(\xi), W(\xi), A(\xi))$ is vector random variable.

Product form events

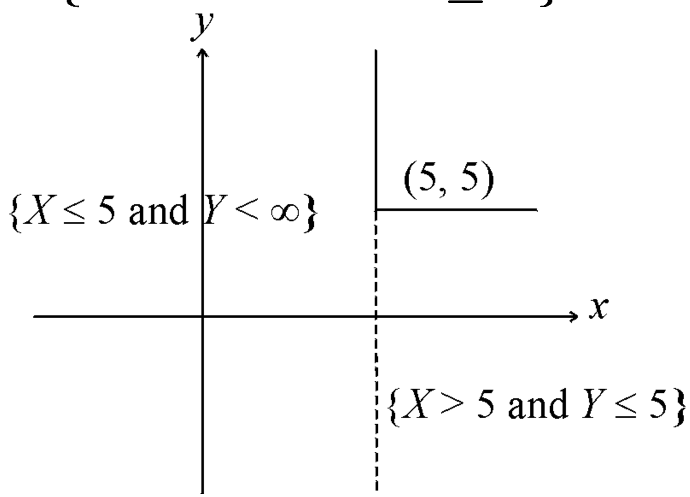
Let $X = (X_1, \cdots, X_n)$, an event of product form is depicted

$$A = \{X_1 \text{ in } A_1\} \cap \{X_2 \text{ in } A_2\} \cap \cdots \cap \{X_n \text{ in } A_n\}$$

where A_k is a one-dimensional event that involves X_k only.

e.g. $B = \{\min(X, Y) \leq 5\}$; B is the union of two product form events.

$B = \{X \leq 5 \text{ and } Y < \infty\} \cup \{X > 5 \text{ and } Y \leq 5\}$.



Pairs of Discrete Random Variables

$\widetilde{X} = (X, Y)$ assumes values from some countable set

$$S = \{(x_j, y_k), j = 1, 2, \cdots, k = 1, 2, \cdots\}.$$

The joint probability mass function of $\widetilde{X}$ specifies the probability of the product-form event $\{X = x_j\} \cap \{Y = y_k\}$, $j = 1, 2, \cdots, k = 1, 2 \cdots$

$$\begin{aligned} P_{X,Y}(x_j, y_k) &= P[\{X = x_j\} \cap \{Y = y_k\}] \\ &= P[X = x_j, Y = y_k], j = 1, 2, \cdots, k = 1, 2, \cdots. \end{aligned}$$

This can be interpreted as the long-term relative frequency of the joint event $\{X = x_j\} \cap \{Y = y_k\}$ in a sequence of repetitions of the random experiment.

$$P[\widetilde{X} \text{ in } A] = \sum \sum\nolimits_{(x_j, y_k) \in A} P_{X,Y}(x_j, y_k)$$

and

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} P_{X,Y}(x_j, y_k) = 1.$$

Marginal probability mass functions

$$\begin{aligned} P_X(x_j) &= P[X = x_j] \\ &= P[X = x_j, Y = \text{anything}] \\ &= \sum_{k=1}^{\infty} P_{X,Y}(x_j, y_k); \end{aligned}$$

similarly,

$$P_Y(y_k) = \sum_{j=1}^{\infty} P_{X,Y}(x_j, y_k).$$

Marginal pmf's are one-dimensional pmf's; knowledge of the marginal pmf's is insufficient to specify the joint pmf.

