1. Note that for all \( n \),
\[
X_n = \begin{cases} 
1 & \text{if the outcome is } H \\
-1 & \text{if the outcome is } T 
\end{cases}
\]

(a) The only two sample paths:

(b) Given that the coin is fair, we have
\[
P[X_n = 1] = P[\text{outcome is } H] = \frac{1}{2}
\]
\[
P[X_n = -1] = P[\text{outcome is } T] = \frac{1}{2}
\]

(c) \( P[X_n = 1, X_{n+k} = 1] = P[X_n = 1] = \frac{1}{2} \)
\[
P[X_n = -1, X_{n+k} = -1] = P[X_n = -1] = \frac{1}{2}
\]
\[
P[X_n = 1, X_{n+k} = -1] = P[\phi] = 0
\]
\[
P[X_n = -1, X_{n+k} = 1] = P[\phi] = 0
\]

Hence, the joint pmf
\[
P[X_n = i, X_{n+k} = j] = \begin{cases} 
\frac{1}{2}, & i = j \\
0, & i \neq j
\end{cases}
\]

(d) \( P[X_n] = (1)P[X_n = 1] + (-1)P[X_n = -1] = \frac{1}{2} - \frac{1}{2} = 0. \)

\[
C_X(n_1, n_2) = E[(X_{n_1} - E[X_{n_1}])[X_{n_2} - E[X_{n_2}]]] 
= E[X_{n_1}X_{n_2}]
= (1)(1)P[X_{n_1} = 1, X_{n_2} = 1] + (-1)(-1)P[X_{n_1} = -1, X_{n_2} = -1]
= \frac{1}{2} + \frac{1}{2} = 1.
\]
2. (a) \( E[Z(t)] = E[Xt + Y] = tm_X + m_Y \)

\[
C_Z(t_1, t_2) = E[\{(Xt_1 + Y) - (t_1m_X + m_Y)\}\{(Xt_2 + Y) - (t_2m_X + m_Y)\}]
= E[\{(t_1(X - m_X) + (Y - m_Y))\}\{(t_2(X - m_X) + (Y - m_Y))\}]
= t_1t_2E[(X - m_X)^2] + t_1E[(X - m_X)(Y - m_Y)]
+ E[(Y - m_Y)^2] + t_2E[(Y - m_Y)(X - m_X)]
= t_1t_2\sigma_X^2 + (t_1 + t_2)\sigma_X\sigma_Y + \sigma_Y^2.
\]

(b) For joint Gaussian random variables (see Example 4.50, page 244 of textbook), if \( X \) and \( Y \) are jointly Gaussian random variables, then \( Z(t) = Xt + Y \) is also a Gaussian random variable for any fixed \( t \).

By part (a),

\[ m_Z(t) = tm_X + m_Y, \]

\[ \text{VAR}[Z(t)] = C_Z(t, t) = t^2\sigma_X^2 + 2t\sigma_X\sigma_Y + \sigma_Y^2. \]

Hence, the pdf of \( Z(t) \) is

\[
f_{Z(t)}(z) = \frac{1}{\sqrt{2\pi\text{VAR}[Z(t)]}} \exp \left\{ -\frac{1}{2\text{VAR}[Z(t)]}(z - m_Z(t))^2 \right\}.
\]

3. Note that a binomial counting process has independent and stationary increments.

(a) Without loss of generality, we assume \( n' > n \).

\[
P[S_n = j, S_{n'} = i]
= P[S_n = j, S_{n'} - S_n = i - j] \quad \text{for} \quad i \geq j, 0 \leq j \leq n, 0 \leq i \leq n',
= P[S_n = j]P[S_{n'} - S_n = i - j]
= P[S_n = j]P[S_{n'} - n = i - j]
\neq P[S_n = j]P[S_{n'} = i].
\]

(b) Note that \( n_2 > n_1 \Rightarrow S_{n_2} \geq S_{n_1} \).

