MATH246 — Probability and Random Processes

Solution to Homework Three

1. (a) The outcomes of each toss are equiprobable

$\Rightarrow \quad P[X_k = i] = \frac{1}{6}, \quad i = 1, 2, \cdots, 6; \quad k = 1, 2.$

The tosses are independent.

$$\begin{aligned} \Rightarrow \qquad P[X_1 = i, X_2 = j] &= P[X_1 = i]P[X_2 = j] \\ &= \left(\frac{1}{6}\right)\left(\frac{1}{6}\right) = \frac{1}{36}, \quad i, j = 1, 2, \cdots, 6. \end{aligned}$$

(b) Given $X = \min(X_1, X_2), Y = \max(X_1, X_2)$, we may have $X \leq Y$, but $X \not> Y$.

When $i > j$,

$$P[X = i, Y = j] = 0;$$

When $i = j$,

$$\begin{aligned} P[X = i, Y = i] &= P[\min(X_1, X_2) = i, \max(X_1, X_2) = i] \\ &= P[X_1 = i, X_2 = i] = \frac{1}{36}; \end{aligned}$$

when $i < j$,

$$\begin{aligned} P[X = i, Y = i] &= P[\min(X_1, X_2) = i, \max(X_1, X_2) = j] \\ &= P[\{X_1 = i, X_2 = j\} \cup \{X_1 = j, X_2 = i\}] \\ &= P[X_1 = i, X_2 = j] + P[X_i = j, X_2 = i] \\ &= \frac{1}{36} + \frac{1}{36} = \frac{1}{18}. \end{aligned}$$

Hence,

$$P[X = i, Y = j] = \begin{cases} 0, & i > j; j = 1, 2, \cdots, 5 \\ \frac{1}{36}, & i = j, = 1, 2, \cdots, 6 \\ \frac{1}{18}, & i = 1, 2, \cdots, 5; \quad i < j \end{cases}.$$

(c)

$$\begin{aligned} P[X - i] &= \sum_{j=1}^{6} P[X = i, Y = j] \\ &= P[X = i, Y = i] + \sum_{j=i+1}^{6} P[X = i, Y = j] \\ &= \frac{1}{36} + \frac{1}{18}(6 - i) \\ &= \frac{13 - 2i}{36}, \qquad i = 1, 2, \cdots, 6. \end{aligned}$$

$$
\begin{aligned}
P[Y=j] &= \sum_{i=1}^{6} P[X=i, Y=j] \\
&= \sum_{i=1}^{j-1} P[X=i, Y=j] + P[X=i, Y=j] \\
&= \frac{1}{18}(j-1) + \frac{1}{36} \\
&= \frac{2j-1}{36}, \qquad j = 1, 2, \cdots, 6.
\end{aligned}
$$

2.
$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)\ dxdy \\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_2\sigma_2\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x-m_1}{\sigma_1}\right)^2 \right.\right. \\
&\qquad \left.\left. -2\rho\left(\frac{x-m_1}{\sigma_1}\right)\left(\frac{y-m_2}{\sigma_2}\right) + \left(\frac{y-m_2}{\sigma_2}\right)^2\right]\right\}\ dy \\
&= \frac{1}{\sigma_1\sqrt{2\pi}} \exp\left\{-\frac{(x-m_1)^2}{2\sigma_1^2}\right\} X \\
&\quad \int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi}\sigma_2\sqrt{1-\rho}} \exp\left(\frac{-1}{2(\sigma_2\sqrt{1-\rho^2})^2}\left\{y - \left[m_2 + \frac{\rho\sigma_2}{\sigma_1}(x-m_1)\right]\right\}^2\right)}_{g(y)}\ dy
\end{aligned}
$$

Note that $g(y)$ is the pdf of a Gaussian random variable with mean $m_2 + \frac{\rho\sigma_2}{\sigma_1}(x-m_1)$ and variance $\sigma_2\sqrt{1-\rho^2}$. Hence, $\int_{-\infty}^{\infty} g(y)\ dy = 1$. This gives

$$
f_X(x) = \frac{1}{\sigma_1\sqrt{2\pi}} \exp\left\{-\frac{(x-m_1)^2}{2\sigma_1^2}\right\}, \quad x \in \mathbb{R}
$$

Similarly, we have

$$
f_Y(y) = \frac{1}{\sigma_2\sqrt{2\pi}} \exp\left\{-\frac{(y-m_2)^2}{2\sigma_2^2}\right\}, \quad y \in \mathbb{R}.
$$

Therefore, $f_X(x)$ and $f_Y(y)$ are pdf's of Gaussian random variables with means m_1 and m_2 and variances σ_1^2 and σ_2^2, respectively.

