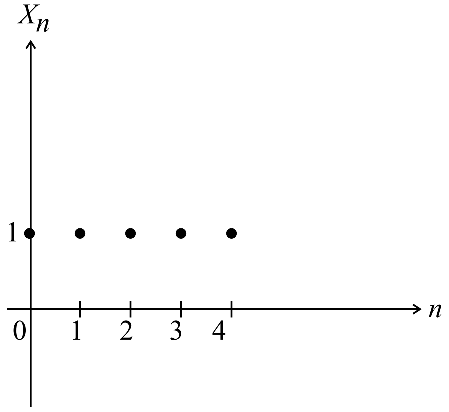
MATH246 — Probability and Random Processes

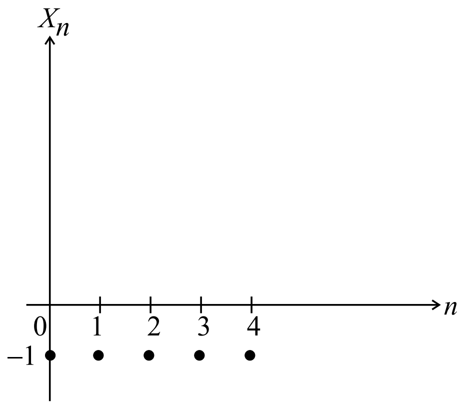
Solution to Homework Four

1. Note that for all n,

$$X_n = \begin{cases} 1 & \text{if the outcome is } H \\ -1 & \text{if the outcome is } T \end{cases}.$$

   (a) The only two sample paths:





   (b) Given that the coin is fair, we have

$P[X_n = 1] = P[\text{outcome is } H] = \frac{1}{2}$

$P[X_n = -1] = P[\text{outcome is } T] = \frac{1}{2}$.

   (c) $P[X_n = 1, X_{n+k} = 1] = P[X_n = 1] = \frac{1}{2}$

$P[X_n = -1, X_{n+k} = -1] = P[X_n = -1] = \frac{1}{2}$

$P[X_n = 1, X_{n+k} = -1] = P[\phi] = 0$

$P[X_n = -1, X_{n+k} = 1] = P[\phi] = 0$

Hence, the joint pmf

$$P[X_n = i, X_{n+k} = j] = \begin{cases} \frac{1}{2}, & i = j \\ 0, & i \neq j \end{cases}.$$

   (d) $P[X_n] = (1)P[X_n = 1] + (-1)P[X_n = -1] = \frac{1}{2} - \frac{1}{2} = 0.$

$$\begin{aligned} C_X(n_1, n_2) &= E[\{X_{n_1} - E[X_{n_1}]\}\{X_{n_2} - E[X_{n_2}]\}] \\ &= E[X_{n_1} X_{n_2}] \\ &= (1)(1)P[X_{n_1} = 1, X_{n_2} = 1] + (-1)(-1)P[X_{n_1} = -1, X_{n_2} = -1] \\ &= \frac{1}{2} + \frac{1}{2} = 1. \end{aligned}$$

2. (a) $E[Z(t)] = E[Xt + Y] = tm_X + m_Y$

$$\begin{aligned} C_Z(t_1, t_2) &= E[\{(Xt_1 + Y) - (t_1 m_X + m_Y)\}\{(Xt_2 + Y) - (t_2 m_X + m_Y)\}] \\ &= E[\{t_1(X - m_X) + (Y - m_Y)\}\{t_2(X - m_X) + (Y - m_Y)\}] \\ &= t_1 t_2 E[\{X - m_X\}^2] + t_1 E[(X - m_X)(Y - m_Y)] \\ &\quad + E[\{Y - m_Y\}^2] + t_2 E[(Y - m_Y)(X - m_X)] \\ &= t_1 t_2 \sigma_X^2 + (t_1 + t_2)\sigma_X \sigma_Y \rho_{XY} + \sigma_Y^2. \end{aligned}$$

(b) For joint Gaussian random variables (see Example 4.50, page 244 of textbook), if X and Y are jointly Gaussian random variables, then $Z(t) = Xt + Y$ is also a Gaussian random variable for any fixed t.

