MATH246 — Probability and Random Processes

Solution to Homework One

1. (a) $P(A \cap B) = P(\phi) = 0$ (mutually exclusive).
   Since $P(A)P(B) > 0$, so $P(A \cap B) \neq P(A)P(B)$.
   Thus, A and B are not independent.
   (b) A and B are independent and $P(A)P(B) > 0$.
   Since $P(A \cap B) = P(A)P(B) > 0$, so A and B are not mutually exclusive.
   (c) ϕ is the impossible event.
   For any event $A \subset \Omega$, $P(A \cap \phi) = P(\phi) = 0 = P(\phi)P(A)$; so the impossible event is independent of any event (including itself).

2. $E_7 = \{(1,6),(2,5),(3,5),(4,3),(5,2),(6,1)\}$
   $F = \{(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\}$
   $T = \{(1,3),(2,3),(3,3),(4,3),(5,3),(6,3)\}$
   (a) $E_7 \cap F = \{(4,3)\}$ so $P(E_7 \cap F) = \dfrac{1}{36}$.
   Since $P(E_7) = \dfrac{6}{36} = \dfrac{1}{6}$ and $P(F) = \dfrac{6}{36} = \dfrac{1}{6}$, so $P(E_7 \cap F) = \dfrac{1}{36} = \left(\dfrac{1}{6}\right)\left(\dfrac{1}{6}\right) = P(E_7)P(F)$.
   Hence, E_7 and F are independent.
   $E_7 \cap T = \{(4,3)\}$, $P(E_7 \cap T) = \dfrac{1}{36}$ and $P(T) = \dfrac{6}{36} = \dfrac{1}{6}$, so $P(E_7 \cap T) = \dfrac{1}{36} = \left(\dfrac{1}{6}\right)\left(\dfrac{1}{6}\right) = P(E_7)P(T)$. Hence, E_7 and T are independent.
   (b) $F \cap T = \{(4,3)\}$
   $E_7 \cap (F \cap T) = \{(4,3)\}$
   $P(E_7 \cap (F \cap T)) = \dfrac{1}{36}$
   $P(F \cap T) = \dfrac{1}{36}$
   $P(E_7 \cap (F \cap T)) = \dfrac{1}{36} \neq \left(\dfrac{1}{6}\right)\left(\dfrac{1}{36}\right) = P(E_7)P(F \cap T)$
   so E_7 is not independent of $F \cap T$.

3. $E = \{\text{operator error occurs}\}$
   $F = \{\text{equipment failure occurs}\}$.
   Given $P(F \cap E^c) = 0.1$, $P(F \cap E) = 0.05$, $P(E) = 0.4$.
   (a) $P(F \cup E) = P(F \cap E^c) + P(E)$
   $= 0.1 + 0.4 = 0.5$.
   (b) $P(E \cap F^c) = P(E) - P(F \cap E)$
   $= 0.4 - 0.05 = 0.35$.
   (c) $P((E \cup F)^c) = 1 - P(E \cup F)$
   $= 1 - 0.5 = 0.5$.
   (d) $P(E|F) = P(E \cap F)/P(F)$
   $= 0.05/\{P(E \cap F) - P(E \cap F^c)\}$

$$= \frac{0.05}{0.5 - 0.35} = \frac{1}{3}$$

(e) $P(E|F^c) = P(E \cap F^c)/P(F^c)$

$$= \frac{0.35}{1 - 0.15} = \frac{7}{17}.$$

E and F are not independent since $P(E|F) \neq P(E)$.

4. Let $R_i, i = 1, 2, 3$ be the event that the plane is in region i; and let E be the event that a search of region 1 is unsuccessful. From Bayes' formula we obtain

$$\begin{aligned} P(R_1|E) &= \frac{P(E \cap R_1)}{P(E)} \\ &= \frac{P(E|R_1)P(R_1)}{\sum_{i=1}^{3} P(E|R_i)P(R_i)} \\ &= \frac{(\alpha_1)(1/3)}{(\alpha_1)(1/3) + (1)1/3 + (1)(1/3)} \\ &= \frac{\alpha_1}{\alpha_1 + 2}. \end{aligned}$$

For $j = 2, 3$

$$\begin{aligned} P(R_j|E) &= \frac{P(E|R_j)P(R_j)}{P(E)} \\ &= \frac{(1)(1/3)}{(\alpha_1)1/3 + 1/3 + 1/3} \\ &= \frac{1}{\alpha_1 + 2}, \qquad j = 2, 3. \end{aligned}$$

Thus, for instance, if $\alpha_1 = 0.4$ then the conditional probability that the plane is in region 1 given that a search of that region fails is equal to $1/6$.

