Expected value of functions of random variables

Let $Z = g(X, Y)$, expected value of Z is given by

$$E[Z] = \begin{cases} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{XY}(x,y)\ dxdy & X, Y \text{ jointly continuous} \\ \sum_i \sum_j g(x_i, y_j) P_{XY}(x_i, y_j) & X, Y \text{ discrete} \end{cases}.$$

Sum of random variables

$$\begin{aligned} E[X+Y] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x'+y') f_{XY}(x', y')\ dx'dy' \\ &= \int_{-\infty}^{\infty} x' f_X(x')\ dx' + \int_{-\infty}^{\infty} y' f_Y(y')\ dy' = E[X] + E[Y]. \end{aligned}$$

In general, $E[X_1 + X_2 + \cdots + X_n] = E[X_1] + E[X_2] + \cdots E[X_n]$.

The random variables do not have to be independent in order that the above formula holds.

Independent random variables

Suppose X and Y are independent random variables, and $g(X, Y)$ is separable where $g(X, Y) = g_1(X)\ g_2(Y)$, then

$$\begin{aligned} E[g_1(X)g_2(Y)] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(x')g_2(y')f_X(x')f_Y(y')\ dx'dy' \\ &= \left[\int_{-\infty}^{\infty} g_1(x')f_X(x')\ dx'\right]\left[\int_{-\infty}^{\infty} g_2(y')f_Y(y')\ dy'\right] \\ &= E[g_1(X)]E[g_2(Y)]. \end{aligned}$$

In general, if $X_1, \cdots, X_n$ are independent random variables, then

$$E[g_1(X_1)g_2(X_2)\ \cdots g_n(X_n)] = E[g_1(X_1)]E[g_2(X_2)]\cdots E[g_n(X_n)].$$

Correlation and covariance of a pair of random variables

The jk^{th} joint moment of X and Y is defined by

$$E[X^j Y^k] = \begin{cases} \displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^j y^k f_{XY}(x,y)\; dxdy, & \text{X and Y jointly continuous} \\ \displaystyle\sum_i \sum_n x_i^j y_n^k P_{XY}(x_i, y_n) & X, Y \text{ discrete} \end{cases}.$$

When $E[XY] = 0$, then X and Y are *orthogonal*.

The jk^{th} **central moment of** X **and** Y is defined as

$$E[(X - E[X])^j (Y - E[Y])^k].$$

When $j = 2, k = 0$, it gives VAR(X); and when $j = 0, k = 2$, it gives VAR(Y). When $j = k = 1$, it gives COV$(X, Y) = E[(X - E[X])(Y - E[Y])]$.

Covariance

$$\begin{aligned}
\text{COV}(X,Y) &= E[XY - XE[Y] - YE[X] + E[X]E[Y]] \\
&= E[XY] - 2E[X]E[Y] + E[X]E[Y] \\
&= E[XY] - E[X]E[Y].
\end{aligned}$$

1. $\text{COV}[X,Y] = E[XY]$ if either of the random variables has mean zero.

2. When X and Y are independent, then $E[XY] = E[X]E[Y]$ so that

$$\text{COV}(X,Y) = 0.$$

3. $\text{COV}(X, \alpha Y_1 + Y_2) = \alpha\text{COV}(X, Y_1) + \text{COV}(X, Y_2)$

   Covariance is linear with respect to one of the arguments.

Correlation coefficient of X and Y

$$\rho_{XY} = \frac{\text{COV}(X,Y)}{\sigma_X \sigma_Y} = \frac{E[XY] - E[X]E[Y]}{\sigma_X \sigma_Y},$$