By part (a),

$$\begin{aligned} & m_Z(t) = tm_X + m_Y \\ & \text{VAR}[Z(t)] = C_Z(t,t) = t^2\sigma_X^2 + 2t\sigma_X\sigma_Y\rho_{XY} + \sigma_Y^2 \end{aligned}$$

Hence, the pdf of $Z(t)$ is

$$f_{Z(t)}(z) = \frac{1}{\sqrt{2\pi \text{VAR}[Z(t)]}} \exp\left\{\frac{-1}{2\text{VAR}[Z(t)]}(z - m_Z(t))^2\right\}.$$

3. Note that a binomial counting process has independent and stationary increments.

(a) Without loss of generality, we assume $n' > n$.

$$\begin{aligned} & P[S_n = j, S_{n'} = i] \\ &= P[S_n = j, S_{n'} - S_n = i - j] \quad \text{for} \quad i \geq j, 0 \leq j \leq n, 0 \leq i \leq n' \\ &= P[S_n = j]P[S_{n'} - S_n = i - j] \\ &= P[S_n = j]P[S_{n'-n} = i - j] \\ &\neq P[S_n = j]P[S_{n'} = i]. \end{aligned}$$

(b) Note that $n_2 > n_1 \Rightarrow S_{n_2} \geq S_{n_1}$.

When $i > j$,

$$P[S_{n_2} = j | S_{n_1} = i] = P[\phi] = 0.$$

When $i \leq j$,

$$\begin{aligned} P[S_{n_2} = j | S_{n_1} = i] &= P[S_{n_2} - S_{n_1} = j - i | S_{n_1} = i] \\ &= P[S_{n_2} - S_{n_1} = j - i] \\ &= P[S_{n_2 - n_1} = j - i] \\ &= C_{j-i}^{n_2 - n_1} p^{j-i}(1-p)^{n_2 - n_1 - j + i}. \end{aligned}$$

(c) We only need to prove the case when $j \geq i \geq k \geq 0$, otherwise, the probabilities on both sides are zero.

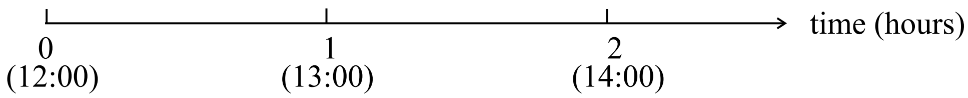
For $n_2 > n_1 > n_0, j \geq i \geq k \geq 0$,

$$
\begin{aligned}
& P[S_{n_2} = j | S_{n_1} = i, S_{n_0} = k] \\
&= \frac{P[S_{n_2} = j, S_{n_1} = i, S_{n_0} = k]}{P[S_{n_1} = i, S_{n_0} = k]} \\
&= \frac{P[S_{n_2} - S_{n_1} = j - i, S_{n_1} - S_{n_0} = i - k, S_{n_0} = k]}{P[S_{n_1} - S_{n_0} = i - k, S_{n_0} = k]} \\
&= \frac{P[S_{n_2} - S_{n_1} = j - i] P[S_{n_1} - S_{n_0} = i - k] P[S_{n_0} = k]}{P[S_{n_1} - S_{n_0} = i - k] P[S_{n_0} = k]} \\
&= P[S_{n_2} - S_{n_1} = j - i] \\
&= \frac{P[S_{n_2} - S_{n_1} = j - i] P[S_{n_1} = i]}{P[S_{n_1} = i]} \\
&= \frac{P[S_{n_2} - S_{n_1} = j - i, S_{n_1} = i]}{P[S_{n_1} = i]} \\
&= \frac{P[S_{n_2} = j, S_{n_1} = i]}{P[S_{n_1} = i]} = P[S_{n_2} = j | S_{n_1} = i].
\end{aligned}
$$

4. Let $N(t)$ = number of cars passing the intersection in $[0, t]$
   $X(t)$ = number of cars disregarding the stop-sign in $[0, t]$.
   Given $\lambda = 40$ per hour,

$$P[N(t) = k] = \frac{(40t)^k}{k!} e^{-40t}, \quad k = 0, 1, 2, \cdots.$$

Set the reference time point at 12:00, ie.,



P[at least 1 car disregarding the stop-sign between 12:00 and 13:00] $= P[X(1) \geq 1]$
Let p = probability that a car will disregard the stop-sign = 0.8%.
Note that $\{X(t) | N(t) = k\}$ has a binomial distribution with parameters k and p, that is,