5. Let $I = \{$selected person is in good risks' class$\}$,

$II = \{$selected person is in average risks' class$\}$,

$III = \{$selected person is in bad risks' class$\}$,

$B = \{$selected person will be involved in an accident over 1-year span$\}$.

Given $P(I) = 0.2, \quad P(II) = 0.5, \quad P(III) = 0.3,$

$P(B|I) = 0.05, \quad P(B|II) = 0.15, \quad P(B|III) = 0.3.$

We have

$$\begin{aligned} P(B) &= P(B \cap I) + P(B \cap II) + P(B \cap III) \\ &= P(B|I)P(I) + P(B|II)P(II) + P(B|III)P(III) \\ &= (0.05)(0.2) + (0.15)(0.5) + (0.3)(0.3) \\ &= 0.175; \end{aligned}$$

$$P(I|B^c) = \frac{P(I \cap B^c)}{P(B^c)} = \frac{P(B^c|I)P(I)}{1 - P(B)} = \frac{(1 - 0.05)(0.2)}{1 - 0.175} = \frac{19}{82.5} = 0.2303;$$

$$P(II|B^c) = \frac{P(II \cap B^c)}{P(B^c)} = \frac{P(B^c|II)P(II)}{1 - P(B)} = \frac{(1 - 0.15)(0.5)}{1 - 0.175} = \frac{17}{33} = 0.5152.$$

6. Let A be the prisoner who asks the question, B and C be the other two prisoners.
Let P_A, P_B and P_C be the respective probabilities that they will be executed, where $P_A + P_B + P_C = 1$.
Since the executed person is chosen at random, then $P_A = P_B = P_C = \dfrac{1}{3}$. Let E_A, E_B and E_C be the events that A, B and C are executed, respectively. Suppose the jailer informs A that C will be set free (the event is denoted by $E_{\widetilde{C}}$), then

$$\begin{aligned} P(E_A|E_{\widetilde{C}}) &= \frac{P(E_{\widetilde{C}}|E_A)P_A}{P(E_{\widetilde{C}}|E_A)P_A + P(E_{\widetilde{C}}|E_B)P_B + P(E_{\widetilde{C}}|E_C)P_C} \\ &= \frac{\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)}{\left(\frac{1}{2}\right)\left(\frac{1}{3}\right) + (1)\left(\frac{1}{3}\right) + (0)\left(\frac{1}{3}\right)} = \frac{1}{3} = P(E_A). \end{aligned}$$

Therefore, the probability that A will be chosen to be executed will not be altered by revealing the information that B or C is set free. Note that $P(E_{\widetilde{C}}|E_B) = 1$. This is because when Prisoner B will be executed, then Prisoner C will be set free. Also, you should not get confused with "set free" as being equivalent to "non-executed". Here, "set free" refers to the action taken by the jailer to set free one of the two prisoners: B or C.

7. $Y =$ no. of heads $-$ no. of tails
   (a) When n is even, $S_Y = \{-n, -n+2, \cdots, 0, \cdots, n-2, n\}$.
   When n is odd, $S_Y = \{-n, -n+2, \cdots, -1, 1, \cdots, n-2, n\}$.
   (b) Equivalent event for $\{Y = 0\}$
      (i) When n is odd, $\{Y = 0\} = \phi$.
      (ii) When n is even, $\{Y = 0\} = \{$outcomes that have $n/2$ heads and $n/2$ tails$\}$. There are ${}_nC_{n/2}$ combinations.
   (c) Equivalent event for $\{Y \le k\}$
   If $n =$ even,

$$\begin{aligned} &K \text{is even} : \{Y \le k\} = \{Y = -n, -n+2, \cdots, 0, 2, 4, \cdots, k\} \\ &K \text{is odd} : \{Y \le k\} = \{Y = -n, -n+2, \cdots, 0, 2, \cdots, k-1\}. \end{aligned}$$

   If $n =$ odd,

$$\begin{aligned} &K \text{is even} : \{Y \le k\} = \{Y = -n, -n+2, \cdots, -1, 1, \cdots, k-1\} \\ &K \text{is odd} : \{Y \le k\} = \{Y = -n, -n+2, \cdots, -1, 1, \cdots, k\} \end{aligned}$$

      (i) If n is even, for any fixed even number $i \le k$.
      Let $A_i = \{Y = i\} = \left\{\text{outcomes that have } \dfrac{n+i}{2} \text{ heads and } \dfrac{n-i}{2} \text{ tails}\right\}$. There are ${}_nC_{\frac{n-i}{2}}$ combinations.
      When K is even, $\{Y \le k\} = \bigcup_{i=-n}^{k} A_i$.
      When K is odd, $\{Y \le k\} = \bigcup_{i=-n}^{k-1} A_i$.
      (ii) If n is odd, for any fixed odd number $i \le k$.