When \( i > j \),

\[
P[S_{n_2} = j | S_{n_1} = i] = P[\emptyset] = 0.
\]

When \( i \leq j \),

\[
P[S_{n_2} = j | S_{n_1} = i]
= P[S_{n_2} - S_{n_1} = j - i | S_{n_1} = i]
= P[S_{n_2} - S_{n_1} = j - i]
= P[S_{n_2} - n_1 = j - i]
= C_{n_2-n_1}^{n_2-i}p^{n_2-i}(1-p)^{n_2-n_1-j+i}.
\]

(c) We only need to prove the case when \( j \geq i \geq k \geq 0 \), otherwise, the probabilities on both sides are zero.
For \( n_2 > n_1 > n_0, j \geq i \geq k \geq 0, \)

\[
\begin{align*}
P[S_{n_2} = j|S_{n_1} = i, S_{n_0} = k] &= \frac{P[S_{n_2} = j, S_{n_1} = i, S_{n_0} = k]}{P[S_{n_1} = i, S_{n_0} = k]} \\
&= \frac{P[S_{n_2} - S_{n_1} = j - i, S_{n_1} - S_{n_0} = i - k, S_{n_0} = k]}{P[S_{n_1} - S_{n_0} = i - k]P[S_{n_0} = k]} \\
&= \frac{P[S_{n_2} - S_{n_1} = j - i]P[S_{n_1} = i]}{P[S_{n_1} = i]} \\
&= \frac{P[S_{n_2} - S_{n_1} = j - i]}{P[S_{n_1} = i]} = P[S_{n_2} = j|S_{n_1} = i].
\end{align*}
\]

4. Let \( N(t) = \) number of cars passing the intersection in \([0, t]\)
\( \ X(t) = \) number of cars disregarding the stop-sign in \([0, t]\).
Given \( \lambda = 40 \) per hour,

\[
P[N(t) = k] = \frac{(40t)^k}{k!} e^{-40t}, \quad k = 0, 1, 2, \ldots
\]

Set the reference time point at 12:00, ie.,

\[
P[\text{at least 1 car disregarding the stop-sign between 12:00 and 13:00}] = P[X(1) \geq 1]
\]

Let \( p = \) probability that a car will disregard the stop-sign = 0.8%.

Note that \( \{X(t)|N(t) = k\} \) has a binomial distribution with parameters \( k \) and \( p \), that is,

\[
P[X(t) = i|N(t) = k] = C_k^i p^i (1 - p)^{k-i}.
\]

By the rule of total probabilities, we have

\[
P[X(t) = i] = \sum_{k=0}^{\infty} P[X(t) = i|N(t) = k]P[N(t) = k]
\]

\[
= \sum_{k=0}^{\infty} C_k^i p^i (1 - p)^{k-i} \frac{(40t)^k}{k!} e^{-40t}
\]

\[
P[X(1) = 0] = \sum_{k=0}^{\infty} C_k^0 p^0 (1 - p)^k \frac{40^k}{4!} e^{-40}
\]

\[3\]
5. (a) Note that $N(t) = N_1(t) + N_2(t)$, we have
\[
\{N_1(t) = j, N_2(t) = k | N(t) = k + j\} 
\Leftrightarrow \{N_1(t) = j | N(t) = k + j\}.
\]
This is because
\[
\{N_1(t) = j, N_2(t) = k | N(t) = k + j\} 
\Leftrightarrow \{N_1(t) = j, N(t) - N_1(t) = k | N(t) = k + j\} 
\Leftrightarrow \{N_1(t) = j, N_1(t) = N(t) - k | N(t) = k + j\} 
\Leftrightarrow \{N_1(t) = j, N(t) = k + j - k | N(t) = k + j\} 
\Leftrightarrow \{N_1(t) = j, N_1(t) = j | N(t) = k + j\} 
\Leftrightarrow \{N_1(t) = j | N(t) = k + j\}.
\]
Since $p$ is the probability of a head showing up and $N_1(t)$ is the number of heads recorded up to time $t$, $\{N_1(t) | N(t) = k + j\}$ has a binomial distribution with parameters $k + j$ and $p$, we have
\[
P[N_1(t) = j, N_2(t) = k | N(t) = k + j] 
= P[N_1(t) = j | N(t) = k + j] 
= C_{k+j}^{j} p^j (1-p)^{k}.
\]
(b) Note that for an integer $n \neq k + j$,
\[
P[N_1(t) = j, N_2(t) = k | N(t) = n] 
= P[N_1(t) = j, N_2(t) = k | N_1(t) + N_2(t) = n] 
= P[n] = 0.
\]
By the rule of total probabilities, we obtain
\[
P[N_1(t) = j, N_2(t) = k] 
= \sum_{n=0}^{\infty} P[N_1(t) = j, N_2(t) = k | N(t) = n] P[N(t) = n] 
= P[N_1(t) = j, N_2(t) = k | N(t) = k + j] P[N(t) = k + j] 
+ \sum_{n \neq k+j}^{\infty} P[N_1(t) = j, N_2(t) = k | N(t) = n] P[N(t) = n] 
= P[N_1(t) = j, N_2(t) = k | N(t) = k + j] 
= C_{k+j}^{j} p^j (1-p)^{k} \cdot \frac{(\lambda t)^{k+j}}{(k+j)!} e^{-\lambda t} 
= \frac{(k + j)!}{j! k!} p^j (1-p)^{k} \cdot \frac{(\lambda t)^{k+j}}{(k+j)!} e^{-\lambda t[p+(1-p)]} 
= \frac{(p\lambda)^j}{j!} e^{-\lambda t} \frac{[(1-p)\lambda t]^k}{k!} e^{-(1-p)\lambda t}.
\]
We then have

