MATH 246 — Probability and Random Processes

Solution to Mid-term Test

Fall 1999 **Course Instructor:** *Prof. Y. K. Kwok*

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1. (a) Since $F = E \cup (E^C \cap F)$ and $E \cap (E^C \cap F) = \phi$,
   so $P[F] = P[E] + P[E^C \cap F] \geq P[E] \quad$ since $\quad P[E^C \cap F] \geq 0$.

   (b) Since $(E \cup F) \subset S$, so $P[E \cup F] \leq P[S] = 0 \quad$ by (a).
   On the other hand,
   $$P[E \cup F] = P[E] + P[F] - P[E \cap F] \leq 1$$
   so $\quad P[E \cap F] \geq P[E] + P[F] - 1$.

   (c) E and F are independent
   $\Rightarrow \quad P[F|E] = P[F]$
   $\Rightarrow \quad 1 - P[F|E] = 1 - P[F]$
   $\Rightarrow \quad P[F^c|E] = P[F^c]$
   $\Rightarrow \quad E$ and F^c are independent.

2. Define $A = \{\text{a person has cancer}\}$ and $T = \{\text{test result indicates a person has cancer}\}$.
   Given $\quad P[A] = 0.004, \qquad P[T|A] = 0.95,$
   $\qquad\quad P[A^C] = 0.996, \qquad P[T|A^C] = 0.05.$
   Note that $\{A, A^C\}$ forms a partition of the sample space. By Bayes's theorem,
   $$\begin{aligned}\text{required probability} &= P[A|T] \\ &= \frac{P[T|A]P[A]}{P[T|A]P[A] + P[T|A^C]P[A^C]} \\ &= \frac{(0.95)(0.004)}{(0.95)(0.004) + (0.05)(0.996)}.\end{aligned}$$

3. (a) Since $\displaystyle\int_{-\infty}^{\infty} f_X(x)\, dx = 1$
   $$\begin{aligned}\text{so} \quad 1 &= \int_0^1 Cx(1-x)\, dx \\ &= C\left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Bigg|_0^1 = \frac{C}{6}\end{aligned}$$
   $\Rightarrow \quad C = 6.$
   $$\text{i.e. } f_X(x) = \begin{cases} 6x(1-x), & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}.$$
   $$P\left[\frac{1}{2} \leq X \leq \frac{3}{4}\right] = \int_{\frac{1}{2}}^{\frac{3}{4}} 6x(1-x)\, dx$$

$$= (3x^2 - 2x^3)\Big|_{\frac{1}{2}}^{\frac{3}{4}}$$
$$= \left(3 \times \frac{9}{16} - 2 \times \frac{27}{64}\right) - \left(3 \times \frac{1}{4} - 2 \times \frac{1}{8}\right)$$
$$= \frac{11}{32}.$$

(b) $F_X(x) = \int_{-\infty}^{x} f_X(t)\ dt$

(i) when $x < 0$, $F_X(x) = \int_{-\infty}^{x} 0\ dt = 0$;

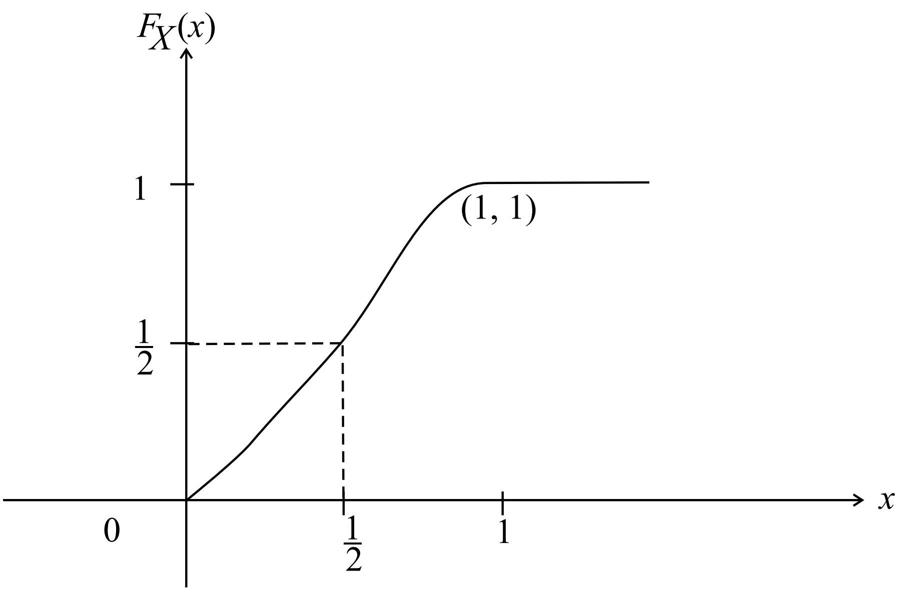
(ii) when $0 \leq x \leq 1$,

$$\begin{aligned} F_X(x) &= \int_{-\infty}^{0} 0\ dt + \int_{0}^{x} 6t(1-t)\ dt \\ &= (3t^2 - 2t^3)\Big|_{0}^{x} \\ &= 3x^2 - 2x^3; \end{aligned}$$

(iii) when $x > 1$,

$$\begin{aligned} F_X(x) &= \int_{-\infty}^{0} 0\ dt + \int_{0}^{1} 6t(1-t)\ dt + \int_{1}^{x} 0\ dt \\ &= (3t^2 - 2t^3)\Big|_{0}^{1} \\ &= 1. \end{aligned}$$

Hence, $F_X(x) = \begin{cases} 0, & x < 0 \\ 3x^2 - 2x^3, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$.



4. (a) First,
$$\begin{aligned} P[N > m] &= \sum_{k=m+1}^{\infty} (1-p)^{k-1} p \\ &= p(1-p)^m \sum_{k=0}^{\infty} (1-p)^k \\ &= p(1-p)^m \frac{1}{1-(1-p)} \\ &= (1-p)^m. \end{aligned}$$

Now,
$$\begin{aligned} P[N = k | N \leq m] &= \frac{P[N = k \cap N \leq m]}{1 - P[N > m]} \\ &= \frac{P[N = k \cap N \leq m]}{1-(1-p)^m} \end{aligned}$$

(i) If $1 \leq k \leq m, \{N = k\} \cap \{N \leq m\} = \{N = k\}$

so
$$\begin{aligned} P[N = k | N \leq m] &= \frac{P[N = k]}{1-(1-p)^m} \\ &= \frac{(1-p)^{k-1} p}{1-(1-p)^m} \end{aligned}$$

(ii) If $k > m, \{N = k\} \cap \{N \leq m\} = \phi$ so $P[N = k | N \leq m] = 0$.