$$P[X(t) = i | N(t) = k] = C_i^k p^i (1 - p)^{k-i}.$$

By the rule of total probabilities, we have

$$
\begin{aligned}
P[X(t) = i] &= \sum_{k=0}^{\infty} P[X(t) = i | N(t) = k] P[N(t) = k] \\
&= \sum_{k=0}^{\infty} C_i^k p^i (1 - p)^{k-i} \frac{(40t)^k}{k!} e^{-40t} \\
P[X(1) = 0] &= \sum_{k=0}^{\infty} C_0^k p^0 (1 - p)^k \frac{40^k}{4!} e^{-40}
\end{aligned}
$$

$$
\begin{aligned}
&= e^{-40}\sum_{k=0}^{\infty}\frac{[(1-p)40]^k}{k!} \\
&= e^{-40}\cdot e^{(1-p)40} \\
&= e^{-40p} = e^{-40\times 0.8\%} = e^{-0.32}.
\end{aligned}
$$

Hence,
$$
\begin{aligned}
P[X(1)\geq 1] &= 1-P[X(1)=0] \\
&= 1-e^{-0.32} \\
&= 0.2739.
\end{aligned}
$$

5. (a) Note that $N(t)=N_1(t)+N_2(t)$, we have

$$
\begin{aligned}
&\{N_1(t)=j, N_2(t)=k|N(t)=k+j\} \\
\Leftrightarrow\quad &\{N_1(t)=j|N(t)=k+j\}.
\end{aligned}
$$

This is because

$$
\begin{aligned}
&\{N_1(t)=j, N_2(t)=k|N(t)=k+j\} \\
\Leftrightarrow\quad &\{N_1(t)=j, N(t)-N_1(t)=k|N(t)=k+j\} \\
\Leftrightarrow\quad &\{N_1(t)=j, N_1(t)=N(t)-k|N(t)=k+j\} \\
\Leftrightarrow\quad &\{N_1(t)=j, N_1(t)=k+j-k|N(t)=k+j\} \\
\Leftrightarrow\quad &\{N_1(t)=j, N_1(t)=j|N(t)=k+j\} \\
\Leftrightarrow\quad &\{N_1(t)=j|N(t)=k+j\}
\end{aligned}
$$

Since p is the probability of a head showing up and $N_1(t)$ is the number of heads recorded up to time t, $\{N_1(t)|N\ (t)=k+j\}$ has a binomial distribution with parameters $k+j$ and p, we have

$$
\begin{aligned}
&P[N_1(t)=j, N_2(t)=k|N(t)=k+j] \\
&= P[N_1(t)=j|N(t)=k+j] \\
&= C_j^{k+j}p^j(1-p)^k.
\end{aligned}
$$

(b) Note that for an integer $n\neq k+j$,

$$
\begin{aligned}
&P[N_1(t)=j, N_2(t)=k|N(t)=n] \\
&= P[N_1(t)=j, N_2(t)=k|N_1(t)+N_2(t)=n] \\
&= P[\phi]=0.
\end{aligned}
$$

By the rule of total probabilities, we obtain

$$
\begin{aligned}
P[N_1(t)=j, N_2(t)=k] &= \sum_{n=0}^{\infty}P[N_1(t)=j, N_2(t)=k|N(t)=n]P[N(t)=n] \\
&= P[N_1(t)=j, N_2(t)=k|N(t)=k+j]P[N(t)=k+j] \\
&\quad+\sum_{n\neq k+j}^{\infty}P[N_1(t)=j, N_2(t)=k|N(t)=n]P[N(t)=n] \\
&= P[N_1(t)=j, N_2(t)=k|N(t)=k+j]P[N(t)=k+j] \\
&= C_j^{k+j}p^j(1-p)^k\cdot\frac{(\lambda t)^{k+j}}{(k+j)!}e^{-\lambda t} \\
&= \frac{(k+j)!}{j!k!}p^j(1-p)^k\frac{(\lambda t)^k(\lambda t)^j}{(k+j)!}e^{-\lambda t[p+(1-p)]} \\
&= \frac{(p\lambda t)^j}{j!}e^{-p\lambda t}\frac{[(1-p)\lambda t]^k}{k!}e^{-(1-p)\lambda t}. \qquad (1)
\end{aligned}
$$