Example A urn contains 3 red, 4 white and 5 blue balls. Now, 3 balls are drawn. Let X and Y be the number of red and white balls chosen, respectively, find the joint probability mass function of X and Y,

$$P_{X,Y}(j,k) = P[\{X = j\} \cap \{Y = k\}].$$

j \ k	0	1	2	3	$P_X(j)$
0	10/220	40/220	30/220	4/220	84/220
1	30/220	60/220	18/220	0	108/220
2	15/220	12/220	0	0	27/220
3	1/220	0	0	0	1/220
$P_Y(k)$	56/220	112/220	48/220	4/220	

$$P_{X,Y}(1,1) = \frac{{}_3C_1\ {}_4C_1\ {}_5C_1}{{}_{12}C_3} = \frac{60}{220}$$

$$P_{X,Y}(2,1) = \frac{{}_3C_2\ {}_4C_1}{{}_{12}C_3} = \frac{12}{220}$$

Example The number of bytes N in a message has a geometric distribution with parameter p and range $S_N = \{0, 1, 2, \cdots\}$. Messages are broken into packets of maximum length M bytes. Let Q be the number of full packets in a message, and R be the number of bytes left over.

Find the joint pmf and the marginal pmf's of Q and R.

Solution

Q is the quotient of division of N by M, and R is the remaining bytes in the above division. Q takes on values in $\{0, 1, \cdots\}$; that is, all non-negative integers.

R takes on values in $\{0, 1, \cdots, M-1\}$.

Interestingly, the joint pmf is relatively easier to compute

$$P[Q = q, R = r] = P[N = qM + r] = (1-p)p^{qM+r}.$$

Marginal pmf of Q is given by

$$\begin{aligned} P[Q=q] &= P[N \text{ in } \{qM, qM+1, \cdots, qM+(M-1)\}] \\ &= \sum_{k=0}^{M-1} (1-p)p^{qM+k} \\ &= (1-p)p^{qM}\frac{1-p^M}{1-p} = (1-p^M)(p^M)^q, \quad q = 0, 1, 2, \cdots. \end{aligned}$$

The probability of achieving one full packet $= p^M$, so the probability of having q full packets $= (p^M)^q(1-p^M)$. The marginal pmf of Q is a geometric distribution with parameter p^M.

Marginal pmf of R is found to be

$$\begin{aligned} P[R=r] &= P[N \text{ in } \{r, M+r, 2M+r, \cdots\}] \\ &= \sum_{q=0}^{\infty} (1-p)p^{qM+r} = \frac{1-p}{1-p^M}p^r, \quad r = 0, 1, \cdots, M-1. \end{aligned}$$

R can be considered as a truncated geometric distribution.

Joint cdf of X and Y

Defined as the probability of the product-form event $\{X \leq x_1\} \cap \{Y \leq y_1\}$

$$F_{X,Y}(x_1, y_1) = P[X \leq x_1, Y \leq y_1]$$

Properties

(i) $F_{X,Y}(x_1, y_1) \leq F_{X,Y}(x_2, y_2), \qquad x_1 \leq x_2$ and $y_1 \leq y_2$

This is because $\{X \leq x_1\} \cap \{Y \leq y_1\}$ is a subset of $\{X \leq x_2\} \cap \{Y \leq y_2\}$.

(ii) $F_{X,Y}(-\infty, y_1) = F_{X,Y}(x_1, -\infty) = 0$

This is because $\{X \leq -\infty\} \cap \{Y \leq y_1\}$ and $\{X \leq x_1\} \cap \{Y \leq -\infty\}$ are impossible events.

(iii) $F_{X,Y}(\infty, \infty) = 1$

This is because $\{X < \infty\} \cap \{Y < \infty\}$ is the sure event.

(iv) marginal cumulative distribution functions

$$\begin{aligned} F_X(x) &= F_{X,Y}(x,\infty) = P[X \le x, Y < \infty] = P[X \le x] \\ F_Y(y) &= F_{X,Y}(\infty, y) = P[Y \le y] \end{aligned}$$

(v) joint cdf is continuous from the 'north' and the 'east'

$$\begin{aligned} \lim_{x \to a^+} F_{X,Y}(x,y) &= F_{X,Y}(a,y) \\ \lim_{y \to b^+} F_{X,Y}(x,y) &= F_{X,Y}(x,b) \end{aligned}$$

This is a generalization of the right continuity property of the one-dimensiona cdf.