3. (a)
$$
\begin{aligned}
P[X=k, Y \le y] &= P[Y \le y | X=k]P[X=k] \\
&= P[N \le y-k | X=k]P[X=k] \\
&= P[N \le y-k]P[X=k]
\end{aligned}
$$
since X and N are independent.

Note that $P[X=1] = P[X=-1] = \frac{1}{2}$.

For $k=-1$,

$$
\begin{aligned}
P[X=-1, Y \le y] &= P[N \le y+1]P[X=-1] \\
&= \frac{1}{2}\int_{-\infty}^{y+1} \frac{1}{2}\alpha e^{-\alpha|z|}\ dz = \frac{1}{4}\int_{-\infty}^{y+1} \alpha e^{-\alpha|z|}\ dz.
\end{aligned}
$$

When $y+1 \le 0$,

$$P[X=-1, Y \le y] = \frac{1}{4}\int_{-\infty}^{y+1} \alpha e^{\alpha z}\ dz = \frac{1}{4}e^{\alpha(y+1)};$$

when $y+1 > 0$

$$\begin{aligned} p[X=-1, Y \le y] &= \frac{1}{4}\int_{-\infty}^{0} \alpha e^{\alpha z}\ dz + \frac{1}{4}\int_{0}^{y+1} \alpha e^{-\alpha z}\ dz \\ &= \frac{1}{4} - \left(\frac{1}{4}e^{-\alpha(y+1)} - \frac{1}{4}\right) \\ &= \frac{1}{2} - \frac{1}{4}e^{-\alpha(y+1)}. \end{aligned}$$

For $k=1$,

$$\begin{aligned} P[X=1, Y \le y] &= P[N \le y-1]P[X=1] \\ &= \frac{1}{4}\int_{-\infty}^{y-1} \alpha e^{-\alpha|z|}\ dz. \end{aligned}$$

When $y-1 \le 0$,

$$P[X=1, Y \le y] = \frac{1}{4}\int_{-\infty}^{y-1} \alpha e^{\alpha z}\ dz = \frac{1}{4}e^{\alpha(y-1)};$$

when $y-1 > 0$,

$$\begin{aligned} P[X=1, Y \le y] &= \frac{1}{4}\int_{-\infty}^{0} \alpha e^{\alpha z}\ dz + \frac{1}{4}\int_{0}^{y-1} \alpha e^{-\alpha z}\ dz \\ &= \frac{1}{4} - \left(\frac{1}{4}e^{-\alpha(y-1)} - \frac{1}{4}\right) \\ &= \frac{1}{2} - \frac{1}{4}e^{-\alpha(y-1)}. \end{aligned}$$

Hence,

$$P[X=-1, Y \le y] = \begin{cases} \frac{1}{4}e^{\alpha(y+1)}, & y \le -1 \\ \frac{1}{2} - \frac{1}{4}e^{-\alpha(y+1)}, & y > -1 \end{cases};$$

$$P[X=1, Y \le y] = \begin{cases} \frac{1}{4}e^{\alpha(y-1)}, & y \le 1 \\ \frac{1}{2} - \frac{1}{4}e^{-\alpha(y-1)}, & y > 1 \end{cases}.$$