\[ P[N_1(t) = j] = \sum_{k=0}^{\infty} P[N_1(t) = j, N_2(t) = k] \]
\[ = \sum_{k=0}^{\infty} \frac{(p\lambda t)^j}{j!} e^{-p\lambda t} \frac{[(1-p)\lambda t]^k}{k!} e^{-(1-p)\lambda t} \]
\[ = \frac{(p\lambda t)^j}{j!} e^{-p\lambda t} \sum_{k=0}^{\infty} \frac{[(1-p)\lambda t]^k}{k!} \]
\[ = \frac{(p\lambda t)^j}{j!} e^{-p\lambda t} \cdot e^{-(1-p)\lambda t} \cdot e^{(1-p)\lambda t} \]
\[ = \frac{(p\lambda t)^j}{j!} e^{-p\lambda t} \]  

which indicates that \( N_1(t) \) is a Poisson random variable with rate \( p\lambda \). Similarly, we can obtain

\[ P[N_2(t) = k] = \frac{[(1-p)\lambda t]^k}{k!} e^{-(1-p)\lambda t} \]  

and so \( N_2(t) \) is a Poisson random variable with rate \( (1-p)\lambda \). Finally, from equations (1), (2) and (3), we can see that

\[ P[N_1(t) = j, N_2(t) = k] = P[N_1(t) = j]P[N_2(t) = k]. \]

Hence, \( N_1(t) \) and \( N_2(t) \) are independent.

6. Let \( N(t) \) be the number of soft drinks dispensed up to time \( t \), and \( X(t) \) be the number of customer arrivals up to time \( t \).

\[ P[N(t) = k] = \sum_{n=k}^{\infty} P[N(t) = k|X(t) = n]P[X(t) = n] \]
\[ = \sum_{n=k}^{\infty} e^{\lambda t} p^k (1-p)^{n-k} \left[ \frac{e^{-\lambda t}(\lambda t)^n}{n!} \right] \]
\[ = \sum_{n=k}^{\infty} e^{\lambda t} \cdot \frac{\lambda^m (\lambda t)^m}{(m+k)!} \quad \text{set } n = m + k \]
\[ = e^{-\lambda t} \sum_{m=0}^{\infty} \frac{[\lambda t(1-p)]^m}{m!} \frac{(\lambda pt)^k}{k!} \]
\[ = e^{-\lambda t} e^{\lambda t(1-p)} \frac{(\lambda pt)^k}{k!} = e^{-\lambda pt} \frac{(\lambda pt)^k}{k!}, \quad k = 0, 1, 2, \ldots. \]

7. (a) We need to show that

“\( Y(t) \) is a random telegraph signal” \((*)\)

If (*) holds, together with the fact that the random telegraph signal is equally likely to be \( \pm 1 \) at any time \( t > 0 \), we have