Example $F_{X,Y}(x,y) = \begin{cases} (1-e^{-\alpha x})(1-e^{-\beta y}), & x \ge 0, y \ge 0 \\ 0 & \text{otherwise} \end{cases}$

then $F_X(x) = \lim_{y \to \infty} F_{X,Y}(x,y) = 1 - e^{-\alpha x}, x \ge 0$

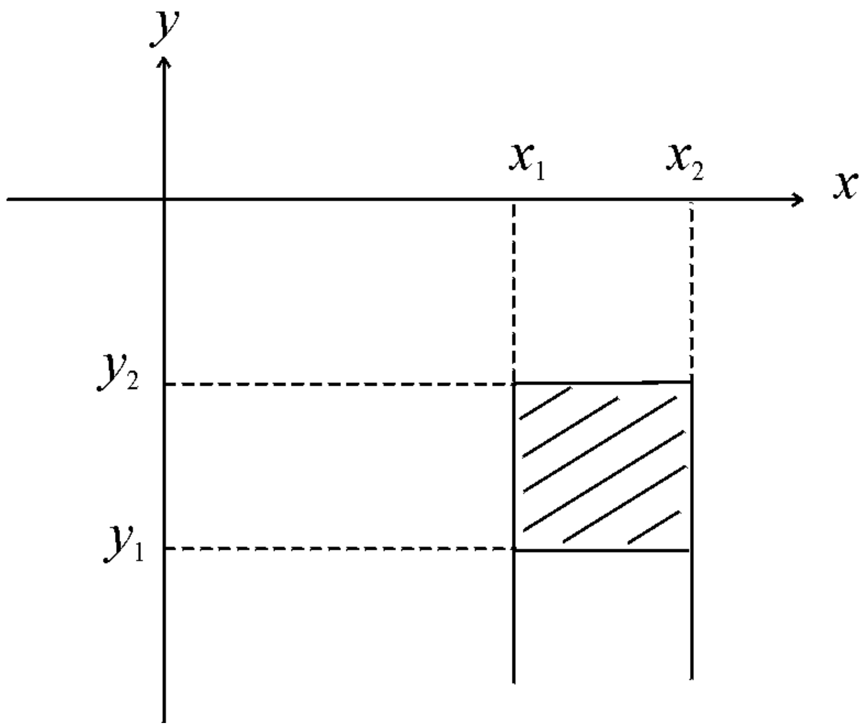
$F_Y(y) = \lim_{x \to \infty} F_{X,Y}(x,y) = 1 - e^{-\beta y}, y \ge 0.$

X and Y are exponentially distributed with respective parameter α and β.

The cdf can be used to find the probability of events that can be expressed as the union and intersection of semi-infinite rectangles.

In particular,

$$\begin{aligned} F_{X,Y}(x_2, y_2) = \quad & P[x_1 < X \le x_2, y_1 < Y \le y_2] \\ & + F_{X,Y}(x_2, y_1) - F_{X,Y}(x_1, y_1) + F_{X,Y}(x_1, y_2). \end{aligned}$$



Example

Given $F_{X,Y}(x,y) = \begin{cases} (1-e^{-\alpha x})(1-e^{-\beta y}), & x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$

$$\begin{aligned} & P[1 < X \leq 3, 2 < Y \leq 5] \\ = \; & F_{X,Y}(3,5) - F_{X,Y}(3,2) - F_{X,Y}(1,5) + F_{X,Y}(1,2) \\ = \; & (1-e^{-3\alpha})(1-e^{-5\beta}) - (1-e^{-3\alpha})(1-e^{-2\beta}) \\ & -(1-e^{-\alpha})(1-e^{-5\beta}) + (1-e^{-\alpha})(1-e^{-2\beta}). \end{aligned}$$

Joint pdf of two jointly continuous random variables

Joint cdf: $F_{X,Y}(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(x',y')\ dx'dy'$.

