Random experiments

A random experiment is a process characterized by the following properties:

(i) It is performed according to some set of rules,
(ii) It can be repeated arbitrarily often,
(iii) The result of each performance depends on *chance* and cannot be predicted uniquely.

Example: Tossing of a coin

The outcome of a trial can be either head or tail showing up.

Sequential random experiments –
performing a sequence of simple random sub-experiments

eg First toss a coin, then throw a dice.

Sometimes, the second sub-experiment depends on the outcome of the first; eg Toss a coin first, if it is a head, then throw a dice.

A random experiment may involve a continuum of measurements. Say, the height of a student takes some value between 1.4m to 2m.

Sample space S of a random experiment is defined as the set of *all possible* outcomes.

Outcomes are *mutually exclusive* in the sense that they cannot occur simultaneously.

A sample space can be finite, countably infinite or uncountably infinite.

1. Toss a coin two times

   $S_1 = \{(H, H), (H, T), (T, H), (T, T)\}$

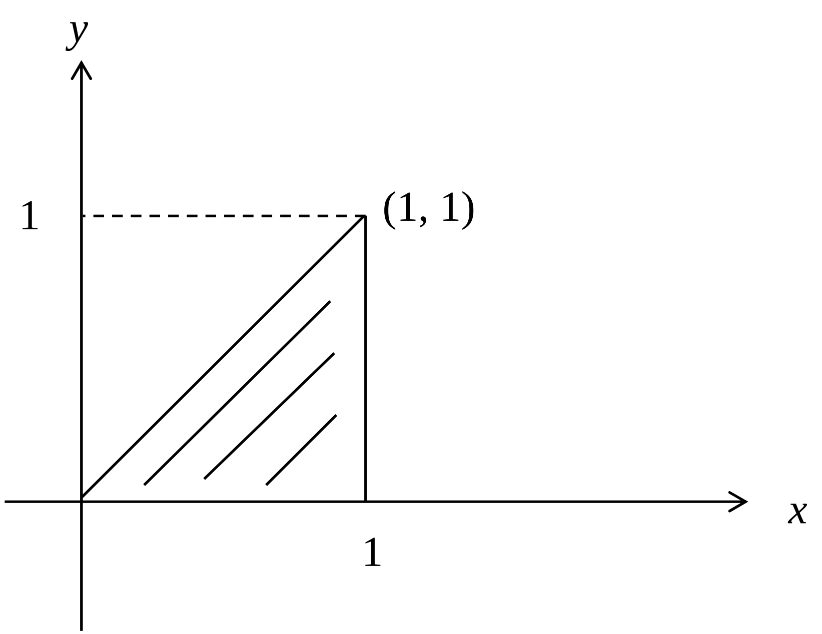
   S_1 is countable, S_1 is called a *discrete* sample space. Define $B = \{H, T\}$, then $S_1 = B \times B$.

2. Toss a dice until a ‘six’ appears and count the number of times the dice was tossed.

$S_2 = \{1, 2, 3, \ldots\}$;
S_2 is discrete and countably infinite (one-to-one correspondence with positive integers)

3. Pick a number X at random between zero and one, then pick a number Y at random between zero and X.

$S_3 = \{(x, y): 0 \leq y \leq x \leq 1\}$;
S_3 is a continuous sample space.



- An *event* or event set is a set of possible outcomes of an experiment, so an event is a subset of sample space S.

- The whole sample space is an event and is called the *sure* event.

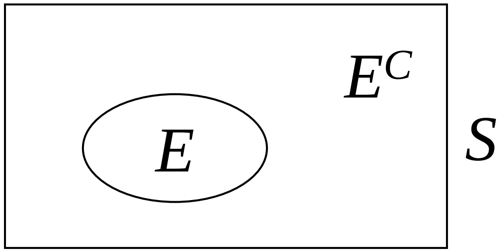
- The empty set ϕ is called the *impossible event*.

Example Tossing of a dice

Event E: dice turns up an even number; $E = \{2, 4, 6\}$, which is a subset of the sample space $S = \{1, 2, 3, 4, 5, 6\}$.

E^C – complement of E in S: defined as the set of elements not in E.

$E^C = \{1, 3, 5\}$, the dice turns up an odd number.



Suppose A and B are events in S, the following events are called derived events

(i) $A \cup B$ (either A or B or both)
(ii) $A \cap B$ (both A and B)
(*iii*) $A - B$ (A but not B)

Two events A and B are mutually exclusive if both cannot occur simultaneously, that is, $A \cap B = \phi$.

$\boldsymbol{A \subset B}$

Event A is a subset of event B, then event B will occur whenever event A occurs.

