Review of Topics — Random Processes

1. Random Process
   (a) Definition

   A random process is an indexed family of random variables

$$X(t) = X(e,t), \quad t \in I.$$

   Equivalently, a random process is a function of e and t.

   (b) Interpretation of a random process $X(t)$
   - e and t are variables:
     $X(t)$ is a family of functions $X(e,t)$.
   - e is fixed, t is a variable:
     $X(t)$ is a single time function, called sample path.
   - e is a variable, t is fixed:
     $X(t)$ is a random variable
   - e and t are fixed:
     $X(t)$ is a number.

   (c) Let S be the sample space of $X(t)$. Then $X(t)$ is called
   - a discrete-valued process if S is discrete;
     a continuous-valued process if S is continuous;
   - a discrete-time process if I is discrete;
     a continuous-time process if I is continuous.

   (d) Examples of random processes
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A binomial random process		A binomial random variable
$Y(n) = Y_n, n = 1, 2, \cdots,$	versus	Y
$\begin{pmatrix}\text{discrete-time,} \\ \text{discrete-valued}\end{pmatrix}$		(discrete random variable)

   - Let $Z(t)$ be the balance in your bank account at time t, then $Z(t), t \geq 0$, is a continuous-time, continuous-valued random process.

2. Specifying a random process by joint CDF. A random process can be described by specifying the collection of k^{th}-order joint CDF's

$$F_{X_1 \cdots X_k}(x_1, \cdots, x_k) = P[X_1 \leq x_1, \cdots, X_k \leq x_k]$$

   for all k and all choices at sampling instants $t_1, t_2, \cdots, t_k$.

3. Mean, Autocorrelation and Autocovariance of $X(t)$
   - Mean

$$m_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x f_{X(t)}(x)\, dx$$

- Autocorrelation

$$\begin{aligned} R_X(t_1, t_2) &= E[X(t_1)X(t_2)] \\ &= \int_{-\infty}^{\infty} x_1 x_2 f_{X(t_1)X(t_2)}(x_1, x_2)\ dx_1 dx_2 \end{aligned}$$

- Autocovariance

$$\begin{aligned} C_X(t_1, t_2) &= E[\{X(t_1) - m_X(t_1)\}\{X(t_2) - m_X(t_2)\}] \\ &= R_X(t_1, t_2) - m_X(t_1)m_X(t_2) \end{aligned}$$

- $\mathrm{VAR}[X(t)] = E[\{X(t) - m_X(t)\}^2] = C_X(t, t)$

$$\rho_X(t_1, t_2) = \frac{C_X(t_1, t_2)}{\sqrt{C_X(t_1, t_1) \cdot C_X(t_2, t_2)}}$$

4. Cross-correlation, cross-covariance of $X(t)$ and $Y(t)$
   - Cross-correlation

$$R_{X,Y}(t_1, t_2) = E[X(t_1)Y(t_2)]$$

   - Cross-covariance

$$\begin{aligned} C_{X,Y}(t_1, t_2) &= E[\{X(t_1) - m_X(t_1)\}\{Y(t_2) - m_Y(t_2)\}] \\ &= R_{X,Y}(t_1, t_2) - m_X(t_1)m_Y(t_2) \end{aligned}$$

   $C_{X,Y}(t_1, t_2) = 0$, for all t_1, t_2
   $\Rightarrow X(t)$ and $Y(t)$ are uncorrelated random processes.

5. Independence of random process
   - $X(t)$ is called an independent random process if all subsets of random variables are independent, i.e.,

$$X(t_1), X(t_2), \cdots, X(t_k)$$

     are independent for all k and all choices of $t_1, t_2, \cdots, t_k$.
   - $X(t)$ and $Y(t)$ are said to be independent if

$$[X(t_1), \cdots, X(t_k)] \quad \text{and} \quad [Y(t'_1), \cdots, Y(t'_k)]$$

     are independent random vectors for all k, j and all choices of $t_1, t_2, \cdots, t_k$ and $t'_1, t'_2, \cdots, t'_k$.

6. Independent and identically distributed random process
   Denote m = common mean, σ^2 = common variance
   - $m_X(n) = m$, independent of n
   - $C_X(n, k) = \sigma^2 \delta_{n,k}$
     Example: Bernoulli random process

7. Increments of random process: $X(t+h) - X(t)$, fixed h
   - $X(t+h) - X(t)$ is a random variable
   - $X(t)$ has stationary increments if random variables

$$X(t_1 + h) - X(t_1) = Y_1, X(t_2 + h) - X(t_2) = Y_2$$

     have the same distribution for all t_1, t_2. That is,

$$E[Y_1] = E[Y_2], \quad \text{VAR}[Y_1] = \text{VAR}[Y_2].$$

   - $X(t)$ has independent increments if random variables

$$X(t_2) - X(t_1), \cdots, X(t_k) - X(t_{k-1})$$

     are independent for all k and all choices of $t_1 < t_2 < \cdots < t_k$.

8. Markov Process
   $X(t)$ is said to be Markov if the future of the process given the present is independent of the past.

9. Sum Process, S_n
   - $S_n = \sum_{i=1}^{n} X_i, X_i$'s are iid random variables, $n \geq 1$ and $S_0 = 0$
   - S_n is a discrete-time random process
   - Denote m_x = common means of X_i's, σ_X^2 = common variance of X_i's

$$m_S(n) = n m_X, \quad \text{VAR}[S(n)] = n\sigma_X^2$$
$$C_S(n,k) = \min(n,k)\sigma_X^2$$

   - S_n is a Markov process since

$$\begin{aligned} & P[S_n = \alpha_n | S_{n-1} = \alpha_{n-1}] \\ = & P[S_n = \alpha_n | S_{n-1} = \alpha_{n-1}, \cdots, S_1 = \alpha_1] \end{aligned}$$

   - S_n has independent and stationary increments. That is,

$$f_{S_n, S_k}(y_n, y_k) = f_{S_n}(y_n) f_{S_{n-k}}(y_n - y_k), \quad k > n$$

   - S_n is called the binomial counting process if X_i's are iid Bernoulli random variables.

Poisson Process $N(t)$

10. Definition
    - $N(t)$ = number of event occurrences in $[0, t]$
    - pmf: $P[N(t) = k] = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad k = 0, 1, 2, \cdots$

11. Properties
    - $N(t)$ is a non-decreasing, continuous-time and discrete-valued random process

- $N(t)$ has independent and stationary increments

12. Interevent time T in a Poisson Process
    - T = time between event occurrences
    - Interevent times T_i's are iid exponential random variables with mean $\frac{1}{\lambda}$

13. Occurrence of the n^{th} event, $S_n = \sum_{i=1}^{n} T_i$

14. The sum of independent Poisson random variables is also a Poisson random variable.