Conditional probabilities

Interested in calculating probabilities when some *partial information* about the outcome of the random experiment is available.

Example Tossing 2 dice

Suppose the first die is ‘3’; given this information, what is the probability that the sum of the 2 dice equals 8?

There are 6 possible outcomes, given the first dice is ‘3’:
(3, 1), (3, 2), (3, 3), (3, 4), ***(3, 5)***, and (3, 6).

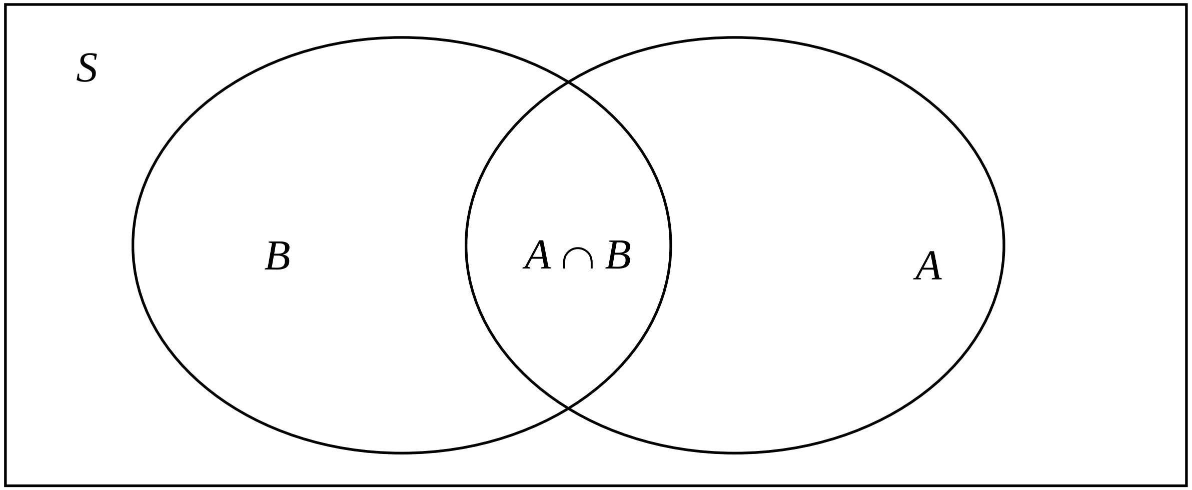
The outcomes are equally probable. Hence, the probability that the sum is 8, given the first dice is ‘3’, is $\frac{1}{6}$.

Definition

The probability of an event A occurring when it is known that some event B has occurred is called the *conditional probability* $P[A|B]$. It is defined by

$$P[A \mid B] = \frac{P[A \cap B]}{P[B]}, P[B] > 0.$$

Remark: $P[A|B]$ is meaningless if $P[B] = 0$.



We may view the experiment as now having the *reduced* sample space B.

Example

Let A = sum of the 2 dice is 8

B = first dice is ‘3’.

