Review of Topics — Single Random Variable

1. Discrete random variable arises when we count the occurrences of A. Different counting rules lead to different types of random variables.

2. Bernoulli random variable

$$\begin{aligned} \text{count} &= \text{number of “success” in a Bernoulli trial} \\ S_X &= \{0, 1\} \\ \text{pmf: } P_X(0) &= 1-p, P_X(1) = p. \end{aligned}$$

3. Binomial random variable

$$\begin{aligned} \text{count} &= \text{number of “successes” in } n \text{ Bernoulli trials} \\ S_X &= \{0, 1, 2, \cdots, n\} \\ \text{pmf: } P_X(k) &= C_k^n p^k (1-p)^{n-k}, \quad k = 0, 1, \cdots, n. \end{aligned}$$

   - Maximum value of $P_X(k)$ is obtained at $k_{max} = [(n+1)p]$.

4. Geometric random variable

$$\begin{aligned} \text{count} &= \text{number of trials required until the } 1^{\text{st}} \text{ “success” appears} \\ S_X &= \{1, 2, \cdots, n\} \\ \text{pmf: } P_X(k) &= (1-p)^{k-1} p, \quad k = 1, 2, \cdots. \end{aligned}$$

   - Memoryless property

$$P[X \geq k + j | X > j] = P[X \geq k], \quad k, j \geq 1.$$

5. Poisson random variable

$$\begin{aligned} \text{count} &= \text{number of “successes” over time period } [0, T] \\ S_X &= \{0, 1, 2, \cdots, n, \cdots\} \\ \text{pmf: } P_X(k) &= \frac{\alpha^k}{k!} e^{-\alpha}, k = 0, 1, 2, \cdots \end{aligned}$$

   - $\frac{\alpha^k}{k!} e^{-\alpha} \approx C_k^n p^k (1-p)^{n-k}, \quad \alpha = np$ (sufficiently accurate when $n \geq 10$ and $p \leq 0.1$)
   - $P_X(k)$ achieves maximum at $k = \begin{cases} 0, & \alpha < 1 \\ [\alpha], & \alpha \geq 1 \end{cases}$

6. $E[X]$ and VAR$[X]$ of a discrete random variable

$$\begin{aligned} &E[X] = \sum_k x_k P_X(x_k) \quad \text{weighted average} \\ &\text{VAR}[X] = E[X^2] - E[X]^2 \\ &\text{STD}[X] = \sqrt{\text{VAR}[X]} \end{aligned}$$

- $E[c] = c$ if c is a constant.
- $E[cX] = cE[X]$, c is a constant.

7. For binomial random variable

$$E[X] = np, \quad \text{VAR}[X] = np(1-p).$$

For geometric random variable

$$E[X] = \frac{1}{p}, \quad \text{VAR}[X] = \frac{1-p}{p^2}.$$

For Poisson random variable

$$E[X] = \text{VAR}[X] = \alpha.$$

8. Uniform random variable on $[a, b]$

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}.$$

For example, X = a number picked at random between 0 and 1.

9. Exponential random variable

$$\begin{aligned} S_X &= [0, \infty) \\ f_X(x) &= \begin{cases} 0, & x < 0 \\ \lambda e^{-\lambda x}, & x \ge 0, \end{cases} \\ F_X(x) &= \begin{cases} 0, & x < 0 \\ 1 - e^{-\lambda x}, & x \ge 0 \end{cases}. \end{aligned}$$

- Memoryless property

$$P[X > t + h | X > t] = P[X > h]$$

i.e., the probability of the 1^{st} arrival of event is independent of how long you have been waiting.

10. The exponential random variable is obtained as the limiting form of the geometric random variable.

11. For a Poisson random variable, the following random times have exponential distribution:
    (i) the waiting time until the 1^{st} success;
    (ii) the time between successes.

12. Normal random variable

$$S_X = (-\infty, \infty)$$

$$F_X(x) = N\left(\frac{x-m}{\sigma}\right), \quad N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-m)^2/2\sigma^2}.$$

- $E[X] = m$, $\text{VAR}[X] = \sigma^2$.

13. Central Limit Theorem

    If X_i are independent and identically distributed random variables with mean μ and variance σ^2, then $S_n = X_1 + X_2 + \cdots + X_n$ is a normal random variable with $E[S_n] = n\mu$ and $\text{VAR}[S_n] = n\sigma^2$.

14. Expectation

$$E[X] = \sum_k x_k P_X(x_k) \text{ for } X \text{ being discrete}$$

$$E[X] = \int_{-\infty}^{\infty} t f_X(t)\, dt \text{ for } X \text{ being continuous}$$

- If $f_X(m - x) = f_X(m + x)$, then $E[X] = m$
- For non-negative random variable X

(i) $E[X] = \int_0^{\infty} [1 - F_X(t)]\, dt$, X is continuous

(ii) $E[X] = \sum_{k=0}^{\infty} P[X > k], \quad S_X = \{0, 1, 2, \cdots\}$.

15. Function of a random variable: $Y = g(X)$
    - Given $y = g(x)$, find $F_Y(y)$ and $f_Y(y)$ using the concept of equivalent events.
    - If $g(x) = y$ has n solutions, $x_1, x_2, \cdots, x_n$, then

$$f_Y(y) = \sum_{k=1}^{n} \left. \frac{f_X(x)}{|dy/dx|} \right|_{x = x_k}.$$

16. Expectation of $Y = g(X)$

$$E[Y] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

$$E[Y] = \sum_k g(x_k) P_X(x_k)$$

- $E[c] = c$
- $E[cX] = cE[X]$
- $E\left[\sum_{k=1}^{n} g_k(X)\right] = \sum_{k=1}^{n} E[g_k(X)]$.