Worked examples — Basic Concepts of Probability Theory

Example 1 A regular tetrahedron is a body that has four faces and, if is tossed, the probability that it lands on any face is 1/4. Suppose that one face of a regular tetrahedron has three colors: red, green, and blue. The three faces each have only one color: red, blue, and green, respectively. We throw the tetrahedron once and let R, G, and B be the events that the face on which it lands contains red, green and blue, respectively. Then, $P(R|G) = 1/2 = P(R), P(R|B) = 1/2 = P(R)$, and $P(B|G) = 1/2 = P(B)$. Thus the events R, B and G are pairwise independent. However, R, B and G are not independent events since $P(R|GB) = 1 \neq P(R)$.

Example 2 A box contains 7 red and 13 blue balls. Two balls are selected at random and are discarded without their colors being seen. If a third ball is drawn randomly and observed to be red, what is the probability that both of the discarded balls were blue?

Solution Let BB, BR, and RR be the events that the discarded balls are blue and blue, blue and red, red and red, respectively. Also, let R be the event that the third ball drawn is red. Since $\{BB, BR, RR\}$ is a partition of the sample space, Bayes' formula can be used to calculate $P(BB|R)$.

$$P(BB|R) = \frac{P(R|BB)P(BB)}{P(R|BB)P(BB) + P(R|BR)P(BR) + P(R|RR)P(RR)}.$$

Now

$$P(BB) = \frac{13}{20} \times \frac{12}{19} = \frac{39}{95}, \quad P(RR) = \frac{7}{20} \times \frac{6}{19} = \frac{21}{190},$$

and

$$P(BR) = \frac{13}{20} \times \frac{7}{19} + \frac{7}{20} \times \frac{13}{19} = \frac{91}{190},$$

where the last equation follows since BR is the union of two disjoint events: namely, the first ball discarded was blue, the second was red, and vice versa. Thus

$$P(BB|R) = \frac{\frac{7}{18} \times \frac{39}{95}}{\frac{7}{18} \times \frac{39}{95} + \frac{6}{18} \times \frac{91}{190} + \frac{5}{18} \times \frac{21}{190}} \approx 0.46.$$

Example 3 Let X be the number of births in a hospital until the first girl is born. Determine the probability function and the distribution function of X. Assume that the probability is 1/2 that a baby born is a girl.

Solution X is a random variable that can assume any positive integer i. $p(i) = P(X = i)$, and $X = i$ occurs if the first $i - 1$ births are all boys and the ith birth is a girl. Thus $p(i) = (1/2)^{i-1}(1/2) = (1/2)^i$ for $i = 1, 2, 3, \cdots$, and $p(x) = 0$ if $x \neq 1, 2, 3, \cdots$. To determine $F(t)$, note that for $t < 1, F(t) = 0$; for $1 \leq t < 2, F(t) = 1/2$; for $2 \leq t < 3, F(t) = 1/2 + 1/4 = 3/4$; for $3 \leq t < 4, F(t) = 1/2 + 1/4 + 1/8 = 7/8$; and in general for $n - 1 \leq t < n$,

$$
\begin{aligned}
F(t) &= \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}} = \sum_{i=1}^{n-1} \left(\frac{1}{2}\right)^i \\
&= \frac{1 - (1/2)^2}{1 - 1/2} - 1 = 1 - \left(\frac{1}{2}\right)^{n-1}.
\end{aligned}
$$

by the partial sum formula for geometric series. Thus

$$
F(t) = \begin{cases} 0 & t < 1 \\ 1 - (1/2)^{n-1} & n - 1 \leq t < n, \quad n = 2, 3, 4, \cdots. \end{cases}
$$

Example 4 Independent trials, consisting of rolling a pair of fair dice, are performed. What is the probability that an outcome of 5 appears before an outcome of 7 when the outcome of a roll is the sum of the dice?