(i) $A \cap B \subset A$ and $A \cap B \subset B$

(ii) $A \subset A \cup B$ and $B \subset A \cup B$

$\boldsymbol{A = B}$ Two events are equal if they contain the same set of outcomes.

Notation

$$\bigcup_{k=1}^{n} A_k = A_1 \cup A_2 \cdots \cup A_n \quad \text{and} \quad \bigcap_{k=1}^{n} A_k = A_1 \cap A_2 \cdots \cap A_n$$

For countably infinite sequence of events, we have

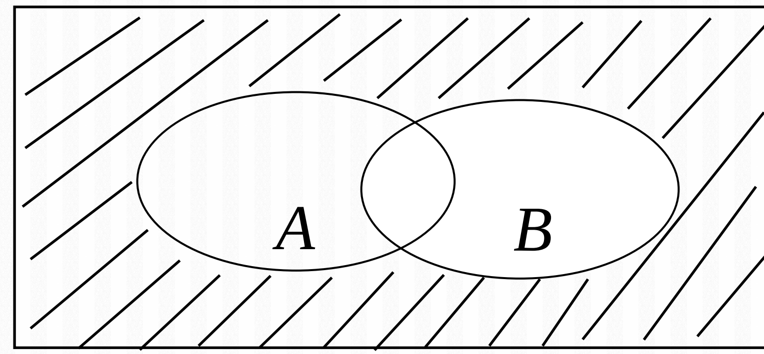
$$\bigcup_{k=1}^{\infty} A_k \qquad \text{and} \qquad \bigcap_{k=1}^{\infty} A_k$$

De Morgan's rules

$$(A \cap B)^c = A^c \cup B^c \qquad \text{and} \qquad (A \cup B)^c = A^c \cap B^c$$

Proof of the second rule:

Suppose $x \in (A \cup B)^c$



$\Leftrightarrow$ x is not contained in any of the events A and B

$\Leftrightarrow$ x is contained in A^c and B^c

$\Leftrightarrow$ $x \in A^c \cap B^c$.

Proof of the first rule:

Based on the second rule, take $A \to A^c$ and $B \to B^c$, we then have

$$(A^c \cup B^c)^c = A \cap B.$$

Taking complement on both sides, we obtain the first rule.

What do we mean by the probability $P[E]$ of an event E? For example, what is the probability of getting a head in the toss of a coin?

Statistically, the probability $P[E]$ is defined as

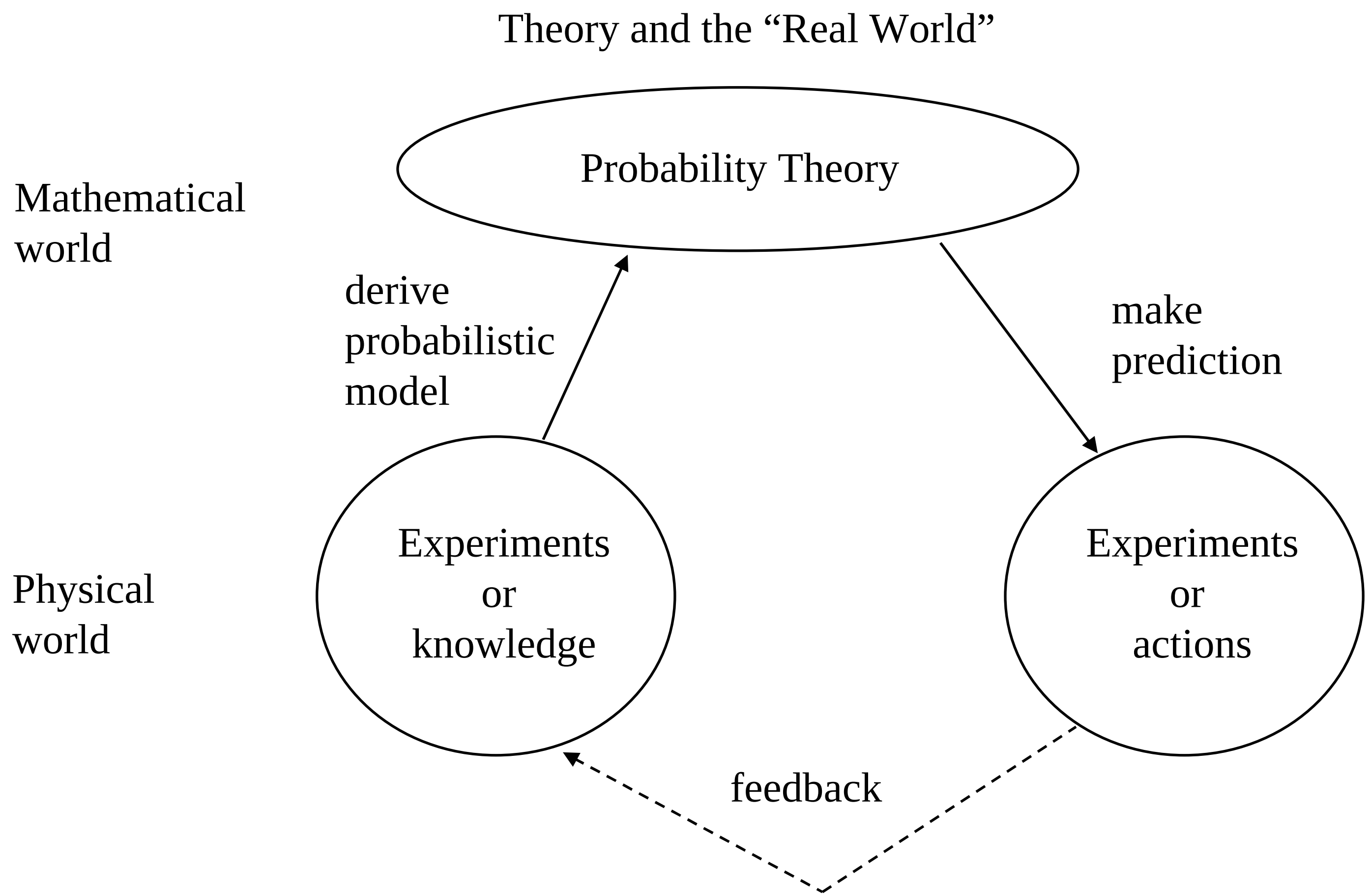
$$P[E] = \lim_{n\to\infty} \frac{f_n[E]}{n},$$