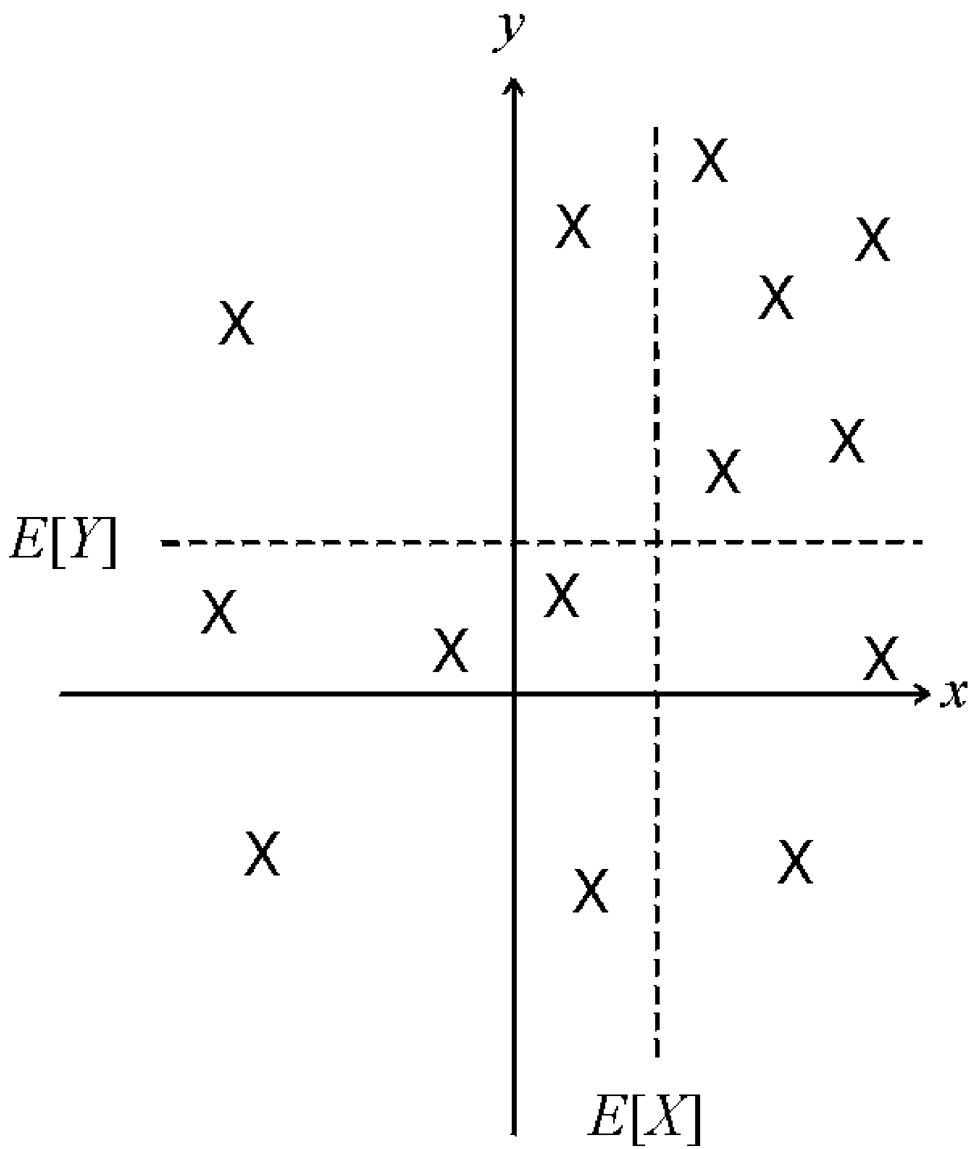
where σ_X and σ_Y are standard deviations of X and Y, respectively.

