Worked examples — Single Random Variables

Example 1 Suppose that jury members decide independently, and that each with probability p $(0 < p < 1)$ makes the correct decision. If the decision of the majority is final, which is preferable: a three-person jury or a single juror?

Solution Let X denote the number of persons who decide correctly among a three-person jury. Then X is a binomial random variable with parameters $(3, p)$. Hence the probability that a three-person jury decide correctly is

$$\begin{aligned} P(X \geq 2) &= P(X = 2) + P(X = 3) \\ &= {}_3C_2 p^2(1-p) + {}_3C_3 p^3 (1-p)^0 \\ &= 3p^2(1-p) + p^3 = 3p^2 - 2p^3. \end{aligned}$$

Since the probability is p that a single juror decides correctly, a three-person jury is preferable to a single juror if and only if

$$3p^2 - 2p^3 > p.$$

This is equivalent to $3p - 2p^2 > 1$, so $-2p^2 + 3p - 1 > 0$. But $-2p^2 + 3p - 1 = 2(1-p)(p-1/2)$. Since $1 - p > 0, 2(1-p)(p-1/2) > 0$ if and only if $p > 1/2$. Hence a three-person jury is preferable if $p > 1/2$. If $p < 1/2$, the decision of a single juror is preferable. For $p = 1/2$ there is no difference.

Example 2 Let p be the probability that a randomly chosen person is against abortion, and let X be the number of persons against abortion in a random sample of size n. Suppose that, in a particular random sample of n persons, k are against abortion. Show that $P(X = k)$ is a maximum for $\widehat{p} = k/n$. That is, $\widehat{p}$ is the value of p that makes the outcome $X = k$ *most probable*.

Solution By definition of X:

$$P(X = k) = {}_nC_k p^k (1-p)^{n-k}.$$

This gives

$$\begin{aligned} \frac{d}{dp} P(X = k) &= {}_nC_k [kp^{k-1}(1-p)^{n-k} - (n-k)p^k(1-p)^{n-k-1}] \\ &= {}_nC_k p^{k-1}(1-p)^{n-k-1}[k(1-p) - (n-k)p]. \end{aligned}$$

Letting $\frac{d}{dp}P(X=k)=0$, we obtain $p=k/n$. Now since $\frac{d^2}{dp^2}P(X=k)<0, \widehat{p}=k/n$ is the maximum of $P(X=k)$, and hence it is an estimate of p that makes the outcome $x=k$ most probable.

Example 3 Ten percent of the tool produced in a certain manufacturing process turn out to be defective. Find the probability that in a sample of 10 tools chosen at random exactly two will be defective, by using (a) the binomial distribution, (b) the Poisson approximation to the binomial distribution.

Solution

(a) The probability of a defective tools is $p=0.1$. Let X denote the number of defective tool out of 10 chosen. Then according to the binomial distribution

$$P(X=2)={}_{10}C_2(0.1)^2(0.9)^8=0.1937.$$

(b) We have $\lambda=np=(10)(0.1)=1$. Then according to the Poisson distribution

$$P(X=x)=\frac{\lambda^x e^{-\lambda}}{x!} \quad \text{or} \quad P(X=2)=\frac{(1)^2e^{-1}}{2!}=0.1839.$$

In general the approximation is good if $p\le 0.1$ and $\lambda=np\le 5$.

Example 4 If the probability that an individual suffers a bad reaction from injection of a given serum is 0.001, determine the probability that out of 2000 individuals (a) exactly 3, (b) more than 2, individuals will suffer a bad reaction.

Solution Let X denote the number of individuals suffering a bad reaction. Since bad reactions are assumed to be rare events, we can suppose that X is Poisson distributed, i.e.

$$P(X=x)=\frac{\lambda^x e^{-\lambda}}{x!} \quad \text{where} \quad \lambda=np=(2000)(0.001)=2.$$

(a) $P(X=3)=\frac{2^3e^{-2}}{3!}=0.180.$

(b) $$\begin{aligned}P(X>2)&=1-[P(X=0)+P(X=1)+P(X=2)]\\&=1-\left[\frac{2^0e^{-2}}{0!}+\frac{2^1e^{-2}}{1!}+\frac{2^2e^{-2}}{2!}\right]\\&=1-5e^{-2}=0.323.\end{aligned}$$

An exact evaluation of the probabilities using the binomial distribution would require much more labor.

Example 5 Suppose that the length of a phone call in minutes is an exponential random variable with parameter $\lambda = \dfrac{1}{10}$. If someone arrives immediately ahead of you at a public telephone booth, find the probability that you will have to wait

(a) more than 10 minutes;

(b) between 10 and 20 minutes.

Solution Letting X denote the length of the call made by the person in the booth, the desired probabilities are

(a) $P\{X > 10\} = 1 - F(10)$
$= e^{-1} \approx 0.368.$

(b) $P\{10 < X < 20\} = F(20) - F(10)$
$= e^{-1} - e^{-2} \approx 0.233.$

Example 6 If X has a probability density f_X, then $Y = |X|$ has a density function that is obtained as follows: For $y \geq 0$,

$$\begin{aligned} F_Y(y) &= P[Y \leq y] \\ &= P[|X| \leq y] \\ &= P[-y \leq X \leq y] \\ &= F_X(y) - F_X(-y). \end{aligned}$$

Hence, on differentiation, we obtain

$$f_Y(y) = f_X(y) + f_X(-y) \quad y \geq 0.$$

Example 7 Let X be a continuous nonnegative random variable with density function f, and let $Y = X^n$. Find f_Y, the probability density function of Y.

