Review of Topics — Random Processes

1. Random Process
   (a) Definition
   A random process is an indexed family of random variables
   \[ X(t) - X(\varepsilon, t), \quad t \in I. \]
   Equivalently, a random process is a function of \( \varepsilon \) and \( t \).
   (b) Interpretation of a random process \( X(t) \)
   • \( \varepsilon \) and \( t \) are variables:
     \( X(t) \) is a family of functions \( X(\varepsilon, t) \).
   • \( \varepsilon \) is fixed, \( t \) is a variable:
     \( X(t) \) is a single time function, called sample path.
   • \( \varepsilon \) is a variable, \( t \) is fixed:
     \( X(t) \) is a random variable
   • \( \varepsilon \) and \( t \) are fixed:
     \( X(t) \) is a number.
   (c) Let \( S \) be the sample space of \( X(t) \). Then \( X(t) \) is called
   • a discrete-valued process if \( S \) is discrete;
     a continuous-valued process if \( S \) is continuous;
   • a discrete-time process if \( I \) is discrete;
     a continuous-time process if \( I \) is continuous.
   (d) Examples of random processes
   • A binomial random process \( Y(n) = Y_n, n = 1, 2, \ldots \) versus \( Y \)
     \[ \begin{pmatrix} \text{discrete-time} \\ \text{discrete-valued} \end{pmatrix} \quad \text{(discrete random variable)} \]
   • Let \( Z(t) \) be the balance in your bank account at time \( t \), then \( Z(t), t \geq 0 \), is a continuous-time, continuous-valued random process.

2. Specifying a random process by joint CDF. A random process can be described by specifying the collection of \( k \)th-order joint CDF’s
   \[ F_{X_1,\ldots,X_k}(x_1, \ldots, x_k) = P[X_1 \leq x_1, \ldots, X_k \leq x_k] \]
   for all \( k \) and all choices at sampling instants \( t_1, t_2, \ldots, t_k \).

3. Mean, Autocorrelation and Autocovariance of \( X(t) \)
   • Mean
   \[ m_X(t) = E[X(t)] - \int_{-\infty}^{\infty} x f_X(x) \, dx \]
• Autocorrelation

\[ R_X(t_1, t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty} x_1 x_2 f_{X(t_1), X(t_2)}(x_1, x_2) \, dx_1 dx_2 \]

• Autocovariance

\[ C_X(t_1, t_2) = E[(X(t_1) - m_X(t_1))(X(t_2) - m_X(t_2))] = R_X(t_1, t_2) - m_X(t_1)m_X(t_2) \]

• \( \text{VAR}[X(t)] = E[(X(t) - m_X(t))^2] = C_X(t, t) \)

\[ \rho_X(t_1, t_2) = \frac{C_X(t_1, t_2)}{\sqrt{C_X(t_1, t_1) \cdot C_X(t_2, t_2)}} \]

4. Cross-correlation, cross-covariance of \( X(t) \) and \( Y(t) \)

• Cross-correlation

\[ R_{X,Y}(t_1, t_2) = E[X(t_1)Y(t_2)] \]

• Cross-covariance

\[ C_{X,Y}(t_1, t_2) = E[(X(t_1) - m_X(t_1))(Y(t_2) - m_Y(t_2))] = R_{X,Y}(t_1, t_2) - m_X(t_1)m_Y(t_2) \]

\( C_{X,Y}(t_1, t_2) = 0, \text{ for all } t_1, t_2 \)

\( \Rightarrow X(t) \text{ and } Y(t) \text{ are uncorrelated random processes.} \)

5. Independence of random process

• \( X(t) \) is called an independent random process if all subsets of random variables are independent, i.e.,

\[ X(t_1), X(t_2), \ldots, X(t_k) \]

are independent for all \( k \) and all choices of \( t_1, t_2, \ldots, t_k \).

• \( X(t) \) and \( Y(t) \) are said to be independent if

\[ [X(t_1), \ldots, X(t_k)] \text{ and } [Y(t'_1), \ldots, Y(t'_k)] \]

are independent random vectors for all \( k, j \) and all choices of \( t_1, t_2, \ldots, t_k \) and \( t'_1, t'_2, \ldots, t'_k \).

6. Independent and identically distributed random process

Denote \( m \) = common mean, \( \sigma^2 \) = common variance

• \( m_X(n) = m, \text{ independent of } n \)

• \( C_X(n, k) = \sigma^2 \delta_{n,k} \)

Example: Bernoulli random process
7. Increments of random process: \( X(t + h) - X(t) \), fixed \( h \)
   - \( X(t + h) - X(t) \) is a random variable
   - \( X(t) \) has stationary increments if random variables
     \[
     X(t_1 + h) - X(t_1) = Y_1, X(t_2 + h) - X(t_2) = Y_2
     \]
     have the same distribution for all \( t_1, t_2 \). That is,
     \[
     E[Y_1] = E[Y_2], \quad \text{VAR}[Y_1] = \text{VAR}[Y_2].
     \]
   - \( X(t) \) has independent increments if random variables
     \[
     X(t_2) - X(t_1), \ldots, X(t_k) - X(t_{k-1})
     \]
     are independent for all \( k \) and all choices of \( t_1 < t_2 < \cdots < t_k \).

8. Markov Process
   \( X(t) \) is said to be Markov if the future of the process given the present is independent of the past.

9. Sum Process, \( S_n \)
   - \( S_n = \sum_{i=1}^{n} X_i, X_i's \) are iid random variables, \( n \geq 1 \) and \( S_0 = 0 \)
   - \( S_n \) is a discrete-time random process
   - Denote \( m_x = \) common means of \( X_i's, \sigma_X^2 = \) common variance of \( X_i's \)
     \[
     m_S(n) = nm_X, \quad \text{VAR}[S(n)] = n\sigma_X^2
     \]
     \[
     C_S(n, k) = \min(n, k)\sigma_X^2
     \]
   - \( S_n \) is a Markov process since
     \[
     P[S_n - \alpha_n|S_{n-1} - \alpha_{n-1}] = P[S_n - \alpha_n|S_{n-1} - \alpha_{n-1}, \ldots, S_1 - \alpha_1]
     \]
   - \( S_n \) has independent and stationary increments. That is,
     \[
     f_{S_n, S_k}(y_n, y_k) = f_{S_n}(y_n)f_{S_{n-k}}(y_n - y_k), \quad k > n
     \]
   - \( S_n \) is called the binomial counting process if \( X_i's \) are iid Bernoulli random variables.

Poisson Process \( N(t) \)

10. Definition
     - \( N(t) \) - number of event occurrences in \([0, t]\)
     - pmf: \( P[N(t) = k] = \frac{(\lambda t)^k}{k!}e^{-\lambda t}, \quad k = 0, 1, 2, \ldots \)

11. Properties
     - \( N(t) \) is a non-decreasing, continuous-time and discrete-valued random process
• $N(t)$ has independent and stationary increments

12. Inter-event time $T$ in a Poisson Process
• $T$ – time between event occurrences
• Inter-event times $T_i$'s are i.i.d. exponential random variables with mean $\frac{1}{\lambda}$

13. Occurrence of the $n^{\text{th}}$ event, $S_n = \sum_{i=1}^{n} T_i$

14. The sum of independent Poisson random variables is also a Poisson random variable.