(b)

$$\begin{aligned} F_Y(y) &= P[Y \le y] \\ &= P[Y \le y, X=-1] + P[Y \le y, X=1] \\ &= \begin{cases} \frac{1}{4}[e^{\alpha(y-1)} + e^{\alpha(y+1)}], & y \le -1 \\ \frac{1}{4}e^{\alpha(y-1)} + \frac{1}{2} - \frac{1}{4}e^{-\alpha(y+1)}, & -1 < y \le 1 \\ 1 - \frac{1}{4}[e^{-\alpha(y-1)} + e^{-\alpha(y+1)}], & y \ge 1 \end{cases} \\ f_Y(y) &= \frac{d}{dy}F_Y(y) \\ &= \begin{cases} \frac{\alpha}{4}[e^{\alpha(y-1)} + e^{\alpha(y+1)}], & y \le -1 \\ \frac{\alpha}{4}[e^{\alpha(y-1)} + e^{-\alpha(y+1)}], & -1 < y \le 1 \\ \frac{\alpha}{4}[e^{-\alpha(y-1)} + e^{-\alpha(y+1)}], & y \ge 1 \end{cases} \end{aligned}$$

(c) Compare the magnitudes of conditional probabilities, $P[X=1|Y>0]$ and $P[X=-1|Y>0]$:

$$\begin{aligned}
& P[X=1,Y>0]-P[X=-1,Y>0] \\
&= 1-P[X=1,Y\le 0]-1+P[X=-1,Y\le 0] \\
&= -\frac{1}{4}e^{\alpha(0-1)}+\frac{1}{2}-\frac{1}{4}e^{-\alpha(0+1)} \\
&= \frac{1}{2}(1-e^{-\alpha})>0 \quad \text{as} \quad \alpha>0.
\end{aligned}$$

Hence,

$$\begin{aligned}
& P[X=1,Y>0] > P[X=-1,Y>0] \\
\Rightarrow\ & \frac{P[X=1,Y>0]}{P[Y>0]} > \frac{P[X=-1,Y>0]}{P[Y>0]} \\
\Rightarrow\ & P[X=1|Y>0] > P[X=-1|Y>0]
\end{aligned}$$

so $X=1$ is more likely to occur given $Y>0$.

4. Assume that X and Y are continuous random variables.

(a) $$\begin{aligned} P[\{a<X\le b\}\cap\{Y\le d\}] &= P[a<X\le b]P[Y\le d] \quad \text{(since } X \text{ and } Y \text{ are independent)} \\ &= [F_X(b)-F_X(a)]F_Y(d). \end{aligned}$$

(b) $$\begin{aligned} P[\{a\le X\le b\}\cap\{c\le Y\le d\}] &= P[a\le X\le b]P[c\le Y\le d] \\ &= [F_X(b)-F_X(a)][F_Y(d)-F_Y(c)] \end{aligned}$$

(c) $$\begin{aligned} P[\{|X|<a\}\cap\{c\le Y\le d\}] &= P[\{X>a\}\cap\{X<-a\}]P[c\le Y\le d] \\ &= \{P[X>a]+P[X<-a]\}P[c\le Y\le d] \\ &= [1-F_X(a)+F_X(-a)][F_Y(d)-F_Y(c)] \end{aligned}$$

5. (a) Given that N is the number of successes in the first n Bernoulli trials and M is the number of successes in the next m Bernoulli trials. Since all $m+n$ Bernoulli trials are independent, N and M should be independent.

(b) Note that N and M are binomial random variables.

The marginal pmf's:

$$\begin{aligned} P[N=k] &= C_k^n p^k(1-p)^{n-k}, \quad k=0,1,\cdots,n \\ P[M=r] &= C_r^m p^r(1-p)^{m-r}, \quad r=0,1,\cdots,m. \end{aligned}$$

The joint pmf:

$$\begin{aligned} P[N=k,M=r] &= P[N=k]P[M=r] \qquad \text{by independence} \\ &= C_k^n C_r^m p^{k+r}(1-p)^{m+n-k-r}, \quad k=0,1,\cdots,n; r=0,1,\cdots,m. \end{aligned}$$

(c) Let $Z=$ the total number of successes in the $n+m$ trials, then Z is also a binomial random variable. Hence,

$$P[Z=z]=C_z^{n+m}p^z(1-p)^{n+m-z}, \quad z=0,1,\cdots,n+m.$$

6. Note that $X^2 + Y^2 = \cos^2\theta + \sin^2\theta = 1$, hence the sample space of (X, Y) is the unit circle.
Now, fix $X = x$, we have

$$x^2 + Y^2 = 1 \quad \Rightarrow \quad Y = \sqrt{1 - x^2} \quad \text{or} \quad Y = -\sqrt{1 - x^2}.$$

