Review of Topics — Multiple Random Variables

1. For a pair of discrete random variables X and Y:
   - conditional pmf
   $$P_Y(y_j|x_k) = \frac{P_{X,Y}(x_k, y_j)}{P_X(x_k)}, \quad P_X(x_k) > 0$$
   - $P[Y \text{ in } A|X = x_k] = \sum_{y_j \text{ in } A} P_Y(y_j|x_k)$
   - X and Y are independent $\Longleftrightarrow$
   $$P_Y(y_j|x_k) = P_Y(y_j).$$

2. For a pair of continuous random variables X and Y:
   - conditional CDF
   $$\begin{aligned} F_Y(y|x) &= \lim_{h\to 0} F_Y(y|x < X \le x+h) \\ &= \frac{\int_{-\infty}^{y} f_{X,Y}(x,t)\ dt}{f_X(x)} \end{aligned}$$
   - conditional PDF
   $$f_Y(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}, \quad f_X(x) > 0$$
   - $P[Y \text{ in } A|X = x] = \int_{y \text{ in } A} f_Y(y|x)\ dy$
   - X and Y are independent $\Longleftrightarrow$
   $$\begin{aligned} & F_Y(y|x) = F_Y(y) \\ \text{or} \quad & f_Y(y|x) = f_Y(y). \end{aligned}$$

3. Theorem on Total Probabilities
   - Interpretation:
     To find $P[Y \text{ in } A]$, we first find $P[Y \text{ in } A|X = x]$, then average over x.
   - X discrete, Y discrete or continuous
   $$P[Y \text{ in } A] = \sum_{\text{all } x_k} P[Y \text{ in } A|X = x_k]P_X(x_k)$$
   - X continuous, Y discrete or continuous
   $$P[Y \text{ in } A] = \int_{-\infty}^{\infty} P[Y \text{ in } A|X] f_X(x)\ dx.$$

4. Conditional Expectation of Y
   - $E[Y|x] = \int_{-\infty}^{\infty} y f_Y(y|x)\ dy$, X and Y are continuous
   - $E[Y|x_i] = \sum_{j=1}^{\infty} y_j P_X(y_j|x_i)$, X and Y are discrete.

5. Properties of Conditional Expectation
   - $E[Y] = E[E[Y|X]]$
   - $E[g(Y)] = E[E[g(Y)|X]]$.

6. Expectation of $Z = g(X,Y)$

$$E[Z] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\ dxdy$$

$$E[Z] = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} g(x_i,y_j) P_{X,Y}(x_i,y_j)$$

   - $E[X+Y] = E[X] + E[Y]$
   - $E[g(X)h(Y)] = E[g(X)] \cdot E[h(Y)]$ if X and Y are independent.

7. Covariance of X and Y

$$\begin{aligned} \mathrm{COV}[X,Y] &= E[(X - E[X])(Y - E[Y])] \\ &= E[XY] - E[X]E[Y] \end{aligned}$$

8. Correlation coefficient of X and Y

$$\rho_{XY} = \frac{\mathrm{COV}(X,Y)}{\sigma_X \sigma_Y} = \frac{E[XY] - E[X]E[Y]}{\sigma_X \sigma_Y}$$

   - $-1 \le \rho_{XY} \le 1$
   - X and Y are independent $\Rightarrow \mathrm{COV}[X,Y] = 0 \Rightarrow \rho_{X,Y} = 0$ (uncorrelated)
   - Zero correlation does not imply independence.

9. CDF and PDF of $Z = g(X,Y)$
   - General approach

$$\begin{aligned} F_Z(z) &= P[Z \le z] = P[g(X,Y) \le z] \\ &= \iint_{g(x,y)<z} f_{X,Y}(x,y)\ dxdy \\ f_Z(z) &= \frac{d}{dz} F_Z(z). \end{aligned}$$

   - Using conditional PDF of Z

$$f_Z(z) = \int_{-\infty}^{\infty} f_Z(z|y) f_Y(y)\ dy.$$