$P[A|B] = P[A \cap B]/P[B] = 1/6.$

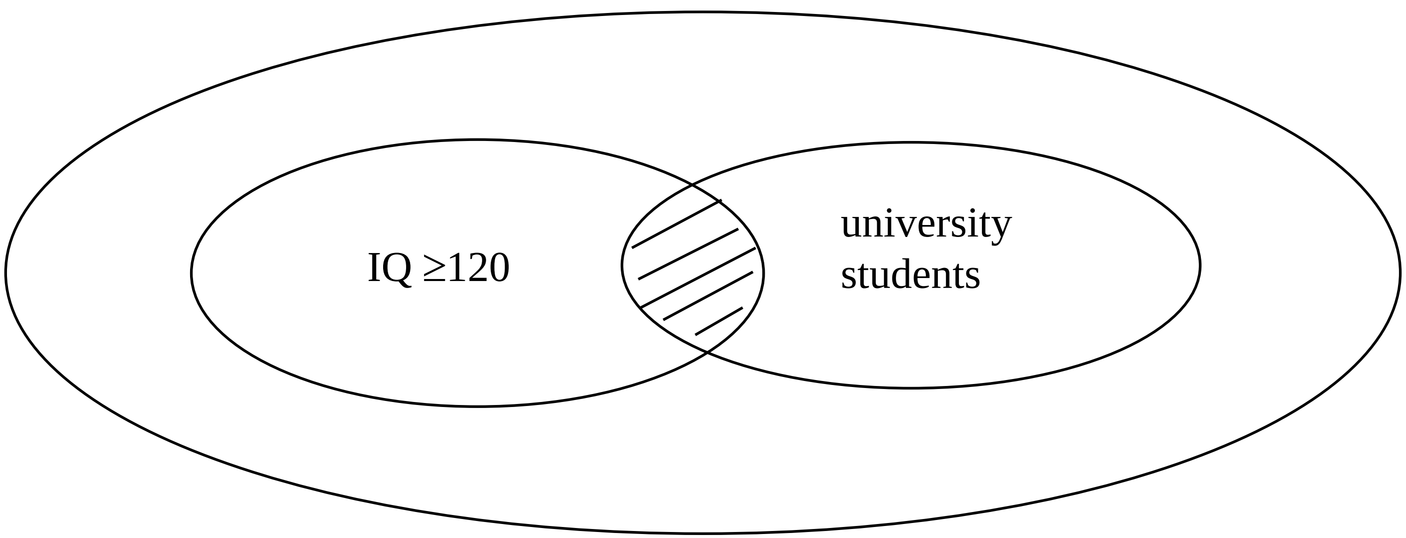
Example

S = set of all human beings;

A = set of people with IQ $\geq$ 120;

B = set of people currently attending universities

$$P[A \mid B] = \frac{P[A \cap B]}{P[B]} = \frac{\text{number of university students with IQ} \geq 120}{\text{total number of university students}}$$



Example

From the set of all families with 2 children, a family is selected at random and is found to have a girl. What is the probability that the other child of the family is a girl? Assume that in two-child families all sex distributions are equally probable.

Solution

A = family has 2 girls
B = family has a girl; $P(B) = 3/4$ and $P(A \cap B) = 1/4$

so $$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{3/4} = 1/3.$$

The reduced sample space $B = \{(g, b), (b, g), (g, g)\}$.

Example

Let A be the event that a married man watches the show,
B be the event that a married woman watches the show.

Given $P(A) = 0.4$ and $P(B) = 0.5$; also

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = 0.7.$$

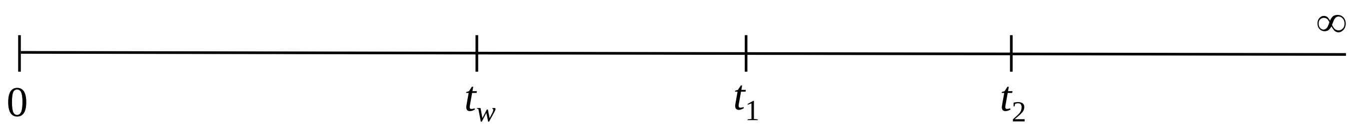
(a) Probability that a married couple watch the show
$= P(A \cap B) = P(B)\,P(A|B) = (0.5)(0.7) = 0.35.$

(b) Probability that a wife watches the show given that her husband does

$$= P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{0.35}{0.4} = 0.875.$$

(c) Probability that at least 1 person watches the show
$= P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.4 + 0.5 - 0.35 = 0.55.$

Example The probability that a component lasts for at least t hours before it fails is $\exp(-at^2)$. Knowing that the component is working until time t_w, find the probability that it will fail within the time interval $[t_1, t_2]$, where $t_w < t_1 < t_2$.



Let A_1 = failure in time interval (t_1, ∞); $P(A_1) = \exp(-at_1^2)$
A_2 = failure in time interval (t_2, ∞); $P(A_2) = \exp(-at_2^2)$
A_{12} = failure in time interval $(t_1, t_2]$.

Events A_{12} and A_2 are mutually exclusive and $A_2 \cup A_{12} = A_1$.

Hence, $P[A_2 \cup A_{12}] = P[A_2] + P[A_{12}] = P[A_1]$;

we then have $P[A_{12}] = P[A_1] - P[A_2] = \exp(-at_1^2) - \exp(-at_2^2)$.

The conditional event B = failure in time interval $[t_w, \infty)$, whose probability is

$$P[B] = \exp(-at_w^2).$$

It is obvious that $A_{12} \cap B = A_{12}$ since $A_{12} \subset B$.