\[ P[Y(t) = \pm 1] = \frac{1}{2}. \]
The proof of (*) goes below.
Assume $X(0)$ and $Y(0)$ have the same distribution. Let $N_X(t)$ be the Poisson process of rate $\alpha$ such that $N_X(t)$ is corresponding to the random telegraph signal $X(t)$.
Consider $N_Y(t)$ = number of times that $Y(t)$ has changed the polarity over $[0, t]$.
Then (*) holds if and only if $N_Y(t)$ is a Poisson random process.
Since $Y(t)$ changes the polarity with probability $p$ if $X(t)$ changes polarity, the conditional random process $\{N_Y(t)|N_X(t) = n\}$ is a binomial random variable with parameters $n$ and $p$, i.e.,
\[
P[N_Y(t) = k|N_X(t) = n] = \binom{n}{k} p^k (1-p)^n - k, \quad n = 0, 1, 2, \ldots; k = 0, 1, \ldots, n.
\]
where $N_X(t)$ = number of times that $X(t)$ has changed the polarity over $[0, t]$.
In general, for $0 \leq t_1 < t_2 < \infty$, we have
\[
P[N_Y(t_2) - N_Y(t_1) = k|N_X(t_2) - N_X(t_1) = n] = \binom{n}{k} p^k (1-p)^{n-k}, \quad n = 0, 1, 2, \ldots; k = 0, 1, \ldots, n.
\]
By the rule of total probabilities, we have
\[
P[N_Y(t) = k] = \sum_{n=0}^{\infty} P[N_Y(t) = k|N_X(t) = n]P[N_X(t) = n]
\]
\[= \sum_{n=0}^{k-1} P[N_Y(t) = k|N_X(t) = n]P[N_X(t) = n]
\]
\[+ \sum_{n=k}^{\infty} P[N_Y(t) = k|N_X(t) = n]P[N_X(t) = n]
\]
\[= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \frac{(\alpha t)^n}{n!} e^{-\alpha t}
\]
\[= \sum_{n=k}^{\infty} \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \frac{(\alpha t)^n}{n!} e^{-\alpha t}
\]
\[= \frac{(\text{pata})^k}{k!} e^{-\alpha t} \sum_{n=k}^{\infty} \frac{[(1-p)\alpha t]^{n-k}}{(n-k)!}
\]
\[= \frac{(\text{pata})^k}{k!} e^{-\alpha t} \sum_{m=0}^{\infty} \frac{[(1-p)\alpha t]^{m}}{m!} \quad \text{by } m = n - k
\]
\[= \frac{(\text{pata})^k}{k!} e^{-\alpha t} e^{(1-p)\alpha t}
\]
\[= \frac{(\text{pata})^k}{k!} e^{-\alpha t}
\]
which indicates that $N_Y(t)$ is a Poisson random variable with parameter $p\alpha$. Thus $\{N_Y(t), t \geq 0\}$ is a Poisson random process.
(b) Recall that $C_X(t_1, t_2) = e^{-2\alpha|t_2-t_1|}$.
For $t_1 < t_2$,
\[
C_Y(t_1, t_2) = E[Y(t_1)Y(t_2)] - E[Y(t_1)]E[Y(t_2)].
\]
Now,
\[ E[Y(t)] = (1)P[Y(t) = 1] + (-1)P[Y(t) = -1] \]
\[ = (1) \left( \frac{1}{2} \right) + (-1) \left( \frac{1}{2} \right) \]
\[ = 0 \]

so

\[ C_Y(t_1, t_2) = E[Y(t_1)Y(t_2)] - E[Y(t_1)]E[Y(t_2)] \]
\[ = (1)P[Y(t_1)Y(t_2) = 1] + (-1)P[Y(t_1)Y(t_2) = -1] \]
\[ = P[Y(t_1) = Y(t_2)] - P[Y(t_1) \neq Y(t_2)] \]
\[ = P[N_Y(t_2) - N_Y(t_1) = \text{even number}] \]
\[ - P[N_Y(t_2) - N_Y(t_1) = \text{odd number}] \]
\[ = P[N_Y(t_2 - t_1) = \text{even number}] - P[N_Y(t_2 - t_1) = \text{odd number}] \]
\[ = \sum_{k=0}^{\infty} P[N_Y(t_2 - t_1) = 2k] - \sum_{k=0}^{\infty} P[N_Y(t_2 - t_1) = 2k + 1] \]
\[ = e^{-p_0(t_2-t_1)} \left\{ \sum_{k=0}^{\infty} \frac{p_0(t_2-t_1)^{2k}}{(2k)!} - \sum_{k=0}^{\infty} \frac{p_0(t_2-t_1)^{2k+1}}{(2k+1)!} \right\} \]
\[ = e^{-p_0(t_2-t_1)} \left\{ \frac{1}{2} \left[ e^{p_0(t_2-t_1)} + e^{-p_0(t_2-t_1)} \right] \right\} \]
\[ = e^{-2p_0(t_2-t_1)}. \]

Similarly,

\[ C_Y(t_1, t_2) = e^{-2p_0(t_1-t_2)} \text{ for } t_1 > t_2. \]

Hence, in general for any \( t_1, t_2, \)
\[ C_Y(t_1, t_2) = e^{-2p_0|t_2-t_1|} = [C_X(t_1, t_2)]^p. \]