The pdf can be obtained from the cdf by differentiation:

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}.$$

$$P[a_1 < X \leq b_1, a_2 < Y \leq b_2] = \int_{a_2}^{b_2} \int_{a_1}^{b_1} f_{X,Y}(x',y')\ dx'dy'$$

and $P[x < X \leq x + dx, y < Y \leq y + dy]$

$$= \int_{y}^{y+dy} \int_{x}^{x+dx} f_{X,Y}(x',y')\ dx'dy' \approx f_{X,Y}(x,y)\ dxdy.$$

Since $P[-\infty < X < \infty, -\infty < Y < \infty] = 1$, we have

$$1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x',y')\ dx'dy'.$$

When the random variables X and Y are jointly continuous, the probability of an event involving (X, Y) can be expressed as an integral of a probability density function. For every event A, which is a subset of the plane, we have

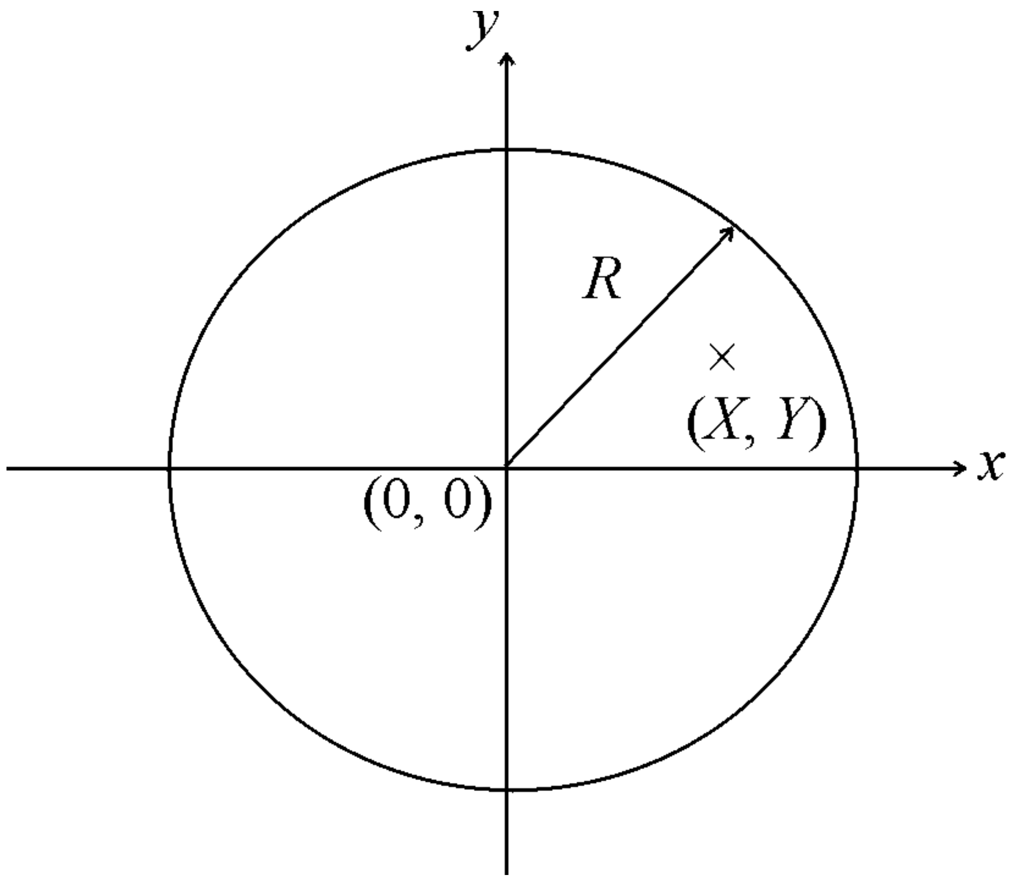
$$P[\widetilde{X} \text{ in } A] = \iint_A \underbrace{f_{X,Y}(x', y')}_{\text{non}-\text{negative}} \; dx'dy'$$

The marginal pdf's $f_X(x)$ and $f_Y(y)$ are obtained by differentiating the marginal cdf's.