Properties of ρ_{XY}

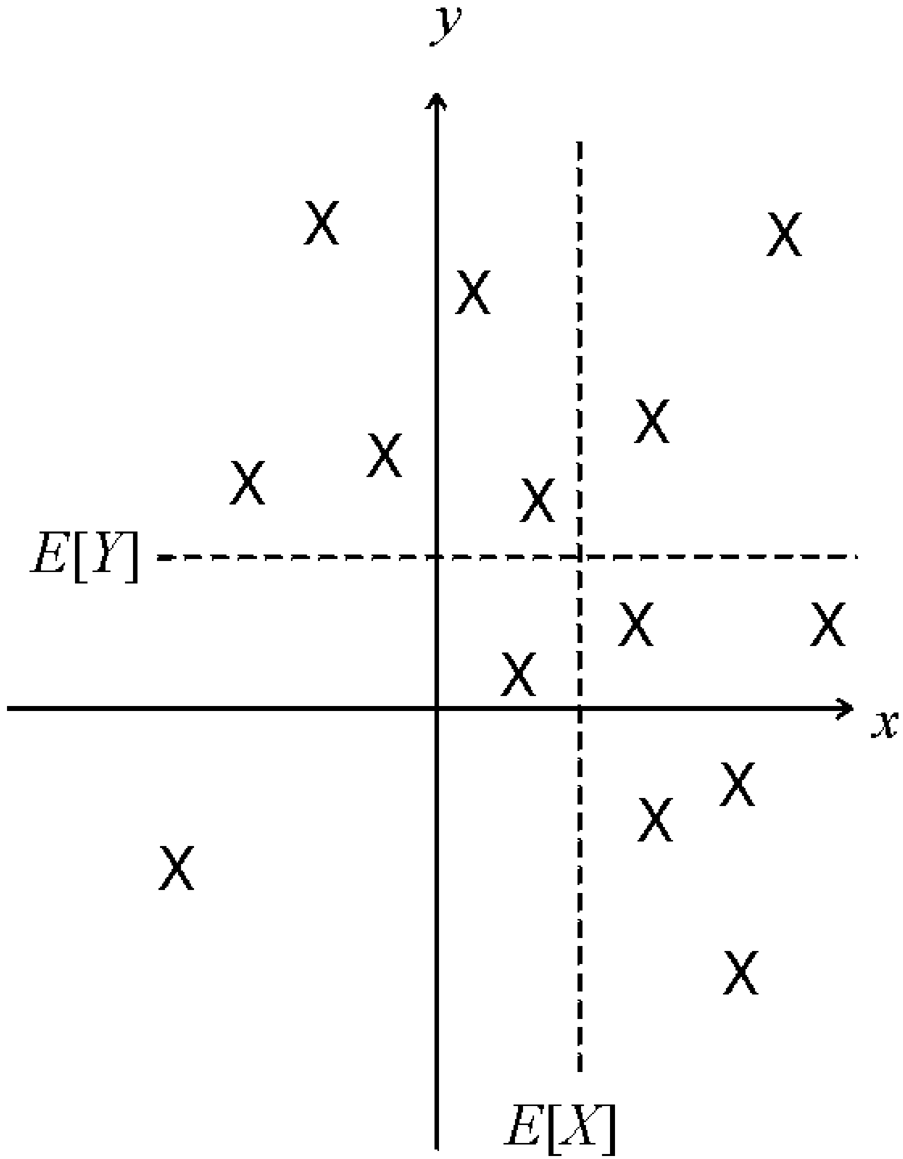
Consider

$$\begin{aligned} 0 &\leq E\left[\left(\frac{X - E[X]}{\sigma_X} \pm \frac{Y - E[Y]}{\sigma_Y}\right)^2\right] \\ &= E\left[\frac{(X - E[X])^2}{\sigma_X^2} \pm 2\frac{(X - E[X])(Y - E[Y])}{\sigma_X \sigma_Y} + \frac{(Y - E[Y])^2}{\sigma_Y^2}\right] \\ &= 1 \pm 2\rho_{XY} + 1 = 2(1 \pm \rho_{XY}), \end{aligned}$$

and so $-1 \leq \rho_{XY} \leq 1$.



positive correlation



negative correlation

The extreme values of ρ_{XY} are achieved when X and Y are related linearly.

When $Y = aX + b$,

$$\text{COV}(X,Y) = \text{COV}(X, aX+b) = a\text{COV}(X,X) + \text{COV}(X,b) = a\text{VAR}(X),$$

so $\rho_{XY} = \dfrac{a\sigma_X^2}{\sigma_X |a| \sigma_X} = \begin{cases} 1 & \text{if } a > 0 \\ -1 & \text{if } a < 0 \end{cases}$.

X and Y are said to be **uncorrelated** if $\rho_{XY} = 0$.

If X and Y are independent, then $\text{COV}(X,Y) = 0$, so $\rho_{XY} = 0$. We have independent $\Rightarrow$ uncorrelated.

How about uncorrelated $\Rightarrow$ independent?

As a special case, when X and Y are jointly Gaussian and $\rho_{XY} = 0$, then X and Y are independent random variables. However, it is possible for X and Y to be uncorrelated but not independent.