Hence,

$$P[A_{12} \mid B] = \frac{P[A_{12} \cap B]}{P[B]} = \frac{\exp(-at_1^2) - \exp(-at_2^2)}{\exp(-at_w^2)}.$$

Properties of conditional probabilities

1. If $B \subset A$, then $P[A|B] = 1$.
2. For a given event B, $P[A|B]$ is a probability measure satisfying the following properties derived from the 3 axioms of probability law:

Axiom I $\quad P[A|B] \geq 0$

Axiom II $\quad P[S|B] = 1 \quad \left(\text{proof} : \dfrac{P[S \cap B]}{P[B]} = 1\right)$

Axiom III $\quad$ If $A_1 \cap A_2 = \phi$, then

$$P[A_1 \cup A_2|B] = P[A_1|B] + P[A_2|B].$$

Remark:

$$P[A_1 \cup A_2 \mid B] = \frac{P[A_1 \cup A_2) \cap B]}{P[B]}$$

$$= \frac{P[A_1 \cap B) \cup (A_2 \cap B)]}{P[B]}$$

$$= \frac{P[A_1 \cap B]}{P[B]} + \frac{P[A_2 \cap B]}{P[B]},$$

since $A_1 \cap B$ and $A_2 \cap B$ are mutually exclusive.

Independence

An event A is said to be independent of an event B if the probability that A occurs is not influenced by whether B has or has not occurred, that is

$$P[A] = P[A|B].$$

It then follows that

$$P[A \cap B] = P[B]\, P[A|B] = P[A]\, P[B].$$

Remark $P[A] = P[A|B]$ would imply $P[B] = P[B|A]$.

Example Suppose a mother gave birth a baby girl as her first child, the event that the second child is a boy is independent of the event that her first child is a girl.

Example A urn contains 5 red balls and 5 blue balls.
R_1 = a red ball is drawn in the first draw
R_2 = a red ball is drawn in the second draw
R_1 and R_2 are independent only if the ball drawn in the first draw is replaced.

Example A fair coin is tossed three times

A = {first toss is head}

B = {second toss is head}

C = {exactly two heads are tossed in consecutive tosses}

which pair(s) of events are independent?

$A = \{HHH, HHT, HTH, HTT\}$

$B = \{HHH, HHT, THH, THT\}$

$C = \{HHT, THH\}$

Since the sample space is equiprobable for a fair coin,

$$P[A] = P[B] = \frac{1}{2} \quad \text{and} \quad P[C] = \frac{1}{4};$$

$A \cap B = \{HHH, HHT\}$, $A \cap C = \{HHT\}$, $B \cap C = \{HHT, THH\}$,

then

$$P[A \cap B] = \frac{1}{4} = P[A]P[B].$$ Hence, A and B are independent.

$$P[A \cap C] = \frac{1}{8} = P[A]P[C].$$ Hence, A and C are independent.

$$P[B \cap C] = \frac{1}{4} \neq P[B]P[C].$$ Hence, B and C are dependent.

Example

A random experiment of giving birth to two children. The sample space is $S = \{BB, BG, GB, GG\}$.

BB	BG
GB	GG

Suppose p = prob of giving birth to a boy, then

$$P[\{BB\}] = p^2, \qquad P[\{BG\}] = P[\{GB\}] = p(1-p),$$
$$P[\{GG\}] = (1-p)^2.$$

There are many events that can be defined, which are subsets of S.
Total number of subsets $= C_0^4 + C_1^4 + C_2^4 + C_3^4 + C_4^4 = 2^4 = 16$.