Let $A_i = \{Y = i\} = \left\{\text{outcomes that have } \frac{n+i}{2} \text{ heads and } \frac{n-i}{2} \text{ tails}\right\}$. There are ${}_nC_{\frac{n+i}{2}}$ combinations.

When k is even, $\{Y \le k\} = \bigcup_{i=-n}^{k-1} B_i$.

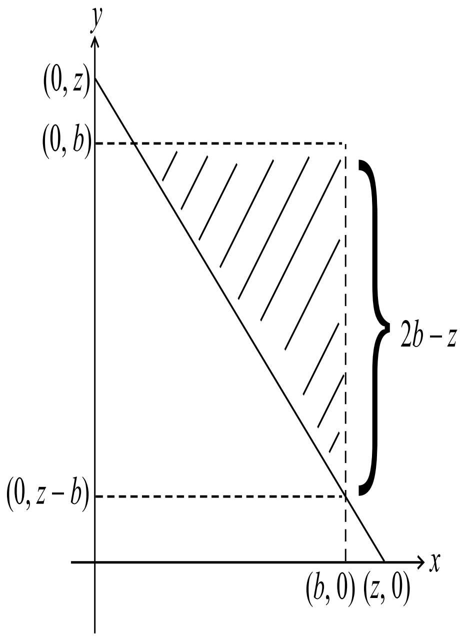
When k is odd, $\{Y \le k\} = \bigcup_{i=-n}^{k} B_i$.

8. (a) The square is $\{(x, y) : 0 \le x \le b, \quad 0 \le y \le b\}$.
   For $Z = X + Y$, we have $S_Z = \{z : 0 \le z \le 2b\}$.

   (b) We divide $(-\infty, \infty)$ into four subintervals:
   (i) $(-\infty, 0]$, (ii) $(0, b]$, (iii) $(b, 2b]$, (iv) $(2b, \infty)$.
   Consider the following cases:
   (1) $z \in (-\infty, 0]$, $\{Z \le z\} = \phi$
   (2) $z \in (0, b]$, $\{Z \le z\} = \{(x, y) : x + y \le z, x > 0, y > 0\}$
   (3) $z \in (b, 2b]$, $\{Z \le z\} = \{(x, y) : x + y \le z, 0 < x \le b, 0 < y \le b\}$
   (4) $z \in (2b, \infty)$, $\{Z \le z\} = \{(x, y) : 0 \le x \le b, 0 \le y \le b\}$.

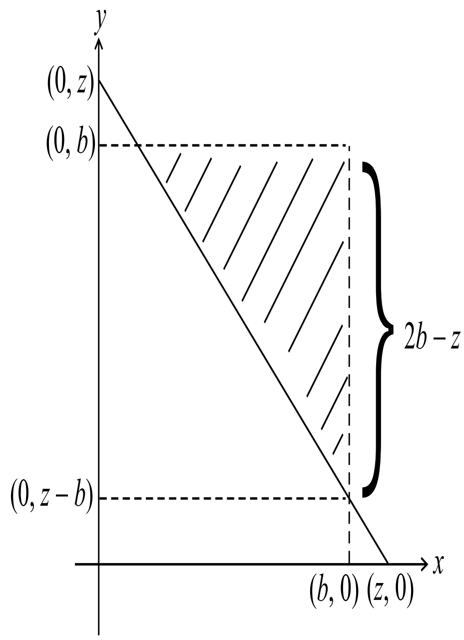
   (c) It is easy to see that when
   (i) $z \in (-\infty, 0]$, $P(Z \le z) = 0$
   (iv) $z \in (2b, \infty)$, $P(Z \le z) = 1$.
   Now, consider cases (ii) and (iii),
   (ii) $z \in (0, b]$

   $$P(Z \le z) = \frac{z^2}{2} \Big/ b^2 = \frac{z^2}{2b^2};$$



   (iii) $z \in (b, 2b]$

   $$P(Z \le z) = \left[b^2 - \frac{(2b - z)^2}{2}\right] \Big/ b^2.$$



Hence, we have

$$P(Z \le z) = \begin{cases} 0 & -\infty < z \le 0 \\ z^2/2b & 0 < z \le b \\ \frac{b^2 - \frac{(2b-z)^2}{2}}{b^2} & b < z \le 2b \\ 1 & 2b < z < \infty \end{cases}.$$