Solution If $g(x) = x^n$, then

$$g^{-1}(y) = y^{1/n}$$

and

$$\frac{d}{dy}\{g^{-1}(y)\} = \frac{1}{n}y^{1/n-1}.$$

Hence, we obtain

$$f_Y(y) = \frac{1}{n}y^{1/n-1}F(y^{1/n}).$$

Example 8 Find the probability density of the random variable $U = X^2$, where X is the random variable defined by

$$f(x) = \begin{cases} x^2/81 & -3 < x < 6 \\ 0 & \text{otherwise} \end{cases}.$$

Solution The interval $-3 < x \leq 3$ corresponds to $0 \leq u \leq 9$ while $3 < x < 6$ corresponds to $9 < u < 36$. The distribution function for U is

$$G(u) = P[U \leq u] = P[X^2 \leq u] = P[-\sqrt{u} \leq X \leq \sqrt{u}] = \int_{-\sqrt{u}}^{\sqrt{u}} f(x)\,dx.$$

Since $f(x) = 0$ for $x < -3$, therefore

$$G(u) = \begin{cases} \int_{-\sqrt{u}}^{\sqrt{u}} f(x)\,dx & 0 \leq u \leq 9 \\ \int_{-3}^{\sqrt{u}} f(x)\,dx & 9 < u < 36 \\ 1 & u \geq 36 \end{cases}.$$

Since the density function $g(u)$ is the derivative of $G(u)$, we have

$$g(u) = \begin{cases} \frac{f(\sqrt{u})+f(-\sqrt{u})}{2\sqrt{u}} & 0 \leq u \leq 9 \\ \frac{f(\sqrt{u})}{2\sqrt{u}} & 9 < u < 36 \\ 0 & \text{otherwise} \end{cases}.$$

Using the given definition of $f(x)$ this becomes

$$g(u) = \begin{cases} \sqrt{u}/81 & 0 \leq u \leq 9 \\ \sqrt{u}/162 & 9 < u < 36 \\ 0 & \text{otherwise} \end{cases}.$$

Check: $\displaystyle\int_0^9 \frac{\sqrt{u}}{81}du + \int_9^{36} \frac{\sqrt{u}}{162}du = \left.\frac{2u^{3/2}}{243}\right|_0^9 + \left.\frac{u^{3/2}}{243}\right|_9^{36} = 1.$

Example 9 The density function of X is given by

$$f_X(x) = \begin{cases} 1 & \text{if} \quad 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

Find $E[e^X]$.

Solution Let $Y = e^X$. We start by determining F_Y, the probability distribution function of Y. Now, for $1 \leq x \leq e$,

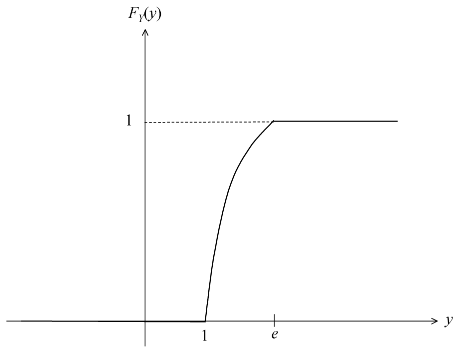
$$\begin{aligned} F_Y(x) &= P[Y \leq x] \\ &= P[e^X \leq x] \\ &= P[X \leq \ln x] \\ &= \int_0^{\ln x} f_X(x')\ dx' \\ &= \ln x. \end{aligned}$$

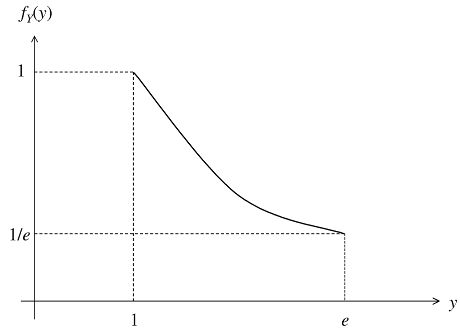
By differentiating $F_Y(x)$, the probability density function of Y is given by

$$f_Y(x) = \begin{cases} \frac{1}{x} & 1 < x < e \\ 0 & \text{if } x < 1 \text{ or } x > e \end{cases}.$$

Hence,

$$\begin{aligned} E[e^X] = E[Y] &= \int_{-\infty}^{\infty} x f_Y(x)\ dx \\ &= \int_1^e dx \\ &= e - 1. \end{aligned}$$





Alternative method

$$E[X] = \int_{-\infty}^{\infty} g(x) F_X(x)\ dx = \int_0^1 e^x\ dx = e - 1.$$

To compute the cdf of Y, we use $F_Y(y) = \displaystyle\int_{-\infty}^{y} f_Y(y')\ dy'$. We obtain

$$F_Y(y) = \begin{cases} 0 & \text{if} \quad y < 1 \\ \displaystyle\int_1^y \frac{1}{y'}\ dy' = \ln y & \text{if} \quad 1 \le y \le e \\ \displaystyle\int_1^e \frac{1}{y'}\ dy' = 1 & \text{if} \quad y > e \end{cases}.$$