$E_1 = \{GB, GG\}$ (the senior child is a girl)

$E_2 = \{BG, BB\}$ (the senior child is a boy)

$E_3 = \{BB, BG, GB\}$ (at least one child is a boy)

$E_4 = \{GG, GB, BG\}$ (at least one child is a girl)

$E_5 = \{BB\}$ (both children are boys)

$E_6 = \{GG\}$ (both children are girls)

$E_7 = \{GG, BG\}$ (the junior child is a girl)

$E_8 = \{BB, GB\}$ (the junior child is a boy)

E_5 and E_6 are mutually exclusive since $E_5 \cap E_6 = \phi$.
E_3 and E_6 are mutually exclusive.

$$P[E_6 \mid E_4] = \frac{P[E_6 \cap E_4]}{P[E_4]} = \frac{(1-p)^2}{1-p^2} \quad \text{since } E_6 \cap E_4 = \{GG\}$$

Consider E_1 and E_8, $E_1 \cap E_8 = \{GB\}$

$$P[E_1 \mid E_8] = \frac{P[E_1 \cap E_8]}{P[E_8]} = \frac{p(1-p)}{p^2 + p(1-p)} = 1 - p = P[E_1].$$

Since $P[E_1|E_8] = P[E_1]$, E_1 and E_8 are said to be **independent**.
Intuitively, the event that the senior child is a girl
should be **independent** from the event that the junior child is a boy.

Note that $E_1 \cap E_8 = \{GB\} \neq \phi$, so E_1 and E_8 are not mutually exclusive.

Events E_5 and E_6 are mutually exclusive, but

$$P[E_5 \mid E_6] = \frac{P[E_5 \cap E_6]}{P[E_6]} = \frac{P[\phi]}{P[E_6]} = 0 \neq P[E_5],$$

so E_5 and E_6 are **not independent**.

For events E_5 and E_3, we have

$$P[E_5 \mid E_3] = \frac{P[E_5 \cap E_3]}{P[E_3]} = \frac{p^2}{1-(1-p)^2} \neq P[E_5];$$

so E_3 and E_5 are **not independent**.

Lastly, we observe that

$$P[E_3 \mid E_5] = \frac{P[E_3 \cap E_5]}{P[E_5]} = \frac{p^2}{p^2} = 1 \text{ since } E_5 \subset E_3.$$

Lemma

If A and B are independent events, then A and B^C are independent and A^C and B are independent.

Proof

Since events A and B are independent, then

$$P[A \cap B] = P[A]P[B].$$

From set theory, we have

$$A = (A \cap B) \cup (A \cap B^C).$$

Also, $A \cap B$ and $A \cap B^C$ are mutually exclusive events. By Axiom 3 of probability theory

$$P[A] = P[A \cap B] + P[A \cap B^C].$$

$$\begin{aligned} P[A \cap B^C] &= P[A] - P[A \cap B] = P[A] - P[A]\,P[B] \\ &= P[A]\,[1 - P[B]] = P[A]\,P[B^C], \end{aligned}$$

so A and B^C are independent.

Three events A, B and C are *independent* if

(1) The events are *pairwise independent*, that is,

$$P[A \cap B] = P[A]\, P[B],$$
$$P[A \cap C] = P[A]\, P[C],$$
$$P[B \cap C] = P[B]\, P[C].$$

(2) $P[A \cap B \cap C] = P[A]\, P[B]\, P[C]$.

Remark

1. Three events can be pairwise independent, but not independent.
2. The definition requires $P[A \cap B \cap C] = P[A]\, P[B]\, P[C]$ in additional to pairwise independence. This is because independence naturally requires that A should be independent of any event formed from B and C. For example, we may require $P[A \mid B \cap C] = P[A]$. Now, $P[A \mid B \cap C] = \dfrac{P[A \cap B \cap C]}{P[B \cap C]}$ and since B and C are independent so $P[B \cap C] = P[B]\, P[C]$. Hence, we also require

   $$P[A \cap B \cap C] = P[A \mid B \cap C]\, P[B \cap C] = P[A]\, P[B]\, P[C].$$

Example

Let a pair of fair coins be tossed. The sample space $S = \{HH, HT, TH, TT\}$ is an equiprobable sample space. Define the events

$$\begin{aligned} A &= \{\text{head on the first coin}\} &&= \{HH, HT\} \\ B &= \{\text{head on the second coin}\} &&= \{HH, TH\} \\ C &= \{\text{head on exactly one coin}\} &&= \{HT, TH\}. \end{aligned}$$

Show that the events are pairwise independent but not independent.