Example Let Θ be uniformly distributed in $[0, 2\pi]$. Let $X = \cos\Theta$ and $Y = \sin\Theta$. The marginal pdf's of X and Y are arcsine pdf's, where

$$f_X(x) = \frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1; \qquad f_Y(y) = \frac{1}{\pi}\frac{1}{\sqrt{1-y^2}}, \quad -1 < y < 1.$$

The product of the marginals is non-zero in the square

$$\{(x,y) : -1 < x < 1, \quad -1 < y < 1\}.$$

Proof by contradiction

If X and Y were independent, then

$$f_{XY}(x,y) = \frac{1}{\pi^2}\frac{1}{\sqrt{1-x^2}}\frac{1}{\sqrt{1-y^2}} > 0, -1 < x < 1 \text{ and } -1 < y < 1,$$

so that the point(X, Y) would assume all values in this square. However, (X, Y) must lie on the unit circle. Hence, (X, Y) cannot assume all values in the square, so X and Y are dependent. On the other hand, consider

$$E[XY] = E(\sin\Theta\cos\Theta) = \frac{1}{2\pi}\int_0^{2\pi} \sin\phi\cos\phi \; d\phi = 0.$$

Since $E[X] = E[Y] = 0$, we have $\rho_{XY} = \frac{E[XY]}{\sigma_X\sigma_Y} = 0$, so X and Y are uncorrelated.

Example Suppose X and Y have the joint pdf:

$$f_{XY}(x,y) = \begin{cases} 2e^{-x}e^{-y} & 0 \le y \le x < \infty \\ 0 & \text{otherwise} \end{cases} .$$

The marginal pdf's are found to be

$$f_X(x) = \int_0^x 2e^{-x}e^{-y}\ dy = 2e^{-x}(1 - e^{-x}), \qquad 0 \le x < \infty,$$

$$f_Y(y) = \int_y^\infty 2e^{-x}e^{-y}\ dx = 2e^{-2y}, \qquad 0 \le y < \infty.$$

$$\begin{aligned} E[X] &= \int_0^\infty 2xe^{-x}(1 - e^{-x})\ dx = \frac{3}{2}, \\ \text{VAR}[X] &= \int_0^\infty 2x^2e^{-x}(1 - e^{-x})\ dx - E[X]^2 = \frac{5}{4}; \\ E[Y] &= \frac{1}{2} \quad \text{and} \quad \text{VAR}(Y) = \frac{1}{4}. \end{aligned}$$

$$\begin{aligned} E[XY] &= \int_0^\infty \int_0^x 2xye^{-x}e^{-y}\ dydx \\ &= \int_0^\infty 2xe^{-x}(1 - e^{-x} - xe^{-x})\ dx = 1. \end{aligned}$$

Correlation coefficient, $\rho_{XY} = \dfrac{1 - \frac{3}{2} \cdot \frac{1}{2}}{\sqrt{\frac{5}{4}}\sqrt{\frac{1}{4}}} = \dfrac{1}{\sqrt{5}}.$

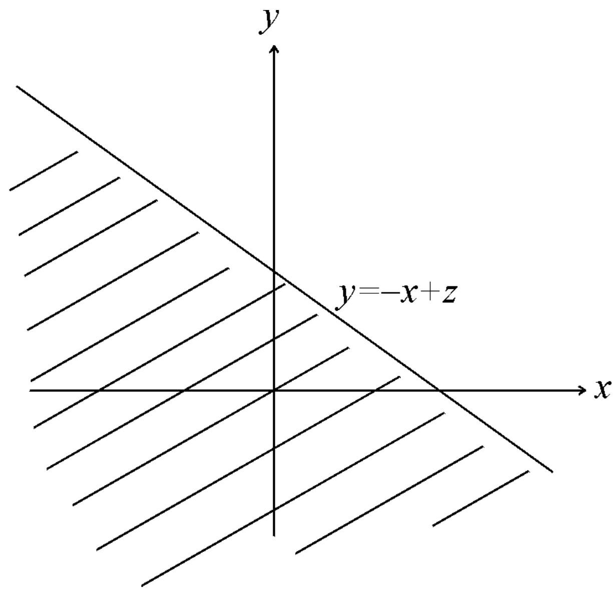
Function of several random variables

Suppose $Z = g(X_1, X_2, \cdots, X_n)$, where $X_1, \cdots, X_n$ are random variables. The cdf of Z is found by first finding the equivalent event of $\{Z \leq z\}$, i.e. the set $R_Z = \{\mathbf{x} = (x_1, \cdots, x_n)$ such that $g(\mathbf{x}) \leq z\}$. Now,

$$\begin{aligned} F_Z(z) &= P[\mathbf{x} \text{ in } R_Z] \\ &= \int_{\mathbf{x} \text{ in } R_Z} \cdots \int f_{X_1 X_2 \cdots, X_n}(x'_1, \cdots, x'_n)\, dx'_1, \cdots dx'_n. \end{aligned}$$

The pdf of Z is $f_Z(z) = \dfrac{d}{dz} F_Z(z)$.

