MATH 246 — Probability and Random Processes

Test Two

Fall 2003 **Course Instructor:** *Prof. Y. K. Kwok*

Time allowed: 75 minutes

[points]

1. (a) Suppose X has a probability density function f_X and define $Y = |X| + 1$. Express the probability density f_Y of the random variable Y in terms of f_X.

   Hint: In your answer for $f_Y(y)$, distinguish between $y \geq 1$ and $y < 1$. **[3]**

   (b) Let X be the standard Gaussian random variable with zero mean and unit standard deviation. Using (a) or otherwise, find the density function of $|X| + 1$. Specify the density function over the whole range $(-\infty, \infty)$. **[2]**

2. Let X and Y be continuous random variables with joint density function:

$$f_{XY}(x,y) = \begin{cases} e^{-y} & \text{for } 0 < x < y < \infty \\ 0 & \text{otherwise} \end{cases}.$$

   (a) Compute the marginal density functions $f_X(x)$ and $f_Y(y)$. Specify the density functions over the whole range $(-\infty, \infty)$. **[2]**

   (b) Are X and Y independent? Explain. **[1]**

   (c) Determine the conditional density $f_X(x|y)$. Be careful that $f_X(x|y)$ takes different forms over different regions in the x-y plane. **[1]**

   (d) Compute $E[X|y]$.

   Hint: $E[X|y] = \int_{-\infty}^{\infty} x f_X(x|y)\, dx$. Be careful that for certain range of x, $f_X(x|y) = 0$. **[2]**

3. (a) Show that the correlation coefficient ρ_{XY} between a pair of random variables X and Y observes

$$-1 \leq \rho_{XY} \leq 1.$$

   Hint: Consider $E\left[\left(\frac{X - E[X]}{\sigma_X} \pm \frac{Y - E[Y]}{\sigma_Y}\right)^2\right]$. **[4]**

   (b) Let X be an exponential random variable with parameter $\lambda > 0$. Define $Y = aX + b$, where a and b are constants. Find the pdf of Y. Find the condition on a and b such that Y remains exponential.

   Hint: For $Y = aX + b$, $f_Y = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$. **[3]**

4. Consider a sequence of $n+m$ independent Bernuolli trials with probability of success p in each trial. Let N be the number of successes in the first n trials and A be the number of successes in **all** $m+n$ trials.

   (a) Find the joint pmf of N and A. **[4]**

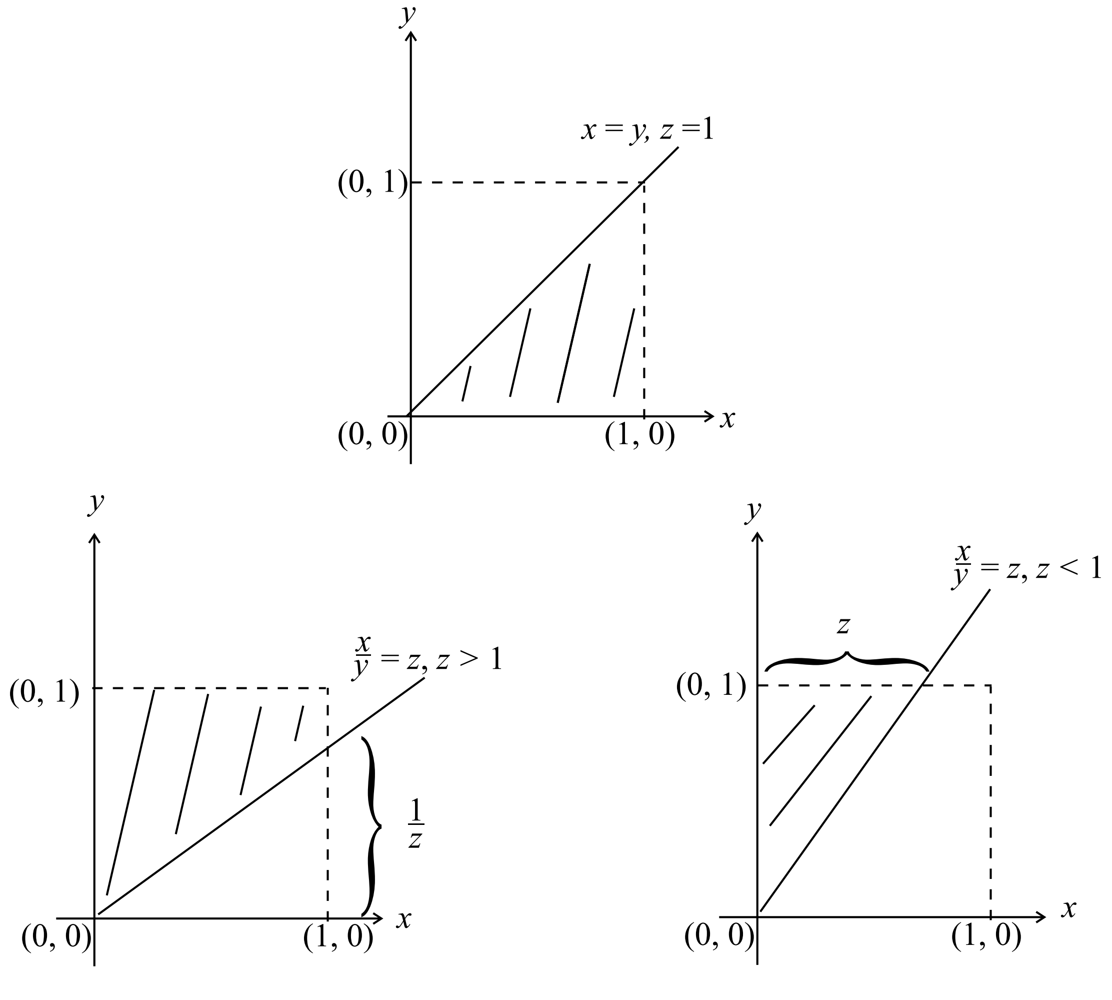
   (b) Find the marginal pmf's of N and A? **[2]**

   (c) Are N and A independent random variables? Give you reasoning. **[1]**

   Hint: The number of successes in the first n trials cannot be greater than the number of successes in *all* $m+n$ trials.

5. Let X and Y be independent and uniformly distributed over $(0,1)$, compute $P[X \geq Y]$. Also, find the cdf of $Z = X/Y$. **[5]**

   Hint: Consider the following figures:



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