From $F_X(x) = F_{X,Y}(x, \infty)$ and $F_Y(y) = F_{X,Y}(\infty, y)$, we have

$$\begin{aligned} f_X(x) &= \frac{d}{dx}\int_{-\infty}^{x}\left[\int_{-\infty}^{\infty} f_{X,Y}(x', y') \; dy'\right] \; dx' = \int_{-\infty}^{\infty} f_{X,Y}(x, y') \; dy' \\ f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x', y) \; dx'. \end{aligned}$$

Example Consider a circle of radius R and a point is uniformly distributed within the circle. Let the origin be the center of the circle, and X and Y be the coordinates of the point chosen.



(a) Since the chosen point is uniformly distributed inside the circle, the joint pdf of X and Y is

$$f_{XY}(x,y) = \begin{cases} C & \text{if} \quad x^2+y^2 \leq R^2 \\ 0 & \text{otherwise} \end{cases} .$$

From $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dxdy = \int\int_{x^2+y^2\leq R^2} C\ dxdy = 1$, we obtain $C = \dfrac{1}{\pi R^2}$.

For a fixed x, $f(x,y)$ is non-zero only within $-\sqrt{R^2-x^2} \leq y \leq \sqrt{R^2-x^2}$.

(b) The marginal density functions of X and Y are found to be

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\ dy = \frac{1}{\pi R^2}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy = \frac{2\sqrt{R^2-x^2}}{\pi R^2}, x^2 \leq R^2$$

and $f_X(x) = 0$ for $x^2 > R^2$;

$$f_Y(y) = \begin{cases} \frac{2}{\pi R^2}\sqrt{R^2-y^2}, & y^2 \leq R^2 \\ 0, & y^2 > R^2 \end{cases} .$$

(c) Compute the probability that the distance from the center to the chosen point is less than or equal to a.

Write $D = \sqrt{X^2 + Y^2}$, $F_D(a) = P[\sqrt{X^2 + Y^2} \le a] = \int\int_{x^2+y^2 \le a^2} \frac{1}{\pi R^2}\, dxdy$.

Upon simplification

$$F_D(a) = \frac{1}{\pi R^2} \int\int_{x^2+y^2 \le a^2} dxdy = \frac{\pi a^2}{\pi R^2} = \frac{a^2}{R^2}$$

so the density function for D is $f_D(a) = 2a/R^2, 0 \le a \le R$.

(d) Compute the mean of the distance of the chosen point from the center, $E[D]$

$$E[D] = \int_0^R a f_D(a)\, da = \int_0^R \frac{2a^2}{R^2}\, da = \frac{2R}{3}.$$

Jointly Gaussian random variables

X and Y are Gaussian random variables with zero mean and unit variance.

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-(x^2-2\rho xy+y^2)/2(1-\rho)^2}, \quad -\infty < x, y < \infty.$$

The marginal pdf of X is obtained by

$$\begin{aligned} f_X(x) &= \frac{e^{-x^2/2(1-\rho^2)}}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-(y^2-2\rho xy)/2(1-\rho^2)}\, dy \\ &= \frac{e^{-x^2/2(1-\rho^2)}}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-[(y-\rho x)^2-\rho^2 x^2]/2(1-\rho^2)}\, dy \\ &= \frac{e^{-x^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-(y-\rho x)^2/2(1-\rho^2)}}{\sqrt{2\pi(1-\rho^2)}}\, dy = \frac{e^{-x^2/2}}{\sqrt{2\pi}}. \end{aligned}$$

The last integral is recognized as the Gaussian pdf with mean ρx and variance $1-\rho^2$, so the value of integral is one.

Hence, $f_X(x)$ is the one-dim Gaussian pdf with zero mean and unit variance.

Independence of two random variables

Two events are independent if the knowledge that one has occurred gives no clue to the likelihood that the other will occur.