Hence, $Y|X = x$ is a discrete random variable. Recall that the pdf of a discrete random variable that assumes discrete values $x_1, x_2, \cdots, x_k$ is given by

$$f_X(x) = \sum_{k=1}^{n} P_X(x_k)\delta(x - x_k).$$

Since (X, Y) is uniform, the probability values that Y assumes $\sqrt{1 - x^2}$ or $-\sqrt{1 - x^2}$ are equal. We then have

$$f_Y(y|x) = \frac{1}{2}\delta(y - \sqrt{1 - x^2}) + \frac{1}{2}\delta(y + \sqrt{1 - x^2}), \quad x^2 + y^2 = 1.$$

(i) When $|x| > 1, E[Y|X = x] = 0$ since X lies on the unit circle.

(ii) When $|x| \leq 1$,

$$\begin{aligned} E[Y|X = x] &= \sqrt{1 - x^2} f_Y(\sqrt{1 - x^2}|x) + (-\sqrt{1 - x^2}) f_Y(-\sqrt{1 - x^2}|x) \\ &= \sqrt{1 - x^2}\left(\frac{1}{2}\right) + (-\sqrt{1 - x^2})\left(\frac{1}{2}\right) = 0. \end{aligned}$$

As $E[Y|X = x] = 0$ for all x, we have $E[Y|X] = 0$.

7. Define $D(z) = \{(x, y) : x + y \leq z\}$

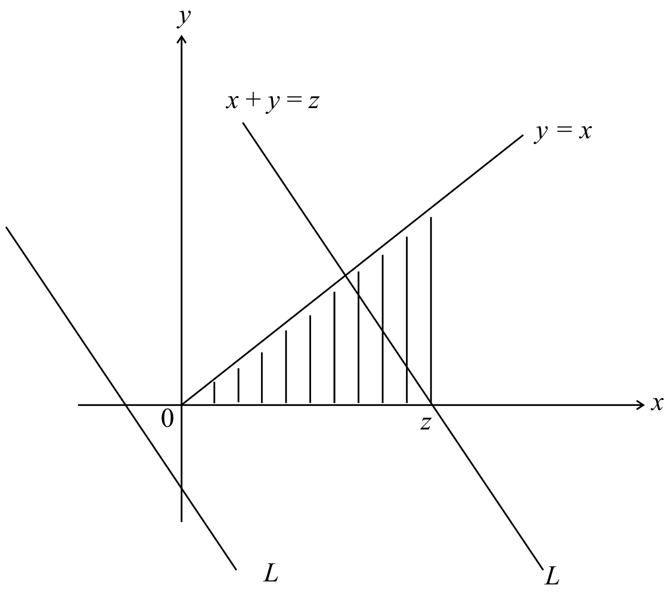
$$\begin{aligned} F_Z(z) &= P[Z \leq z] = P[X + Y \leq z] \\ &= \iint_{D(z)} f_{X,Y}(x, y)\ dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{z-x} f_{X,Y}(x, y)\ dydx \\ f_Z(z) &= \frac{d}{dz}F_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z - x)\ dx \end{aligned}$$

which equals the integration of $f_{X,Y}(x, y)$ with respect to x along the straight line $L : x + y = z$.
When $z < 0$, $f_{X,Y} = 0$ on L, so $f_Z(z) = 0$.
When $z \geq 0$, $f_{XY}(x, z - x)$ is non-zero only when $z - x \geq 0$ and $z - x \leq x$. The two inequalities arises since $f_{X,Y}(x, y)$ is non-zero only when $0 \leq y \leq x < \infty$. Hence, the range of x such that $f_{X,Y}(x, z - x)$ is non-zero is given by $\frac{z}{2} \leq x \leq z$. We then have

$$\begin{aligned} f_Z(z) &= \int_{z/2}^{z} 2e^{-(x+z-x)}\ dx \\ &= 2e^{-z}\int_{z/2}^{z} 1\ dx \\ &= ze^{-z}. \end{aligned}$$

In summary,

$$f_Z(z) = \begin{cases} 0, & z < 0 \\ ze^{-z}, & z \geq 0 \end{cases}.$$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