Example Sum of two random variables



Let $Z = X + Y$

$$F_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-x'} f_{XY}(x', y')\ dy' dx'.$$

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x', z - x')\ dx'$$

If X and Y are independent random variables, then

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x') f_Y(z - x')\ dx'.$$

This is the convolution integral of the marginal pdf of X and Y.

Sum of two Gaussian random variables

Suppose X and Y are two zero-mean unit variance Gaussian random variables with correlation coefficient $\rho = \frac{1}{2}$, then

$$
\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty} f_{XY}(x', z - x')\ dx' \\
&= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left(\frac{-[x'^2 - 2\rho x'(z-x') + (z-x')^2]}{2(1-\rho^2)} \right)\ dx' \\
&= \frac{1}{2\pi(3/4)^{1/2}} \int_{-\infty}^{\infty} e^{-(x'^2 - x'z + z^2)/2(3/4)}\ dx' = \frac{e^{-z^2/2}}{\sqrt{2\pi}}.
\end{aligned}
$$

Therefore, the sum of two non-independent Gaussian random variables remains to be Gaussian.

Example (System with standby redundancy)

The system has a single key component in operation and a duplicate of that component in standby mode. When the first component fails, the second component is put into operation. Find the pdf of the lifetime of the standby system if the components have independent exponentially distributed lifetimes with the same mean.

Solution Let T_1 and T_2 be the lifetimes of the two components, so system lifetime $T = T_1 + T_2$.

$$f_{T_1}(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases} \qquad f_{T_2}(z-x) = \begin{cases} \lambda e^{-\lambda(z-x)} & z-x \geq 0 \\ 0 & x > z \end{cases}$$

$$\begin{aligned} f_T(z) &= \int_0^z \lambda e^{-\lambda x} \lambda e^{-\lambda(z-x)} dx \\ &= \lambda^2 e^{-\lambda z} \int_0^z \, dx = \lambda^2 z e^{-\lambda z}, \quad \text{for } z \geq 0; \\ f_T(z) &= 0, \quad \text{for } z < 0. \end{aligned}$$

The conditional pdf can be used to find the pdf of a function of several random variables.

Let $Z = g(X, Y)$, we first find the pdf of Z given $Y = y$, that is, $f_Z(z|Y = y)$. By the law of total probabilities, the pdf is given by $f_Z(z) = \int_{-\infty}^{\infty} f_Z(z|y')f_Y(y')\ dy'$.

Example Let $Z = X/Y$, find the pdf of Z if X and Y are independent and both are exponentially distributed with mean one.

Hint Recall that for $Y = \alpha X + \beta$, then $f_Y(y) = \frac{1}{|\alpha|} f_X\left(\frac{y-\beta}{\alpha}\right)$.

Solution

Suppose $Y = y, Z = \frac{1}{y}X$. Taking $\alpha = \frac{1}{y}$, we have

$$f_Z(z|y) = |y| f_X(yz|y);$$

$$\begin{aligned} \text{pdf of } Z = f_Z(z) &= \int_{-\infty}^{\infty} |y'| f_X(y'z|y') f_Y(y')\ dy' \\ &= \int_{-\infty}^{\infty} |y'| f_{XY}(y'z, y')\ dy'. \end{aligned}$$

If X and Y are independent and exponentially distributed with mean one, then

$$\begin{aligned} f_Z(z) &= \int_0^{\infty} y' f_X(y'z) f_Y(y')\ dy', & z > 0 \\ &= \int_0^{\infty} y' e^{-y'z} e^{-y'}\ dy' = \frac{1}{(1+z)^2}, & z > 0. \end{aligned}$$