Now $P[A] = P[B] = P[C] = \frac{1}{2}$ and

$$P[A \cap B] = P[\{HH\}] = \frac{1}{4}, P[A \cap C] = P[\{HT\}] = \frac{1}{4}, P[B \cap C] = P[\{TH\}] = \frac{1}{4}.$$

Thus condition (1) is satisfied, that is, the events are pairwise independent. However $A \cap B \cap C = \phi$, so $P[A \cap B \cap C] = 0$; but

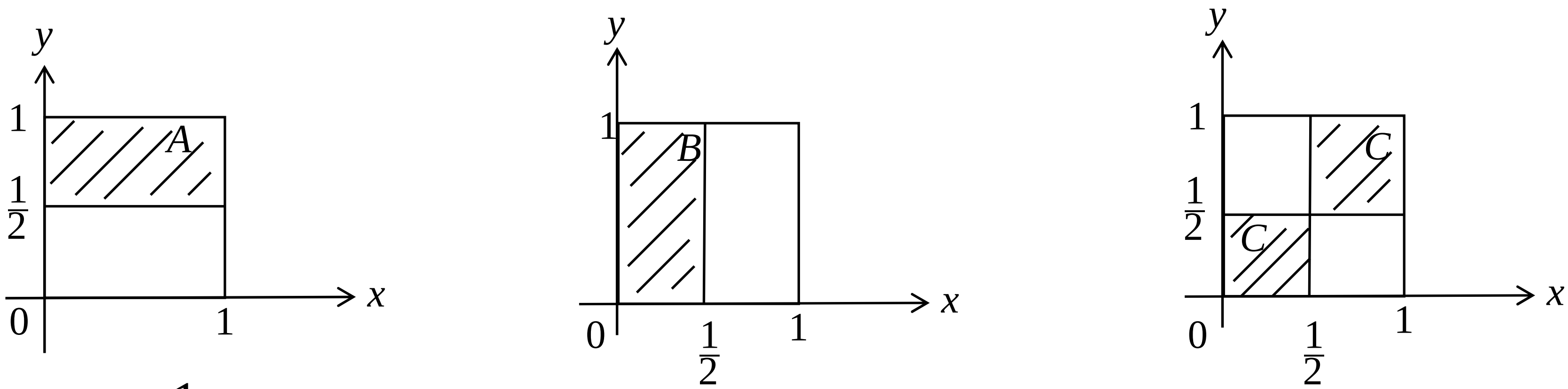
$$P[A]P[B]P[C] = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8} \neq 0.$$

Thus, condition (2) is not satisfied.

Events A, B and C are not independent.

Example Two numbers x and y are selected at random between zero and one. Let events A, B and C be defined by

$$A = \left\{ y > \frac{1}{2} \right\} \qquad B = \left\{ x < \frac{1}{2} \right\} \qquad C = \left\{ x < \frac{1}{2}, y < \frac{1}{2} \right\} \cup \left\{ x > \frac{1}{2}, y > \frac{1}{2} \right\}$$



$$P[A \cap B] = \frac{1}{4} = P[A]P[B]$$

$$P[A \cap C] = \frac{1}{4} = P[A]P[C]$$

$$P[B \cap C] = \frac{1}{4} = P[B]P[C]$$

However, $A \cap B \cap C = \phi$, so

$$P[A \cap B \cap C] = P[\phi] = 0 \neq P[A]P[B]P[C] = \frac{1}{8}.$$

Properties of independence and mutual exclusiveness

(1) If $A \cap B = \phi$, then $P[A|B] = 0$.
A can never occur if B has occurred.

(2) If $A \neq \phi$, $B \neq \phi$ and A and B are independent, then A and B are not mutually exclusive. This is because

$$P[A \cap B] = P[A]\, P[B] \neq 0.$$

Remarks We have seen that if A and B are mutually exclusive, then A and B cannot be independent.

(3) If A and B are independent and $A \cap B = \phi$, then either $A = \phi$ or $B = \phi$ or both. This is because

$$0 = P[A \cap B] = P[A]\, P[B]$$

so that either $P[A] = 0$, $P[B] = 0$ or both are equal.