Let X and Y be discrete random variables. Let A_1 be the event that $X = x, A_2$ be the event that $Y = y$. If X and Y are independent, then A_1 and A_2 are independent.

$$P[A_1 \cap A_2] = P[A_1]P[A_2]$$

or $P_{X,Y}(x, y) = P[X = x \text{ and } Y = y] = P[X = x]P[Y = y] = P_X(x)P_Y(y)$.

For continuous random variables: $f_{XY}(x, y) = f_X(x)f_Y(y)$.

Example

$$\begin{aligned} X_1 &= \text{number of students attending the lecture on a given day} \\ X_2 &= \text{number of tests within that week} \\ X_3 &= \text{number of students having a cold} \\ X_4 &= \text{number of students having hair cut.} \end{aligned}$$

Which pair of random variables are independent?

Definition

Let X and Y be random variables with joint density f_{XY} and marginal densities f_X and f_Y, respectively.

X and Y are independent if and only if

$$f_{XY}(x,y) = f_X(x)f_Y(y), \quad \text{for all } x, y.$$

Remark

By integrating the above equation, we have

$$\int_{-\infty}^{y}\int_{-\infty}^{x} f_{XY}(x',y')\ dx'dy' = \int_{-\infty}^{y} f_Y(y')\ dy' \int_{-\infty}^{x} f_X(x')\ dx'$$

so that

$$F_{XY}(x,y) = F_X(x)F_Y(y) \quad \text{for all } x, y.$$

X and Y are independent if and only if their joint cdf is equal to the product of its marginal cdf's.

Example Consider the jointly distributed Gaussian random variables with the joint pdf:

$$f_{XY}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}}\, e^{-(x^2-2\rho xy+y^2)/2(1-\rho^2)}, \quad -\infty < x, y < \infty.$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}, \quad f_Y(y) = \frac{1}{\sqrt{2\pi}}\, e^{-y^2/2},$$

$$f_X(x) f_Y(y) = \frac{1}{2\pi}\, e^{-(x^2+y^2)/2}, \quad -\infty < x, y < \infty.$$

The product of the marginals equals the joint pdf if and only if $\rho = 0$. Hence, X and Y are independent if and only if $\rho = 0$.

What is the interpretation of ρ? It is related to a concept called *correlation* (to be discussed later).

Example

Let X and Y be zero-mean, unit variance independent Gaussian random variables. Find the value of r for which the probability that (X, Y) falls inside a circle of radius r is $1/2$.

Solution

Since X and Y are independent, the joint pdf for X and Y is

$$f_{XY}(x, y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2}.$$

In polar coordinates: $x = r\cos\theta, y = r\sin\theta$ so that $x^2 + y^2 = r^2$. The Jacobian of transformation is given by $dxdy = rdrd\theta$.

$$\begin{aligned} P[X^2 + Y^2 \leq r^2] &= \int_0^{2\pi} \int_0^r \frac{1}{2\pi} e^{-r'^2/2} r' dr' d\theta' \\ &= \int_0^r r' e^{-r'^2/2} dr' = e^{-r'^2/2}\Big]_r^0 = 1 - e^{-r^2/2}. \end{aligned}$$

Solve for r such that $1 - e^{-r^2/2} = 1/2$; we obtain $r = \sqrt{2\ln 2}$.

Theorem Let X and Y be independent random variables, then the random variables defined by $g(X)$ and $h(Y)$ are also independent.

Proof Consider the one-dimensional events A and B. Let A' be the set of all values of x such that if x is in A', then $g(x)$ is in A; let B' be the set of all values of y such that if y is in B', then $h(y)$ is in B.

That is, A' and B' are the equivalent event of A and B, respectively.

$$\begin{aligned} P[g(X) \text{ in } A, h(Y) \text{ in } B] &= P[X \text{ in } A', Y \text{ in } B'] \\ &= P[X \text{ in } A']P[A \text{ in } B'] \\ &= P[g(X) \text{ in } A]P[h(Y) \text{ in } B] \end{aligned}$$

The last but one equality follows from the independence of X and Y. The last equality is obtained since A and A', and B and B' are pairs of equivalent events.