We then have

$$\begin{aligned}
P[N_1(t)=j] &= \sum_{k=0}^{\infty} P[N_1(t)=j, N_2(t)=k] \\
&= \sum_{k=0}^{\infty} \frac{(p\lambda t)^j}{j!} e^{-p\lambda t} \frac{[(1-p)\lambda t]^k}{k!} e^{-(1-p)\lambda t} \\
&= \frac{(p\lambda t)^j}{j!} e^{-p\lambda t} e^{-(1-p)\lambda t} \sum_{k=0}^{\infty} \frac{[(1-p)\lambda t]^k}{k!} \\
&= \frac{(p\lambda t)^j}{j!} e^{-p\lambda t} \cdot e^{-(1-p)\lambda t} \cdot e^{(1-p)\lambda t} \\
&= \frac{(p\lambda t)^j}{j!} e^{-p\lambda t} \qquad (2)
\end{aligned}$$

which indicates that $N_1(t)$ is a Poisson random variable with rate $p\lambda$. Similarly, we can obtain

$$P[N_2(t)=k] = \frac{[(1-p)\lambda t]^k}{k!} e^{-(1-p)\lambda t} \qquad (3)$$

and so $N_2(t)$ is a Poisson random variable with rate $(1-p)\lambda$. Finally, from equations (1), (2) and (3), we can see that

$$P[N_1(t)=j, N_2(t)=k] = P[N_1(t)=j]P[N_2(t)=k].$$

Hence, $N_1(t)$ and $N_2(t)$ are independent.

6. Let $N(t)$ be the number of soft drinks dispensed up to time t, and $X(t)$ be the number of customer arrivals up to time t.

$$\begin{aligned}
P[N(t)=k] &= \sum_{n=k}^{\infty} P[N(t)=k|X(t)=n]P[X(t)=n] \\
&= \sum_{n=k}^{\infty} {}_cC_k p^k (1-p)^{n-k} \left[\frac{e^{-\lambda t}(\lambda t)^n}{n!}\right] \\
&= \sum_{m=0}^{\infty} {}_{m+k}C_k p^k (1-p)^m \frac{e^{-\lambda t}(\lambda t)^{m+k}}{(m+k)!}, \text{set } n=m+k \\
&= e^{-\lambda t} \left\{ \sum_{m=0}^{\infty} \frac{[\lambda t(1-p)]^m}{m!} \right\} \frac{(\lambda p t)^k}{k!} \\
&= e^{-\lambda t} e^{\lambda t(1-p)} \frac{(\lambda p t)^k}{k!} = \frac{e^{-\lambda p t}(\lambda p t)^k}{k!}, \quad k = 0, 1, 2, \cdots.
\end{aligned}$$

7. (a) We need to show that

"$Y(t)$ is a random telegraph signal" $\qquad (*)$

If (*) holds, together with the fact that the random telegraph signal is equally likely to be ± 1 at any time $t > 0$, we have

$$P[Y(t) = \pm 1] = \frac{1}{2}.$$

The proof of ($*$) goes below.

Assume $X(0)$ and $Y(0)$ have the same distribution. Let $N_X(t)$ be the Poisson process of rate α such that $N_X(t)$ is corresponding to the random telegraph signal $X(t)$.

Consider $N_Y(t)$ = number of times that $Y(t)$ has changed the polarity over $[0, t]$.

Then ($*$) holds if and only if $N_Y(t)$ is a Poisson random process.

Since $Y(t)$ changes the polarity with probability p if $X(t)$ changes polarity, the conditional random process $\{N_Y(t)|N_X(t) = n\}$ is a binomial random variable with parameters n and p, i.e.,

$$P[N_Y(t) = k|N_X(t) = n] = C_k^n p^k (1-p)^{n-k}, \quad n = 0, 1, 2, \cdots; k = 0, 1, \cdots, n.$$

where $N_X(t)$ = number of times that $X(t)$ has changed the polarity over $[0, t]$.