8. (a) Given \( S = \{0, 1, 2\}. \)

\[ P[X_{n+1} = j|X_n = i, X_{n-1} = x_{n-1}, \ldots, X_0 = x_0] \]
\[ = P[\text{There are } (j - i) \text{ more working parts on } (n + 1)\text{th day than those on } n\text{th day}|X_n = i] \]
\[ = P[X_{n+1} = j|X_n = i] \]

so \( X_n \) is a three-state Markov Chain. Note that

\[
\begin{align*}
p_{00} &= P[X_{n+1} = 0|X_n = 0] = (1 - b)^2 \\
p_{01} &= P[X_{n+1} = 1|X_n = 0] = 2b(1 - b) \\
p_{02} &= P[X_{n+1} = 2|X_n = 0] = b^2 \\
p_{10} &= P[X_{n+1} = 0|X_n = 1] = a(1 - b) \\
p_{11} &= P[X_{n+1} = 1|X_n = 1] = ab + (1 - a)(1 - b) \\
p_{12} &= P[X_{n+1} = 2|X_n = 1] = (1 - a)b \\
p_{20} &= P[X_{n+1} = 0|X_n = 2] = a^2 \\
p_{21} &= P[X_{n+1} = 1|X_n = 2] = 2a(1 - a) \\
p_{22} &= P[X_{n+1} = 2|X_n = 2] = (1 - a)^2
\end{align*}
\]
Hence, the one-step transition probability matrix is
\[
P = \begin{bmatrix}
(1 - b)^2 & 2b(1 - b) & b^2 \\
\alpha(1 - b) & \alpha b + (1 - \alpha)(1 - b) & (1 - \alpha)b \\
\alpha^2 & 2\alpha(1 - \alpha) & (1 - \alpha)^2
\end{bmatrix}.
\]

(b) Let \( \mathbf{\pi} = [\pi_{\infty,0} \, \pi_{\infty,1} \, \pi_{\infty,2}] = [p_1 \, p_2 \, p_3] \) be the steady state pmf.
\[
\mathbf{\pi} = \mathbf{\pi} P \Rightarrow [p_1 \, p_2 \, p_3] = [p_1 \, p_2 \, p_3] P
\]

Expanding into individual components, we obtain
\[
p_1 = (1 - b)^2 p_1 + \alpha(1 - b)p_2 + \alpha^2 p_3
\]
\[
p_2 = 2b(1 - b)p_1 + [\alpha b + (1 - \alpha)(1 - b)]p_2 + 2\alpha(1 - \alpha)p_3
\]
\[
p_3 = b^2 p_1 + (1 - \alpha)b p_2 + (1 - \alpha)^2 p_3
\]

We drop the second equation and observe that the sum of probabilities equals one. Hence, we obtain
\[
-a^2 p_3 = (b^2 - 2b)p_1 + \alpha(1 - b)p_2 \\
-b^2 p_1 = (a^2 - 2a)p_3 + (1 - \alpha)p_2 \\
p_1 + p_2 + p_3 = 1
\]

From Eqs (i) and (ii), we have
\[
-b^2 p_1 = b(1 - a)p_2 - \frac{a^2 - 2a}{a^2} [(b^2 - 2b)p_1 + \alpha(1 - b)p_2] \\
ab p_1 + \alpha b(1 - a)p_2 + (2 - a)(b^2 - 2b)p_1 + \alpha(1 - b)p_2 = 0 \\
2(b^2 - ab - 2b)p_1 = (a^2 + ab - 2a)p_2
\]
\[
p_1 = \frac{ab}{2b} p_2.
\]

From Eq. (ii):
\[
\frac{ab}{2} p_2 = (a^2 - 2a)p_3 + b(1 - a)p_2 \Rightarrow p_3 = \frac{b}{2a} p_2.
\]

From Eq. (iii):
\[
\frac{a}{2b} p_2 + p_2 + \frac{b}{2a} p_2 = 1 \Rightarrow p_2 = \frac{2ab}{(a + b)^2}
\]
\[
\text{so} \quad p_1 = \frac{a^2}{(a + b)^2}, \quad p_3 = \frac{b^2}{(a + b)^2}.
\]

Hence, the general form of steady state pmf is given by
\[
\pi_{\infty,i} = C_i^2 \left( \frac{a}{a + b} \right)^i \left( 1 - \frac{b}{a + b} \right)^{2-i}, \quad i = 0, 1, 2.
\]

Therefore, the entries of \( \mathbf{\pi} \) are binomial coefficients with parameter \( p = \frac{b}{a + b} \).

(c) For a machine that consists of \( n \) parts, the steady state pmf should still be binomial with parameters \( n \) and \( p = \frac{b}{a + b} \).