Review of Topics — Single Random Variable

1. Discrete random variable arises when we count the occurrences of A. Different counting rules lead to different types of random variables.

2. Bernoulli random variable

   \[
   \text{count} = \text{number of "success" in a Bernoulli trial} \\
   S_X = \{0, 1\} \\
   \text{pmf: } P_X(0) = 1 - p, P_X(1) = p.
   \]

3. Binomial random variable

   \[
   \text{count} = \text{number of "successes" in } n \text{ Bernoulli trials} \\
   S_X = \{0, 1, 2, \ldots, n\} \\
   \text{pmf: } P_X(k) = \binom{n}{k} p^k (1 - p)^{n-k}, \quad k = 0, 1, \ldots, n.
   \]

   • Maximum value of \( P_X(k) \) is obtained at \( k_{\text{max}} = \lfloor(n + 1)p\rfloor. \)

4. Geometric random variable

   \[
   \text{count} = \text{number of trials required until the 1st "success" appears} \\
   S_X = \{1, 2, \ldots, n\} \\
   \text{pmf: } P_X(k) = (1 - p)^{k-1} p, \quad k = 1, 2, \ldots.
   \]

   • Memoryless property
     \[
     P[X \geq k + j | X > j] = P[X \geq k], \quad k, j \geq 1.
     \]

5. Poisson random variable

   \[
   \text{count} = \text{number of "successes" over time period } [0, T] \\
   S_X = \{0, 1, 2, \ldots, n, \ldots\} \\
   \text{pmf: } P_X(k) = \frac{\alpha^k e^{-\alpha}}{k!}, k = 0, 1, 2, \ldots
   \]

   • \( \frac{\alpha^k e^{-\alpha}}{k!} \approx \binom{n}{k} p^k (1 - p)^{n-k}, \quad \alpha = np \) (sufficiently accurate when \( n \geq 10 \) and \( p \leq 0.1 \))

   • \( P_X(k) \) achieves maximum at \( k = \begin{cases} 0, & \alpha < 1 \\ \lfloor \alpha \rfloor, & \alpha \geq 1 \end{cases} \)

6. \( E[X] \) and \( \text{VAR}[X] \) of a discrete random variable

   \[
   E[X] = \sum_k x_k P_X(x_k) \quad \text{weighted average} \\
   \text{VAR}[X] = E[X^2] - E[X]^2 \\
   \text{STD}[X] = \sqrt{\text{VAR}[X]}
   \]
\begin{itemize}
  \item $E[c] = c$ if $c$ is a constant.
  \item $E[cX] = cE[X]$, $c$ is a constant.
\end{itemize}

7. For binomial random variable

$$E[X] = np, \quad \text{VAR}[X] = np(1 - p).$$

For geometric random variable

$$E[X] = \frac{1}{p}, \quad \text{VAR}[X] = \frac{1 - p}{p^2}.$$ 

For Poisson random variable

$$E[X] = \text{VAR}[X] = \alpha.$$ 

8. Uniform random variable on $[a, b]$

$$f_X(x) = \begin{cases} \frac{1}{b - a}, & a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}.$$ 

For example, $X$ - a number picked at random between 0 and 1.

9. Exponential random variable

$$S_X = (0, \infty)$$

$$f_X(x) = \begin{cases} 0, & x < 0 \\ \lambda e^{-\lambda x}, & x \geq 0 \end{cases}.$$ 

$$F_X(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\lambda x}, & x \geq 0 \end{cases}.$$ 

- Memoryless property

$$P[X > t + h | X > t] = P[X > h]$$

i.e., the probability of the 1st arrival of event is independent of how long you have been waiting.

10. The exponential random variable is obtained as the limiting form of the geometric random variable.

11. For a Poisson random variable, the following random times have exponential distribution:

   (i) the waiting time until the 1st success;

   (ii) the time between successes.

12. Normal random variable

$$S_X = (-\infty, \infty)$$

$$F_X(x) = N \left( \frac{x - m}{\sigma} \right), \quad N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-m)^2/2\sigma^2}.$$ 

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• $E[X] = m$, $\text{VAR}[X] = \sigma^2$.

13. Central Limit Theorem

If $X_i$ are independent and identically distributed random variables with mean $\mu$ and variance $\sigma^2$, then $S_n = X_1 + X_2 + \cdots + X_n$ is a normal random variable with $E[S_n] = n\mu$ and $\text{VAR}[S_n] = n\sigma^2$.

14. Expectation

$$E[X] = \sum_{k} x_k P_X(x_k) \text{ for } X \text{ being discrete}$$

$$E[X] = \int_{-\infty}^{\infty} t f_X(t) \, dt \text{ for } X \text{ being continuous}$$

• If $f_X(m - x) = f_X(m + x)$, then $E[X] = m$
• For non-negative random variable $X$
  (i) $E[X] = \int_{0}^{\infty} [1 - F_X(t)] \, dt$, $X$ is continuous
  (ii) $E[X] = \sum_{k=0}^{\infty} P[X > k]$, $S_X = \{0, 1, 2, \cdots\}$.

15. Function of a random variable: $Y = g(X)$

• Given $y = g(x)$, find $F_Y(y)$ and $f_Y(y)$ using the concept of equivalent events.
• If $g(x) = y$ has $n$ solutions, $x_1, x_2, \ldots, x_n$, then

$$f_Y(y) = \sum_{k=1}^{n} \frac{f_X(x)}{dy/dx} \bigg|_{x=x_k}.$$ 

16. Expectation of $Y = g(X)$

$$E[Y] = \int_{-\infty}^{\infty} g(x) f_X(x) \, dx$$

$$E[Y] = \sum_{k} g(x_k) P_X(x_k)$$

• $E[c] = c$
• $E[cX] = cE[X]$
• $E\left[\sum_{k=1}^{n} g_k(X)\right] = \sum_{k=1}^{n} E[g_k(X)].$