In general, for $0 \le t_1 < t_2 < \infty$, we have

$$\begin{aligned} & P[N_Y(t_2) - N_Y(t_1) = k|N_X(t_2) - N_X(t_1) = n] \\ = & C_k^n p^k (1-p)^{n-k}, \quad n = 0, 1, 2, \cdots; k = 0, 1, \cdots, n. \end{aligned}$$

By the rule of total probabilities, we have

$$\begin{aligned} P[N_Y(t) = k] &= \sum_{n=0}^{\infty} P[N_Y(t) = k|N_X(t) = n]P[N_X(t) = n] \\ &= \sum_{n=0}^{k-1} P[N_Y(t) = k|N_X(t) = n]P[N_X(t) = n] \\ &\quad + \sum_{n=k}^{\infty} P[N_Y(t) = k|N_X(t) = n]P[N_X(t) = n] \\ &= \sum_{n=k}^{\infty} C_k^n p^k (1-p)^{n-k} \frac{(\alpha t)^n}{n!} e^{-\alpha t} \\ &= \sum_{n=k}^{\infty} \frac{n! p^k (1-p)^{n-k}}{k!(n-k)!} \cdot \frac{(\alpha t)^{n-k+k}}{n!} e^{-\alpha t} \\ &= \frac{(p\alpha t)^k}{k!} e^{-\alpha t} \sum_{n=k}^{\infty} \frac{[(1-p)\alpha t]^{n-k}}{(n-k)!} \\ &= \frac{(p\alpha t)^k}{k!} e^{-\alpha t} \sum_{m=0}^{\infty} \frac{[(1-p)\alpha t]^m}{m!} \quad \text{by } m = n-k \\ &= \frac{(p\alpha t)^k}{k!} e^{-\alpha t} e^{(1-p)\alpha t} \\ &= \frac{(p\alpha t)^k}{k!} e^{-p\alpha t} \end{aligned}$$

which indicates that $N_Y(t)$ is a Poisson random variable with parameter $p\alpha$. Thus $\{N_Y(t), t \ge 0\}$ is a Poisson random process.

(b) Recall that $C_X(t_1, t_2) = e^{-2\alpha|t_2 - t_1|}$.

For $t_1 < t_2$,

$$C_Y(t_1, t_2) = E[Y(t_1)Y(t_2)] - E[Y(t_1)]E[Y(t_2)].$$

Now,

$$\begin{aligned} E[Y(t)] &= (1)P[Y(t)=1]+(-1)P[Y(t)=-1] \\ &= (1)\left(\frac{1}{2}\right)+(-1)\left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

so

$$\begin{aligned} C_Y(t_1,t_2) &= E[Y(t_1)Y(t_2)]-E[Y(t_1)]E[Y(t_2)] \\ &= (1)P[Y(t_1)Y(t_2)=1]+(-1)P[Y(t_1)Y(t_2)=-1] \\ &= P[Y(t_1)=Y(t_2)]-P[Y(t_1)\neq Y(t_2)] \\ &= P[N_Y(t_2)-N_Y(t_1)=\text{ even number}] \\ &\quad -P[N_Y(t_2)-N_Y(t_1)=\text{ odd number}] \\ &= P[N_Y(t_2-t_1)=\text{ even number}]-P[N_Y(t_2-t_1)=\text{ odd number}] \\ &= \sum_{k=0}^{\infty}P[N_Y(t_2-t_1)=2k]-\sum_{k=0}^{\infty}P[N_Y(t_2-t_1)=2k+1] \\ &= e^{-p\alpha(t_2-t_1)}\left\{\sum_{k=0}^{\infty}\frac{[p\alpha(t_2-t_1)]^{2k}}{(2k)!}-\sum_{k=0}^{\infty}\frac{[p\alpha(t_2-t_1)]^{2k+1}}{(2k+1)!}\right\} \\ &= e^{-p\alpha(t_2-t_1)}\left\{\frac{1}{2}[e^{p\alpha(t_2-t_1)}+e^{-p\alpha(t_2-t_1)}]\right. \\ &\quad \left.-\frac{1}{2}[e^{p\alpha(t_2-t_1)}-e^{-p\alpha(t_2-t_1)}]\right\} \\ &= e^{-2p\alpha(t_2-t_1)}. \end{aligned}$$

Similarly,

$$C_Y(t_1,t_2)=e^{-2p\alpha(t_1-t_2)} \text{ for } t_1>t_2.$$

Hence, in general for any t_1,t_2,

$$C_Y(t_1,t_2)=e^{-2p\alpha|t_2-t_1|}=[C_X(t_1,t_2)]^p.$$

8. (a) Given $S=\{0,1,2\}$.