BAYES' THEOREM

Suppose the events $A_1, A_2, \ldots, A_n$ form a partition of a sample space S.
A *partition* of set S is a set $\{A_1, A_2, \ldots, A_n\}$ with the following properties:

(i) $A_j \subseteq S$ $\qquad$ $(j = 1, 2, \ldots, n)$

(ii) $A_j \cap A_k = \phi$ $\qquad$ $(j = 1, 2, \ldots, n;\ k = 1, 2, \ldots, n;\ j \neq k)$

(iii) $A_1 \cup A_2 \cup \ \ldots \cup A_n = S$

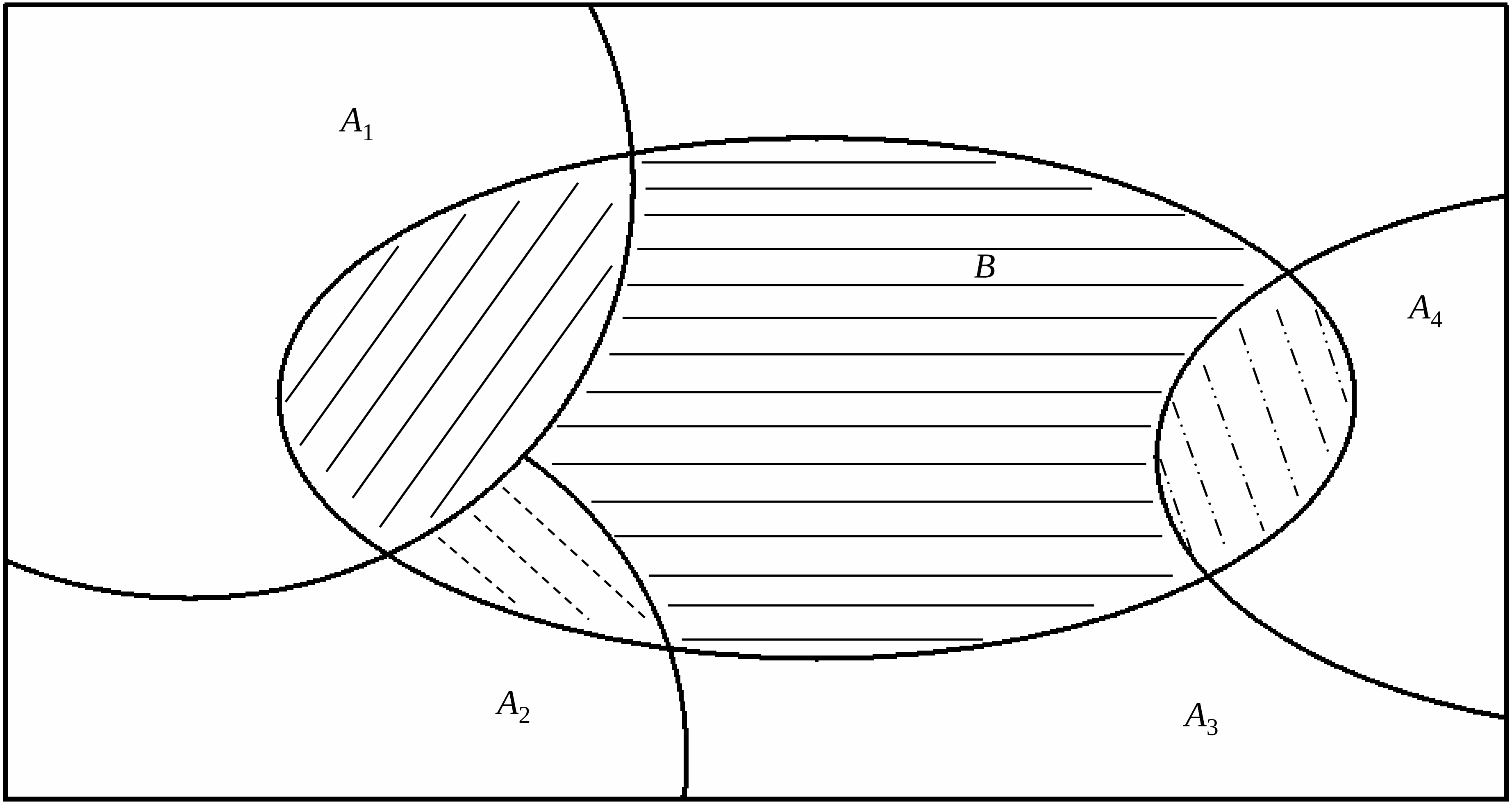
A partition of a set S is a set of subsets of S [property (i)] that are disjoint [property (ii)] and exhaustive [property (iii)]. Every element of S is a member of one and only one of the subsets in the partition.