Solution If we let E_n denote the event that no 5 or 7 appears on the first $n - 1$ trials and a 5 appears on the nth trial, then the desired probability is

$$
P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n).
$$

Now, since $P\{5 \text{ on any trial}\} = \dfrac{4}{36}$ and $P\{7 \text{ on any trial}\} = \dfrac{6}{36}$, we obtain, by the independence of trials

$$
P(E_n) = \left(1 - \frac{10}{36}\right)^{n-1} \frac{4}{36}.
$$

And thus

$$
\begin{aligned}
P\left(\bigcup_{n=1}^{\infty} E_n\right) &= \frac{1}{9} \sum_{n=1}^{\infty} \left(\frac{13}{18}\right)^{n-1} \\
&= \frac{1}{9} \frac{1}{1 - \frac{13}{18}} \\
&= \frac{2}{5}.
\end{aligned}
$$

Alternative approach using conditional probabilities

This result may be obtained by using conditional probabilities. If we let E be the event that 5 occurs before 7, then we can obtain the desired probability, $P(E)$, by conditioning on the outcome of the first trial, as follows: Let F be the event that the first trial results in 5; let G be the event that it results in 7; and let H be the event that the first trial results in neither 5 nor 7. Conditioning on which one of these events occurs gives

$$P(E) = P(E|F)P(F) + P(E|G)P(G) + P(E|H)P(H).$$

However,

$$P(E|F) = 1, \quad P(E|G) = 0, \quad \text{and} \quad P(E|H) = P(E).$$

The first two equalities are obvious. The third follows because if the first outcome results in neither 5 nor 7, then at that point the situation is exactly as when the problem first started; namely, the experimenter will continually roll a pair of fair dice until either 5 or 7 appears. As the trials are independent, the outcome of the first trial will have no effect on subsequent rolls of dice. Since $P(F) = \frac{4}{36}, P(G) = \frac{6}{36}$, and $P(H) = \frac{26}{36}$, we see that

$$P(E) = \frac{1}{9} + P(E)\frac{13}{18} \quad \text{or} \quad P(E) = \frac{2}{5}.$$

Intuitive approach

Students should note that the answer is quite intuitive. That is, since a "5" occurs on any roll with probability $\frac{4}{36}$ and a "7" with probability $\frac{6}{36}$, it seems intuitive that the odds that a "5" appears before "7" should be 6 to 4 against. The probability should be $\frac{4}{10}$, and indeed it is.

The same argument shows that if E and F are mutually exclusive events of an experiment. When the independent trials of this experiment are performed, event E will occur before event F with probability

$$\frac{P(E)}{P(E) + P(F)}.$$

Example 5 Adam tosses a fair coin $n+1$ times, Andrew tosses the same coin n times. What is the probability that Adam gets more heads than Andrew?

Solution Let H_1 and H_2 be the number of heads obtained by Adam and Andrew, respectively. Also let T_1 and T_2 be the number of tails obtained by Adam and Andrew, respectively. Since the coin is fair,

$$P(H_1 > H_2) = P(T_1 > T_2).$$

But

$$P(T_1 > T_2) = P(n + 1 - H_1 > n - H_2) = P(H_1 \le H_2).$$

Therefore, $P(H_1 > H_2) = P(H_1 \le H_2)$. So

$$P(H_1 > H_2) + P(H_1 \le H_2) = 1$$

implies that

$$P(H_1 > H_2) = P(H_1 \le H_2) = \frac{1}{2}.$$

Note that a combinatorial solution to this problem is neither elegent nor easy to handle:

$$\begin{aligned} P(H_1 > H_2) &= \sum_{i=0}^{n} P(H_1 > H_2 | H_2 = i)P(H_2 = i) \\ &= \sum_{i=0}^{n} \sum_{j=i+1}^{n+1} P(H_1 = j)P(H_2 = i) \\ &= \sum_{i=0}^{n} \sum_{j=i+1}^{n+1} \frac{\frac{(n+1)!}{j!(n+1-j)!}}{2^{n+1}} \frac{\frac{n!}{i!(n-i)!}}{2^n} \\ &= \frac{1}{2^{2n+1}} \sum_{i=0}^{n} \sum_{j=i+1}^{n+1} \binom{n+1}{j}\binom{n}{i}. \end{aligned}$$

We then encounter the difficulty of find the sum to the above double summation. However, comparing these two solutions, we obtain the following interesting identity:

$$\sum_{i=0}^{n} \sum_{j=i+1}^{n+1} \binom{n+1}{j}\binom{n}{i} = 2^{2n}.$$

Example 6 An urn contains 10 white and 12 red chips. Two chips are drawn at random and, without looking at their colors, are discarded. What is the probability that a third chip drawn is red?