$$\begin{aligned} &P[X_{n+1}=j|X_n=i,X_{n-1}=x_{n-1},\cdots,X_0=x_0] \\ &= P[\text{There are } (j-i) \text{ more working parts on } (n+1)^{\text{th}} \text{ day than those on } n^{\text{th}} \text{ day}|X_n=i] \\ &= P[X_{n+1}=j|X_n=i] \end{aligned}$$

so X_n is a three-state Markov Chain. Note that

$$\begin{aligned} p_{00} &= P[X_{n+1}=0|X_n=0]=(1-b)^2 \\ p_{01} &= P[X_{n+1}=1|X_n=0]=2b(1-b) \\ p_{02} &= P[X_{n+1}=2|X_n=0]=b^2 \\ p_{10} &= P[X_{n+1}=0|X_n=1]=a(1-b) \\ p_{11} &= P[X_{n+1}=1|X_n=1]=ab+(1-a)(1-b) \\ p_{12} &= P[X_{n+1}=2|X_n=1]=(1-a)b \\ p_{20} &= P[X_{n+1}=0|X_n=2]=a^2 \\ p_{21} &= P[X_{n+1}=1|X_n=2]=2a(1-a) \\ p_{22} &= P[X_{n+1}=2|X_n=2]=(1-a)^2 \end{aligned}$$

Hence, the one-step transition probability matrix is

$$P = \begin{bmatrix} (1-b)^2 & 2b(1-b) & b^2 \\ a(1-b) & ab+(1-a)(1-b) & (1-a)b \\ a^2 & 2a(1-a) & (1-a)^2 \end{bmatrix}.$$

(b) Let $\boldsymbol{\pi} = [\pi_{\infty,0} \quad \pi_{\infty,1} \quad \pi_{\infty,2}] = [p_1 \quad p_2 \quad p_3]$ be the steady state pmf.

$$\boldsymbol{\pi} = \boldsymbol{\pi} P \quad \Rightarrow \quad [p_1 \quad p_2 \quad p_3] = [p_1 \quad p_2 \quad p_3]P$$

Expanding into individual components, we obtain

$$\begin{aligned} p_1 &= (1-b)^2 p_1 + a(1-b)p_2 + a^2 p_3 \\ p_2 &= 2b(1-b)p_1 + [ab + (1-a)(1-b)]p_2 + 2a(1-a)p_3 \\ p_3 &= b^2 p_1 + (1-a)b\ p_2 + (1-a)^2 p_3 \end{aligned}$$

We drop the second equation and observe that the sum of probabilities equals one. Hence, we obtain

$$\begin{aligned} -a^2 p_3 &= (b^2 - 2b)p_1 + a(1-b)p_2 & (i) \\ -b^2 p_1 &= (a^2 - 2a)p_3 + b(1-a)p_2 & (ii) \\ p_1 + p_2 + p_3 &= 1. & (iii) \end{aligned}$$

From Eqs (i) and (ii), we have

$$\begin{aligned} & -b^2 p_1 = b(1-a)p_2 - \frac{a^2 - 2a}{a^2}[(b^2 - 2b)p_1 + a(1-b)p_2] \\ & ab^2 p_1 + ab(1-a)p_2 + (2-a)[(b^2 - 2b)p_1 + a(1-b)p_2] = 0 \\ & 2(b^2 + ab - 2b)p_1 = (a^2 + ab - 2a)p_2 \\ & p_1 = \frac{a}{2b} p_2. \end{aligned}$$

From Eq. (ii): $-\frac{ab}{2}p_2 = (a^2 - 2a)p_3 + b(1-a)p_2 \Rightarrow p_3 = \frac{b}{2a}p_2.$

From Eq. (iii): $\frac{a}{2b}p_2 + p_2 + \frac{b}{2a}p_2 = 1 \Rightarrow p_2 = \frac{2ab}{(a+b)^2}$

$$\text{so} \quad p_1 = \frac{a^2}{(a+b)^2}, \quad p_3 = \frac{b^2}{(a+b)^2}.$$

Hence, the general form of steady state pmf is given by

$$\pi_{\infty,i} = C_i^2 \left(\frac{a}{a+b}\right)^i \left(1 - \frac{b}{a+b}\right)^{2-i}, \quad i = 0, 1, 2.$$

Therefore, the entries of $\boldsymbol{\pi}$ are binomial coefficients with parameter $p = \frac{b}{a+b}$.

(c) For a machine that consists of n parts, the steady state pmf should still be binomial with parameters n and $p = \frac{b}{a+b}$.