Bayes' theorem: Suppose $\{A_1, A_2, \ldots, A_n\}$ is a partition of S and B is any event. Then for any event A_j

$$P[A_j \mid B] = \frac{P[A_j]P[B \mid A_j]}{P[A_1]P[B \mid A_1] + P[A_2]P[B \mid A_2] + \ldots + P[A_n]P[B \mid A_n]}.$$

The theorem is useful when it is relatively easier to compute $P[B|A_j]$, $j = 1, \ldots, n$.

Partitions and Bayes' Theorem



Now let B be any other event. We decompose B as follows:

$$\begin{aligned} B = S \cap B &= (A_1 \cup A_2 \cup \ldots \cup A_n) \cap B \\ &= (A_1 \cap B) \cup \ldots \cup (A_n \cap B). \end{aligned}$$

Since $A_j \cap B$ are also mutually exclusive, we have

$$P[B] = P[A_1 \cap B] + P[A_2 \cap B] + \ldots + P[A_n \cap B];$$

and by multiplication theorem

$$P[B] = P[A_1]P[B \mid A_1] + P[A_2]P[B \mid A_2] + \ldots + P[A_n]P[B \mid A_n]. \qquad \text{(i)}$$

This is called the *law of total probabilities*.

On the other hand,

$$P[A_j \mid B] = \frac{P[A_j \cap B]}{P[B]} \qquad \text{(ii)}$$

For any event A_j, we have

$$P[A_j \cap B] = P[A_j]P[B \mid A_j]. \qquad \text{(iii)}$$

Combining Eqs. (i), (ii) and (iii) together, we obtain the theorem.

Example 1

Three urns contain coloured balls:

Urn	Red	White	Blue
1	3	4	1
2	1	2	3
3	4	3	2

One urn is chosen at random and a ball is withdrawn.

(a) What is the probability that a white ball is drawn?

(b) Suppose a red ball is drawn. What is the probability that it came from urn 2?

Solution

Let E_i be the event that the ith urn is selected, $i = 1, 2, 3$. Let S be the sample space of this experiment – selecting a urn and drawing a ball. Then E_1, E_2 and E_3 form a partition of S. Moreover, since the urn is selected at random, it must be

$$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}.$$

(a) Let W be the event that a white ball is drawn.

$$P(W) = P(E_1)P(W \mid E_1) + P(E_2)P(W \mid E_2) + P(E_3)P(W \mid E_3).$$

$$P(W \mid E_1) = \frac{4}{8}, P(W \mid E_2) = \frac{2}{6} \quad \text{and} \quad P(W \mid E_3) = \frac{3}{9}.$$

Then

$$P(W) = \frac{1}{3}\cdot\frac{4}{8} + \frac{1}{3}\cdot\frac{2}{6} + \frac{1}{3}\cdot\frac{3}{9} = \frac{7}{18}.$$

(b) Let B be the event that the ball withdrawn is red. The probability that the chosen red ball is from urn 2 = $P(E_2|B)$. By Bayes' theorem,

$$P(E_2 \mid B) = \frac{P(E_2)P(B \mid E_2)}{P(E_1)P(B \mid E_1) + P(E_2)P(B \mid E_2) + P(E_3)P(B \mid E_3)}.$$

Now, $P(B|E_i)$ = probability of drawing a red ball given that the ith urn is chosen. Using the information from the table:

$$P(B \mid E_1) = \frac{3}{8}, P(B \mid E_2) = \frac{1}{6} \quad \text{and} \quad P(B \mid E_3) = \frac{4}{9},$$

we obtain

$$P(E_2 \mid B) = \frac{\frac{1}{3}\cdot\frac{1}{6}}{\frac{1}{3}\cdot\frac{3}{8} + \frac{1}{3}\cdot\frac{1}{6} + \frac{1}{3}\cdot\frac{4}{9}} = \frac{12}{71}.$$

Example 2

Suppose we have 3 cards whose sides are coloured as follows

red/red	black/black	red/black

A card is chosen at random and placed on the ground. If the upper side of the card is red, what is the probability that the other side is black?

Define RR = event that the red/red card is drawn

RB = event that the red/black card is drawn

BB = event that the black/black card is drawn

R = event that the upturned card is red

By Bayes' theorem,

$$P[RB/R] = \frac{P[R \mid RB]P[RB]}{P[R \mid RR]P[RR] + P[R \mid RB]P[RB] + P[R \mid BB]P[BB]}$$

$$= \frac{\frac{1}{2}\cdot\frac{1}{3}}{1\cdot\frac{1}{3} + \frac{1}{2}\cdot\frac{1}{3} + 0\cdot\frac{1}{3}} = \frac{1}{3}.$$

Remark

1. At the first glance, it appears that there are two possibilities: the card with two red sides and the card with red/black sides. One might expect the required probability to be ½! Why this is incorrect?

2. This problem is analogous to Example 1, where the 3 cards can be visualized as 3 urns.

Alternative statement of the present problem:

Urn 1 has two red balls, Urn 2 has two black balls, Urn 3 has one red ball and one black ball. One red ball has been drawn, what is the probability that the red ball comes from Urn 3.

Example 3

In a trial, the judge is 65% sure that Susan has committed a crime. Person *F* (friend) and Person *E* (enemy) are two witnesses who know whether Susan is innocent or guilty.

- Person *F* is Susan's friend and will lie with probability 0.25 if Susan is guilty. He will tell the true if Susan is innocent.
- Person *E* is Susan's enemy and will lie with probability 0.30 if Susan is innocent. Person *E* will tell the truth if Susan is guilty.

What is the probability that Person *F* and Person *E* will give conflicting testimony?

Solution

Let I and G be the two mutually exclusive events that Susan is innocent and guilty, respectively. Let C be the event that the two witnesses will give conflicting testimony. Find $P[C]$ based on $P[C|I]$ and $P[C|G]$.

By the law of total probabilities

$$P[C] = P[C \mid I]P[I] + P[C \mid G]P[G]$$
$$= 0.30 \times 0.35 + 0.25 \times 0.65 = 0.2675.$$

Example 4

A judge is 65% sure that a suspect has committed a crime. During the course of the trial, a witness convinces the judge that there is an 85% chance that the criminal is left-handed. If 23% of the population is left-handed and the suspect is also left-handed. With this *new information*, how certain should the judge be of the guilt of the suspect?

Solution

G = event that the suspect is guilty
I = event that the suspect is innocent
L = event that the suspect is left-handed

Since $\{G, I\}$ forms a partition of the sample space, by Bayes' Theorem:

$$P[G \mid L] = \frac{P[L \mid G]P[G]}{P[L \mid G]P[G] + P[L \mid I]P[I]}$$

$$= \frac{0.85 \times 0.65}{0.85 \times 0.65 + 0.23 \times 0.35} = 0.87.$$

Remark

1. How to find $P[L|I]$? When the suspect is innocent, he is considered as an average Joe in the population, so his probability of being left-handed should be the same as that of the whole population.

2. Note that $P[L]$ is the probability that the suspect is left-handed, and its value is given by

   $$P[L] = P[L|G]\,P[G] + P[L|I]\,P[I] = 0.85 \times 0.65 + 0.23 \times 0.35.$$

   This is not the same as $P[L|I]$.

3. The statement "the suspect is also left-handed" does not mean $P[L] = 1$. It just induces us to consider $P[G|L]$, the probability that the suspect is guilty, conditional on L occurs.

4. With the additional information that the suspect is left-handed, what would be the effect on the hypothesis that the suspect is guilty?

$$P[G \mid L] = \frac{P[L \mid G]P[G]}{P[L \mid G]P[G] + P[L \mid I]P[I]}$$

$$P[I \mid L] = \frac{P[L \mid I]P[I]}{P[L \mid G]P[G] + P[L \mid I]P[I]}$$

$$\frac{P[G \mid L]}{P[I \mid L]} = \frac{P[L \mid G]}{P[L \mid I]} \cdot \frac{P[G]}{P[I]} = \frac{0.85}{0.23} \cdot \frac{0.65}{0.35}.$$

The original odds ratio $P[G]/P[I] = 0.65/0.35$ is increased by the factor $P[L|G]/P[L/I] = 0.85/0.23$ to the new odds ratio $P[G|L]/P[I|L]$.