MA246
Homework Four

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1. A discrete-time random process X_n is defined as follows. A fair coin is tossed. If the outcome is heads, $X_n = 1$ for all n; if the outcome is tails, $X_n = -1$ for all n.
   a. Sketch some sample paths of the process.
   b. Find the pmf for X_n.
   c. Find the joint pmf for X_n and X_{n+k}.
   d. Find the mean and autocovariance functions of X_n.

2. The random process $Z(t)$ is defined by
$$Z(t) = Xt + Y,$$
where X and Y are a pair of random variables with means m_X, m_Y, variances σ_X^2, σ_Y^2, and correlation coefficient $\rho_{X,Y}$.
   a. Find the mean and autocovariance of $Z(t)$.
   b. Find the pdf of $Z(t)$ if X and Y are jointly Gaussian random variables.

3. Let S_n denote a binomial counting process.
   a. Show that $P[S_n = j, S_{n'} = i] \neq P[S_n = j]P[S_{n'} = i]$.
   b. Find $P[S_{n_2} = j | S_{n_1} = i]$, where $n_2 > n_1$.
   c. Show that $P[S_{n_2} = j | S_{n_1} = i, S_{n_0} = k] = P[S_{n_2} = j | S_{n_1} = i]$, where $n_2 > n_1 > n_0$.

4. The number of cars which pass a certain intersection daily between 12:00 and 14:00 follows a homogeneous Poisson process with intensity $\lambda = 40$ per hour. Among these, there are 0.8% which disregard the STOP-sign. What is the probability that at least one car disregards the STOP-sign between 12:00 and 13:00?

5. Let $N(t)$ be a Poisson random process with parameter λ. Suppose that each time an event occurs, a coin is flipped and the outcome (heads or tails) is recorded. Let $N_1(t)$ and $N_2(t)$ denote the number of heads and tails recorded up to time t, respectively. Assume that p is the probability of heads.
   a. Find $P[N_1(t) = j, N_2(t) = k | N(t) = k + j]$.
   b. Use part a to show that $N_1(t)$ and $N_2(t)$ are independent Poisson random variables of rates $p\lambda t$ and $(1-p)\lambda t$, respectively:
$$P[N_1(t) = j, N_2(t) = k] = \frac{(p\lambda t)^j}{j!} e^{-p\lambda t} \frac{[(1-p)\lambda t]^k}{k!} e^{-(1-p)\lambda t}.$$

6. Customers arrive at a soft drink dispensing machine according to Poisson process with rate λ. Suppose that each time a customer deposits money, the machine dispenses a soft drink with probability p. Find the pmf for the number of soft dispensed in time t. Assume that the machine holds an infinite number of soft drinks.

7. Let $X(t)$ denote the random telegraph signal, and let $Y(t)$ be a process derived from $X(t)$ as follows: Each time $X(t)$ changes polarity, $Y(t)$ changes polarity with probability p.
   a. Find $P[Y(t) = \pm 1]$.
   b. Find the autocovariance function of $Y(t)$. Compare it to that of $X(t)$.

8. A machine consists of two parts that fail and are repaired independently. A working part fails during any given day with probability a. A part that is not working is repaired by the next day with probability b. Let X_n be the number of working parts in day n.
   a. Show that X_n is a three-state Markov chain and give its one-step transition probability matrix P.
   b. Show that the steady state pmf π is binomial with parameter $p = b/(a + b)$.
   c. What do you expect is steady state pmf for a machine that consists of n parts?