Solution For $i \geq 1$, let R_i be the event that the ith chip drawn is red and W_i be the event that it is white. Intuitively, it should be clear that the two discarded chips provide no information, so $P(R_3) = 12/22$, the same as if it were the first chip drawn from the urn. To prove this mathematically, note that $\{R_2W_1, W_2R_1, R_2R_1, W_2W_1\}$ is a partition of the sample space; therefore,

$$\begin{aligned} P(R_3) = {} & P(R_3|R_2W_1)P(R_2W_1) + P(R_3|W_2R_1)P(W_2R_1) \\ & + P(R_3|R_2R_1)P(R_2R_1) + P(R_3|W_2W_1)P(W_2W_1). \end{aligned}$$

Now

$$\begin{aligned} P(R_2W_1) &= P(R_2|W_1)P(W_1) = \frac{12}{21} \times \frac{10}{22} = \frac{20}{77}, \\ P(W_2R_1) &= P(W_2|R_1)P(R_1) = \frac{10}{21} \times \frac{12}{22} = \frac{20}{77}, \\ P(R_2R_1) &= P(R_2|R_1)P(R_1) = \frac{11}{21} \times \frac{12}{22} = \frac{22}{77}, \\ P(W_2W_1) &= P(W_2|W_1)P(W_1) = \frac{9}{21} \times \frac{10}{22} = \frac{15}{77}. \end{aligned}$$

Substituting these values into the above equation, we get

$$P(R_3) = \frac{11}{20} \times \frac{20}{77} + \frac{11}{20} \times \frac{20}{77} + \frac{10}{20} \times \frac{22}{77} + \frac{12}{20} \times \frac{15}{77} = \frac{12}{22}.$$

Example 7 Two boxes have red, green and blue balls in them; the number of balls of each color is given in Table 1. Our experiment will be to select a box and then a ball from the selected box. One box (number 2) is slightly larger than the other, causing it to be selected more frequently. Let B_2 be the event "select the larger box" while B_1 is the event "select the smaller box." Assume $P(B_1) = \frac{2}{10}$ and $P(B_2) = \frac{8}{10}$. (B_1 and B_2 are mutually exclusive and $B_1 \cup B_2$ is the certain event, since some box must be selected; therefore, $P(B_1) + P(B_2)$ must equal unity.)

Table 1: Numbers of colored balls in two boxes

Solution Now define a discrete random variable X to have values $x_1 = 1, x_2 = 2$, and $x_3 = 3$ when a red, green, or blue ball is selected, and let B be an event equal to either B_1 or B_2. From Table 1:

$$\begin{aligned} P(X=1|B=B_1) &= \frac{5}{100} & P(X=1|B=B_2) &= \frac{80}{150} \\ P(X=2|B=B_1) &= \frac{35}{100} & P(X=2|B=B_2) &= \frac{60}{150} \\ P(X=3|B=B_1) &= \frac{60}{100} & P(X=3|B=B_2) &= \frac{10}{150}. \end{aligned}$$

The conditional probability density $f_X(x|B_1)$ becomes

$$f_X(x|B_1) = \frac{5}{100}\delta(x-1) + \frac{35}{100}\delta(x-2) + \frac{60}{100}\delta(x-3).$$

By direct integration of $f_X(x|B_1)$:

$$F_X(x|B_1) = \frac{5}{100}u(x-1) + \frac{35}{100}u(x-2) + \frac{60}{100}u(x-3).$$

For comparison, we may find the density and distribution of X by determining the probabilities $P(X = 1), P(X = 2)$, and $P(X = 3)$. These are found from the total probability theorem.

$$\begin{aligned} P(X=1) &= P(X=1|B_1)P(B_1) + P(X=1|B_2)P(B_2) \\ &= \frac{5}{100}\left(\frac{2}{10}\right) + \frac{80}{150}\left(\frac{8}{10}\right) = 0.437 \\ P(X=2) &= \frac{35}{100}\left(\frac{2}{10}\right) + \frac{60}{150}\left(\frac{8}{10}\right) = 0.390 \\ P(X=3) &= \frac{60}{100}\left(\frac{2}{10}\right) + \frac{10}{150}\left(\frac{8}{10}\right) = 0.173. \end{aligned}$$

Thus

$$f_X(x) = 0.437\delta(x-1) + 0.390\delta(x-2) + 0.173\delta(x-3)$$

and

$$F_X(x) = 0.437u(x-1) + 0.390u(x-2) + 0.173u(x-3).$$

The distribution functions and density functions are plotted below: