MATH 246 — Probability and Random Processes

Solution to Test Two

Fall 2003 **Course Instructor:** *Prof. Y. K. Kwok*

---

1. (a) For $y \geq 1$,

$$\begin{aligned} F_Y(y) &= P[Y \leq y] \\ &= P[|X| \leq y-1] \\ &= P[-(y-1) \leq X \leq y-1] \\ &= F_X(y-1) - F_X(1-y). \end{aligned}$$

Upon differentiation, we obtain

$$f_Y(y) = f_X(y-1) + f_X(1-y), \quad y \geq 1.$$

For $y < 1$, $f_Y(y) = 0$.

(b) $f_X(x) = \dfrac{1}{\sqrt{2\pi}} e^{-x^2/2}$ so that

$$f_Y(y) = \begin{cases} \frac{2}{\sqrt{2\pi}} e^{-(y-1)^2/2} & \text{if } y \geq 1 \\ 0 & \text{otherwise} \end{cases}.$$

2. (a)

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)\ dy \\ &= \int_x^{\infty} e^{-y}\ dy \\ &= -e^{-y}\Big]_x^{\infty} = e^{-x} \qquad \text{for } x > 0; \\ f_X(x) &= 0 \qquad \text{for } x \leq 0. \\ f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x,y)\ dx \\ &= \int_0^y e^{-y}\ dx = ye^{-y} \qquad \text{for } y > 0; \\ f_Y(y) &= 0 \qquad \text{for } y \leq 0. \end{aligned}$$

(b) Since $f_{XY}(x,y) \neq f_X(x)f_Y(y)$, so X and Y are not independent.

(c)

$$\begin{aligned} f_X(x|y) &= \frac{f_{XY}(x,y)}{f_Y(y)} \\ &= \begin{cases} \frac{1}{y} & \text{for } 0 < x < y \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

(d)

$$\begin{aligned} E[X|y] &= \int_{-\infty}^{\infty} x f_X(x|y)\ dx \\ &= \int_0^y x\left(\frac{1}{y}\right)\ dx = \frac{y}{2} \quad \text{for} \quad y > 0. \end{aligned}$$

By convention, $E[X|y] = 0$ for $y \leq 0$.

3. (a) $$0 \le E\left[\left(\frac{X-E[X]}{\sigma_X} \pm \frac{Y-E[Y]}{\sigma_Y}\right)^2\right]$$
$$= E\left[\frac{(X-E[X])^2}{\sigma_X^2}\right] \pm 2E\left[\frac{(X-E[X])(Y-E[Y])}{\sigma_X\sigma_Y}\right] + E\left[\frac{(Y-E[Y])^2}{\sigma_Y^2}\right]$$
$$= 1 \pm 2\rho_{XY} + 1 = 2(1 \pm \rho_{XY})$$

so that
$$-1 \le \rho_{XY} \le 1.$$

(b) $$f_X(x) = \lambda e^{-\lambda x}, \quad \lambda > 0$$
$$f_Y(y) = \frac{\lambda}{|a|} e^{-\lambda(y-b)/a}$$

If $b = 0$ and $a > 0$, then $f_Y(y) = \dfrac{\lambda}{a} e^{-\frac{\lambda}{a}y}$. In this case, Y is an exponential random variable.

4. (a) $P_{N,A}(k,j) = P[N=k, A=j]$. Note that $P[N=k, A=j] = 0$ if $k > j$. Suppose $k \le j$, then

$$\begin{aligned} P[N=k, A=j] &= P[k \text{ successes in first } n \text{ trials, } j-k \text{ successes in last } m \text{ trials}] \\ &= {}_nC_k p^k (1-p)^{n-k} {}_mC_{j-k} p^{j-k} (1-p)^{m-j+k}. \end{aligned}$$

(b) $P_N(k) = P[N=k] = {}_nC_k p^k (1-p)^{n-k}$,
$P_A(j) = P[A=j] = {}_{n+m}C_j p^j (1-p)^{n+m-j}$.

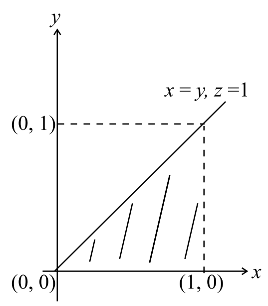
(c) Since $P_{N,A}(k,j) \ne P_N(k)P_A(j)$ so N and A are not independent.

5. Since X and Y are independent, we have

$$f_{XY}(x,y) = f_X(x) f_Y(y) = 1 \quad \text{for} \quad 0 < x < 1, 0 < y < 1$$

and $f_{XY}(x,y) = 0$ if otherwise.

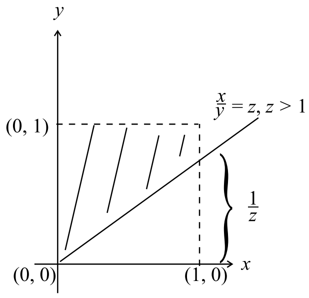
$$\begin{aligned} P[X \ge Y] &= \iint_{x>y} f_{XY}(x,y)\, dxdy \\ &= \text{area of the triangle bounded by} \\ &\quad x = y, y = 0 \text{ and } x = 1 \\ &= \frac{1}{2}. \end{aligned}$$

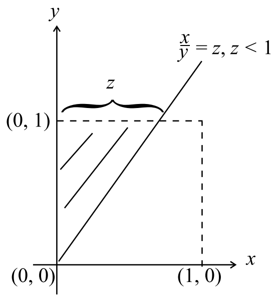


$$F_Z(z) = P\left[\frac{X}{Y} \le z\right]$$

(i) $z \geq 1$

$$F_Z(z) = 1 - \frac{1}{2z}$$





(ii) $0 < z < 1$

$$F_Z(z) = \frac{z}{2}.$$

Hence, the cdf of Z is given by

$$F_Z(z) = \begin{cases} \frac{z}{2} & 0 < z < 1 \\ 1 - \frac{1}{2z} & z \geq 1 \\ 0 & \text{otherwise} \end{cases}.$$