where n is the number of trials and $\frac{f_n[E]}{n}$ is the relative frequency of the occurrence of the event E. This is the *frequency approach*.

Statistical regularity

Averages obtained in long sequences of trials of random experiments consistently yield approximately the same value.

Can we estimate (calculate) the probability from the knowledge of the nature of the experiment?



Axioms of probability

Let E be a random experiment with sample space S. A *probability law* for the experiment E is a rule that assigns to each event A a number $P[A]$, called the probability of A, that satisfies the following axioms:

Axiom I	$0 \le P[A]$
Axiom II	$P[S] = 1$
Axiom III	If $A \cap B = \phi$, then $P[A \cup B] = P[A] + P[B]$
	(A and B are mutually exclusive events)

Corollary 1 $P[A^c] = 1 - P[A]$

As A $\cap$ $A^c = \phi$, from Axiom III, $P[A \cup A^c] = P[A] + P[A^c]$

Since $S = A \cup A^c$, by Axiom II

$$1 = P[S] = P[A \cup A^c] = P[A] + P[A^c].$$

Corollary 2 $P[A] \leq 1$

From Corollary 1, $P[A] = 1 - P[A^c] \leq 1$ since $P[A^c] \geq 0$.

Corollary 3 $P[\phi] = 0$

Let $A = S$, $A^c = \phi$; so $P[\phi] = 1 - P[S] = 0$.

Corollary 4 If $A_1, A_2, \ldots A_n$ are pairwise mutually exclusive, then

$$P\left[\bigcup_{k=1}^{n} A_k\right] = \sum_{k=1}^{n} P[A_k], \qquad\qquad n \geq 2.$$

Proof by mathematical induction. From Axiom III, it is valid for $n = 2$. The trick is to observe that if A_{n+1} and A_j, $j = 1, \ldots, n$ are pairwise mutually exclusive, then

$$\left(\bigcup_{k=1}^{n} A_k\right) \cap A_{n+1} = \bigcup_{k=1}^{n} \left(A_k \cap A_{n+1}\right) = \bigcup_{k=1}^{n} \phi = \phi,$$

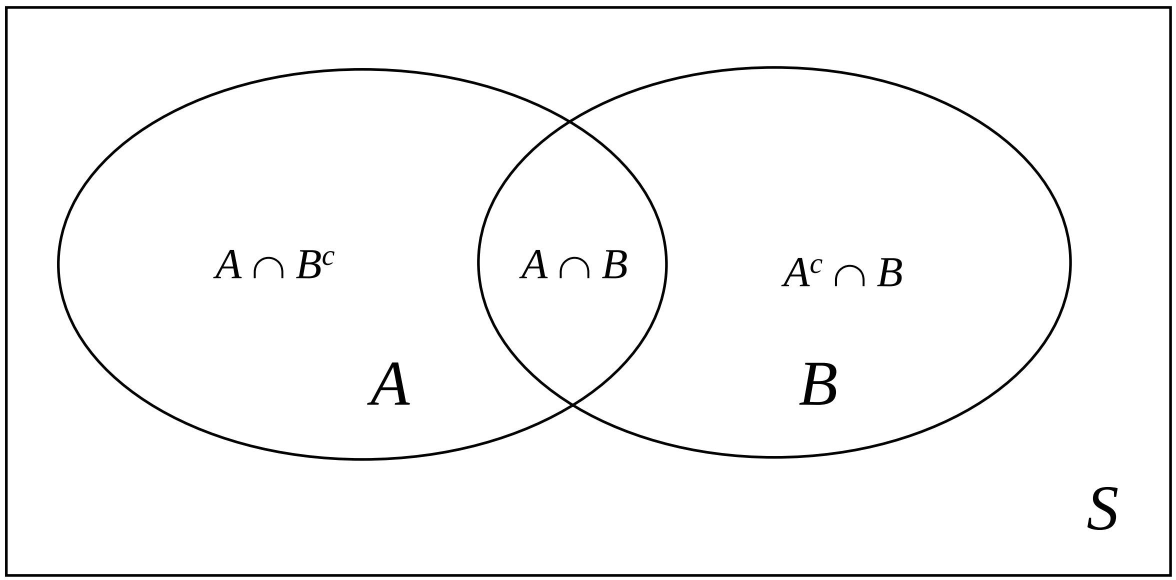
we then have

$$P\left[\bigcup_{k=1}^{n+1} A_k\right] = P\left[\left(\bigcup_{k=1}^{n} A_k\right) \cup A_{n+1}\right] = P\left[\bigcup_{k=1}^{n} A_k\right] + P[A_{n+1}].$$

Corollary 5

$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$

hence $P[A \cup B] \leq P[A] + P[B]$



Since $A \cap B^c$, $A \cap B$ and $A^c \cap B$ are disjoint events, we have

$P[A \cup B] = P[A \cap B^c] + P[B \cap A^c] + P[A \cap B]$

$P[A] = P[A \cap B^c] + P[A \cap B]$

$P[B] = P[B \cap A^c] + P[A \cap B]$.

Generalization $P[A \cup B \cup C] = P[A] + P[B] + P[C] - P[A \cap B]$
$- P[A \cap C] - P[B \cap C] + P[A \cap B \cap C]$

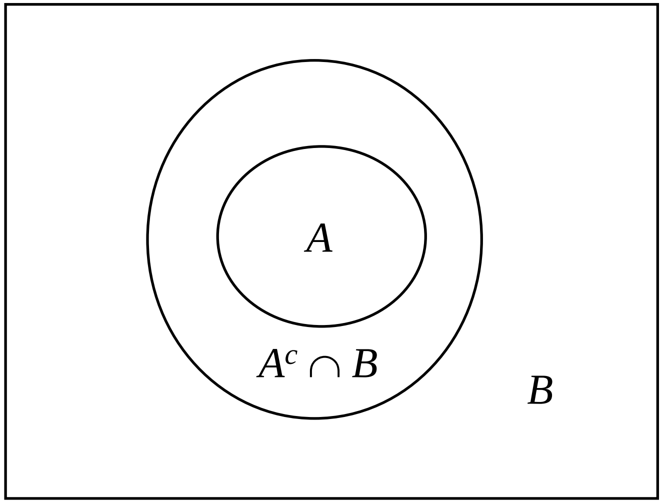
For n events, we have

$$P\left[\bigcup_{k=1}^{n} A_k\right] = \sum_{j=1}^{n} P[A_j] - \sum_{j<k} P[A_j \cap A_k] + \cdots + (-1)^{n+1} P[A_1 \cap \cdots \cap A_n].$$

Corollary 6 If $A \subset B$, then $P[A] \leq P[B]$.

$B = A \cup (A^c \cap B)$
A and $A^c \cap B$ are mutually exclusive

$P[B] = P[A] + P[A^c \cap B] \geq P[A]$



Example Toss a coin three times and observe the sequence of heads and tails. There are 8 possible outcomes:

$$S_3 = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

For a fair coin, the outcomes of S_3 are equiprobable. The outcomes are mutually exclusive, so the probability of each of the above 8 elementary events is $\frac{1}{8}$.

$$\begin{aligned} P[\text{“2 heads in 3 tosses”}] &= P[\{HHT\}, \{HTH\}, \{THH\}] \\ &= P[\{HHT\}] + P[\{HTH\}] + P[\{THH\}] = \frac{3}{8}. \end{aligned}$$

Suppose we count the number of heads in the 3 tosses. The sample space is now $S_4 = \{0, 1, 2, 3\}$.

Are the above outcomes equiprobable?

If yes, then $P[\text{“2 heads in 3 tosses”}] = P[\{2\}] = \frac{1}{4}$, a result contradicting to that of the above.

Similar question

Toss 2 dice and record the sum of face values. Is the chance of getting ‘sum = 2’ the same as that of ‘sum = 3’?